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Statistical methods applicable to analysis of aircraft performance data



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THE PREPARATION OF THIS DATA ITEM

The work on this particular Data Item was monitored and guided by the Performance Committee, which first met in 1946 and at the time of first issue of the Item had the following membership:

Chairman	
Mr W.L. Horsley	— Civil Aviation Authority, Airworthiness Division
Vice-Chairman	
Mr A.N. Page	— British Aerospace (Commercial Aircraft) Ltd, Hatfield
Members	
Mr K.J. Balkwill	— British Aerospace (Military Aircraft) Ltd, Brough
Mr T. Bartup	— British Aerospace (Commercial Aircraft) Ltd, Woodford
Mr T.B.A. Boughton	— Independent
Mr E.N. Brailsford	— Independent
Mr M. Broad	— Independent
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Mr P.D. Feig*	— General Electric Company, Cincinnati, Ohio, USA
Mr N.J. Herniman	— Airbus Industrie
Mr R.G. Humpston	— Rolls-Royce Ltd, Derby Engine Division
Dr L.R. Jenkinson	— Loughborough University of Technology
Mr T.S.R. Jordan	— Independent
Mr R. Kita*	— Grumman Aircraft Systems, Bethpage, NY, USA
Mr W.T. Lewerenz*	— McDonnell Douglas, Long Beach, Calif., USA
Mr E.J. Norris	— Aeroplane and Armament Experimental Establishment
Mr P. Robinson	— Independent
Mr R.C. Shields*	— Boeing Commercial Airplane Company, Seattle, Wash., USA
Mr G.E. Smith	— Independent
Mr A. Stanbrook	— Independent
Prof. E. Torenbeek*	— Delft University of Technology.

* Corresponding Member

The technical work involved in the assessment of the available information and the construction and subsequent development of the Data Item was carried out under contract to ESDU by the then Chairman of the Performance Committee,

Mr W.L. Horsley.

The additional material that constitutes Amendments C and E of this Item was prepared by

Mr K.J. Balkwill.

The person with overall responsibility for the work in this subject area at the time of first issue of the Item was Mr D.J. Mitchell.

STATISTICAL METHODS APPLICABLE TO ANALYSIS OF AIRCRAFT PERFORMANCE DATA

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STATISTICAL METHODS APPLICABLE TO ANALYSIS OF AIRCRAFT PERFORMANCE DATA

1. NOTATION

The only units in this Data Item appear by way of examples in the text.

A	Anderson-Darling test statistic (for Normal distributions)
A^*	modified Anderson-Darling test statistic
a, b	sample regression line parameters
B	Beta distribution
C	function defined by Equation (5.9)
cf	cumulative frequency
d	output error of a system
e	expected frequency
F	F Variance Ratio test statistic, see Section 5.3
f	denotes functional relationships, $f[\dots]$
f	actual frequency
I	number of intervals within a distribution
K	number of samples from same distribution
m	number of component parameters of a variable
n	size of sample
p	size of population
q	probability
R	error sum of squares
r	correlation coefficient of sample
SE	residual error sum of squares
SR	regression sum of squares
s	standard deviation of sample
s^2	variance of sample
s_p	pooled estimate of standard deviation from multiple samples

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$\hat{s}^2$	best estimate of population variance based on small sample
t	Student's t statistic, see Section 4 and Table 12.3
u	specific value of x
V_R	rotation speed (aircraft take-off performance)
v_1, v_2, v_3	transformations defined in Section 9
w	component parameter of variable x
x	statistical variable (independent)
$\bar{x}$	mean value of a sample of variable x
y	statistical variable (dependent)
$\bar{y}$	mean value of a sample of (dependent) variable y
$\hat{y}$	predicted value of y (best estimate)
y_x	regression of y with respect to x (x = independent variable, y = dependent variable)
z	standardised Normal variable, $(x - \mu)/\sigma$, (margin from mean of a probability distribution), see Table 12.1
α, β	regression line parameters of population
ΔV_R	actual rotation speed minus target rotation speed
ζ	function defined in Section 6
η_i	expected value of y_i in regression y_x
μ	mean of population
ν	number of degrees of freedom
ξ_w	influence coefficient, $(w/x)(\partial x/\partial W)$ for effect on x of small change in w
ρ	correlation coefficient of population
σ	standard deviation of population
σ^2	variance of population
$\Phi[x]$	cumulative probability density of variable x , see Section 2.2
$\phi[x]$	probability density of variable x , see Section 2.2
χ^2	“Chi-squared” test statistic, see Section 5.1

Subscripts

a, b	denote values corresponding to sample regression line parameters
I	denotes I th interval within a distribution
i	denotes i th value of any quantity
N	denotes Normal Distribution
w	denotes value due to component parameter w
x	denotes parameter of the distribution of a statistical variable x
α, β	denote values corresponding to population regression line parameters

2. INTRODUCTION

2.1 General

The analysis of aircraft performance data frequently involves the need to resolve scatter. Common problems of this type arise in the use of flight-measured data for the following.

To substantiate an acceptable dispersion of a parameter such as autoland touchdown point.

To confirm or correct an empirical performance estimation method.

To confirm or correct a mathematical model of an aircraft performance characteristic such as climb gradient, landing distance, cruise performance, *etc.*

To establish the effect, or lack of effect, on an aircraft performance characteristic due to some physical modification to the aircraft (such as a change in engine rating or the addition of vortex generators, wing fences or fillets).

To monitor “in-service” performance to get early warning of any performance degradation or the onset of an airframe or engine defect.

Engineers often tackle such problems on the basis of their past experience of similar problems and seek to resolve any residual scatter “by eye”. Where it is appropriate to apply margins this too is usually done intuitively, on the basis of past experience and a knowledge of the likely consequences of being “wrong”, in one direction (over cautious) or the other (over optimistic). This process is so well established that the term “Engineering Judgement” has been coined to describe it and in many cases it is an entirely appropriate and adequate procedure. However, there are occasions where the use of a more objective numerical method is called for and Sections 3 through 8 describe several such methods based on well-established statistical principles. These methods all have direct applications in the analysis of differing types of aircraft performance data, as is illustrated in References 1 to 7, a series of Data Items which provide worked examples covering a variety of aircraft performance problems.

It is important to appreciate that statistical methods do not entirely replace the use of engineering judgement; rather they should be seen as supplementary to it, providing engineers with the ability to be better judges.

Statistical analysis is a large subject and it would not be appropriate here to give a complete coverage. Consequently, the sections that follow give brief summaries of a number of statistical principles and illustrations of their use in the field of aircraft performance but no “proofs” are offered. **The Data Item is not intended as a text book on Statistics; anyone wishing to pursue the subject in greater depth should consult one or more of the many specialist text books.**

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2.2 Probability Distributions

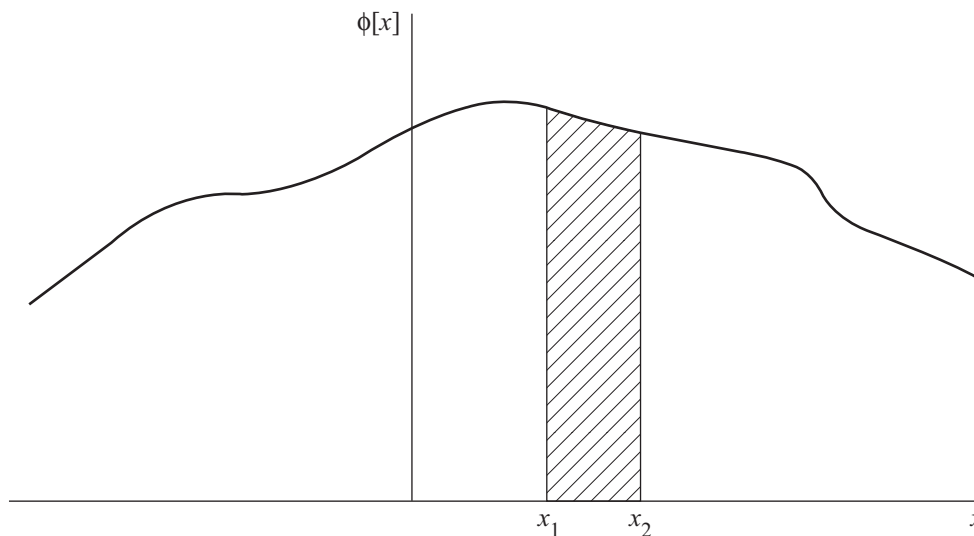
Throughout this Data Item the term “distribution” is used as shorthand for “continuous probability distribution” and all distributions considered here are continuous or are assumed to be continuous for the purposes of practical calculations. In its turn, probability distribution may not be a term that is readily understood so that some description of its meaning should be offered.

In representing the distribution of a statistical variable x , it is usual to plot the probability density, $\phi[x]$, as the ordinate against x (see Sketch 2.1). This quantity is such that the probability, q , of x falling between the two values x_1 and x_2 is given by

$$q = \int_{x_1}^{x_2} \phi[x] dx,$$

i.e. by the shaded area under the curve in Sketch 2.1 (hence the references to “areas in tail” in Table 12.1). The total area under the “distribution” must be unity and the cumulative distribution, $\Phi[x]$, is the way the area under the curve varies with x . The area $\Phi[x]$ gives the probability that $x < x_1$ (say). Note that the probability that $x > x_1$ is given by $(1 - \Phi[x])$.

It is possible for probability distributions to have all kinds of shapes (see Sketch 2.1) but arguably the most useful is the Normal Distribution whose properties are described in Section 3. Some distributions are symmetric – as is the Normal Distribution – and some are skew. Ways of dealing with distributions other than the Normal are outlined in Section 6.



Sketch 2.1 Schematic representation of probability density of statistical variable x

2.3 Statistical Techniques

The statistical analysis techniques described in this Data Item fall into three main categories:

those involving the use of the Normal Distribution and its properties (see Sections 2.3.1, 3, 5 and 6),

those involving the use of Student’s t Distribution (see Sections 2.3.2 and 4), and

those involving the use of regression techniques and the properties of the Normal Distribution (see Section 2.3.3 and Sections 7, 8 and 9).

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2.3.1 Techniques involving use of the Normal Distribution

To the extent that it is possible to make generalised statements about these techniques, the methods used tend to follow the following stages.

- (a) Define the parameters (mean and standard deviation) of a Normal Distribution that give a good fit to the data sample available. Usually this means applying Equations (2.1b) and (2.6b) to calculate $\bar{x}$ and s_x , or plotting the sample data on Normal probability paper (see Section 3.3.2 and Reference 2) to estimate values of $\bar{x}$ and s_x .
- (b) Apply the χ^2 test (see Section 5.1 and Reference 2) to ensure that the assumed distribution is an acceptably good fit to the available data.
- (c) If the χ^2 test shows inadequate fit, apply a transformation to the data (see Section 6 and Reference 2) and apply the χ^2 test again to the transformed distribution. Repeat this process until an acceptable distribution is found.
- (d) When an acceptable distribution has been found, use it and its properties as a means to generalise from the sample data, usually by setting up confidence intervals for the values of individual test points or the population mean.

2.3.2 Techniques involving use of Student's t Distribution

Where the mean and standard deviation of a distribution have to be estimated on the basis of a small sample or a few small samples only and the analysis involves sample means, Student's t Distribution needs to be used instead of the Normal Distribution in working through the stages of Section 2.3.1. In these cases the sample sizes may be too small to permit the use of the χ^2 test.

2.3.3 Techniques involving use of Regression Procedures and properties of the Normal Distribution

The term "regression" is used to describe the relationship between two or more variables. "Regression analysis", using the method of least squares, is frequently applied in conjunction with the statistical analysis of data.

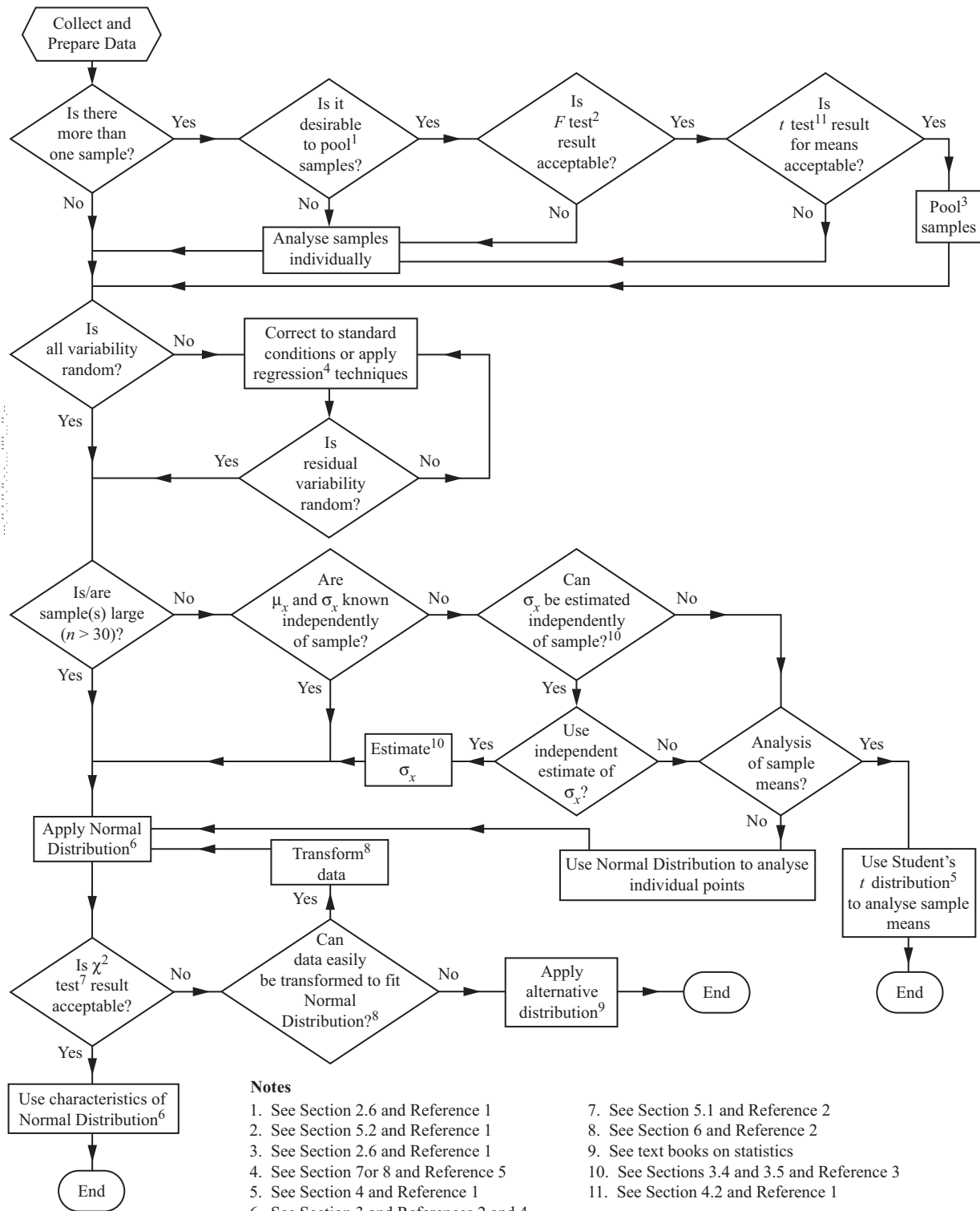
The stages of this type of problem tend to be as follows.

- (a) If the relationship between the variables is expected to be linear, calculate a best straight line regression using the method of least squares (see Section 7 or 8 and References 4 and 5).
- (b) If the relationship is expected to be non-linear, either transform the data so as to produce a linear relationship and proceed as in (a) above using the transformed data (see Reference 4), or use non-linear regression, see Section 7.8.
- (c) Once a regression has been established, check its significance (see Section 3.3 and References 4 and 5) and calculate the correlation coefficient to get a second assessment of the quality of the correlation (see Section 7.6 and Reference 4).
- (d) If an acceptable correlation has been achieved, use the properties of the Normal Distribution to set up confidence intervals about the regression line.

2.3.4 Technique Details

Sections 2.3.1 to 2.3.3 give an idea of some of the stages that are used in many statistical analyses. The Sections that follow and References 1 to 7 address these stages in more detail and apply them to some specific aircraft performance analysis problems. Sketch 2.2 gives a guide to the general logic to be applied in deciding which statistical techniques are applicable to a particular problem and the order of their application.

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Sketch 2.2

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2.4 Measures of Central Tendency

A number of measures of central tendency exist, including,

- the median,
- the mode (or modal value),
- the mean (or arithmetic mean),
- the geometric mean and
- the harmonic mean.

Each of these has its particular uses. The median and the mode are used mainly in *descriptive statistics* – the qualitative evaluation of data. The mean is widely used in quantitative applications whilst the geometric and harmonic means are seldom directly used in such applications. This Data Item deals mainly with the arithmetic mean, with occasional references to the median.

The **median** of a sample, or a population, is the value that is exceeded by one half of the sample, or population, and not exceeded by the other half.

The value of the variable whose frequency is a maximum is called a **mode** or **modal value** of the distribution. When, as is usually the case, there is only one mode, the distribution is described as **unimodal**. A distribution is said to be **bi-modal** if there are two frequency maxima in its range of existence.

The **(arithmetic) mean of a population** is given (for practical purposes) by

$$\mu_x = \frac{\sum_{p} x}{p} . \quad (2.1a)$$

(For a population having a continuous distribution, the right hand side of Equation (2.1a) is replaced by the integral of $x\phi[x]$, with respect to x , from $-\infty + \infty$.)

The **(arithmetic) mean of a sample** is given by

$$\bar{x} = \frac{\sum_{n} x}{n} . \quad (2.1b)$$

The **geometric mean** of a sample with n members is defined as the n th root of the product of the n members.

$$\bar{x}_G = \left(\prod_{i=1}^n x_i \right)^{1/n} . \quad (2.2)$$

Note that, if $y = \ln[x]$ then

$$\bar{x}_G = e^{\bar{y}} . \quad (2.3)$$

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The **harmonic mean** of a sample with n members is defined as the reciprocal of the mean value of the reciprocal of the n members.

$$\bar{x}_H = \left(\frac{1}{n} \sum_{i=1}^n \frac{1}{x_i} \right)^{-1} . \quad (2.4)$$

2.5 Measures of Dispersion

A number of measures of dispersion can be defined: the two considered in this Data Item are,

the variance,
the standard deviation.

The variance is the mean of the squares of the individual differences from the mean, and the standard deviation is the square root of the variance. These definitions may be expressed in equation form, as follows.

The variance of a population is given (for practical purposes) by

$$\sigma_x^2 = \frac{1}{p} \sum_{i=1}^p (x_i - \mu_x)^2 . \quad (2.5a)$$

(For a population having a continuous distribution, the right hand side of Equation (2.5a) is replaced by the integral of $(x - \mu_x)^2 \phi[x]$, with respect to x , from $-\infty$ to $+\infty$.)

The variance of a sample is given by

$$s_x^2 = \frac{1}{n} \sum_{i=1}^n (x_i - \bar{x})^2 . \quad (2.5b)$$

However, where the variance (s_x^2) of a sample is to be used as an estimate of the variance (σ_x^2) of the population, a better (unbiased) estimate results from using the relationship,

$$\sigma_x^2 \approx \hat{s}_x^2 = \frac{1}{(n-1)} \sum_{i=1}^n (x_i - \bar{x})^2 , \quad (2.5c)^\dagger$$

which is referred to as a “sample variance”.

[†] Many electronic calculators are programmed to calculate variance and standard deviation using Equations (2.5c) and (2.6c). Equations (2.5c) and (2.6c) are derived from Equations (2.5b) and (2.6b) respectively by multiplying them by Bessel's Correction ($n/(n-1)$). The need to apply this correction arises from the fact that Equations (2.5b) and (2.6b) are subject to a bias which causes them to tend to underestimate the variance and standard deviation of the population, particularly for small samples. Clearly, Bessel's Correction becomes unimportant for large samples.

The standard deviation of a population is given (for practical purposes) by

$$\sigma_x = \left(\frac{\sum_{i=1}^p (x_i - \mu_x)^2}{p} \right)^{1/2} \quad (2.6a)$$

(For a population having a continuous distribution, the quantity in square brackets in Equation (2.6a) is replaced by the integral of $(x - \mu_x)^2 \phi[x]$, with respect to x , from $-\infty$ to $+\infty$.)

The standard deviation of a sample is given by

$$s_x = \left(\frac{\sum_{i=1}^n (x_i - \bar{x})^2}{n} \right)^{1/2} \quad (2.6b)$$

However, as for variance, where the standard deviation (s_x) of a sample is to be used as an estimate of the standard deviation (σ_x) of a population, a better estimate results from using the relationship,

$$\sigma_x \approx \hat{s}_x = \left(\frac{\sum_{i=1}^n (x_i - \bar{x})^2}{(n-1)} \right)^{1/2} \quad \dagger (2.6c)$$

2.6 Combining Values of Standard Deviation of Samples

Where several samples are available from the same population[‡], the standard deviation of the population may be estimated by combining the values of standard deviations of the samples.

[†] Many electronic calculators are programmed to calculate variance and standard deviation using Equations (2.5c) and (2.6c). Equations (2.5c) and (2.6c) are derived from Equations (2.5b) and (2.6b) respectively by multiplying them by Bessel's Correction ($n/(n-1)$). The need to apply this correction arises from the fact that Equations (2.5b) and (2.6b) are subject to a bias which causes them to tend to underestimate the variance and standard deviation of the population, particularly for small samples. Clearly, Bessel's Correction becomes unimportant for large samples.

[‡] If there is any doubt whether the samples are from the same population see Section 5.3 – F Variance Ratio test and Section 4, Student's t distribution.

Following from Equation (2.6b), if K samples are available of $n_1, n_2, n_3, \dots, n_K$ data points with standard deviations $s_1, s_2, s_3, \dots, s_K$, the standard deviation of the population may be estimated as

$$\sigma_x \approx \left(\frac{\sum_{i=1}^K n_i s_i^2}{\left(\sum_{i=1}^K n_i \right) - K} \right)^{1/2} \quad (2.7)$$

The quantity, $(\sum_{i=1}^K n_i) - K$, in Equation (2.7) is often referred to as the number of degrees of freedom[†], ν .

2.7 Hypothesis Testing

The general theory of hypothesis testing is part of the study of Mathematical Statistics. An **Hypothesis Test** is a formal framework within which the truth of an hypothesis concerning a population parameter is established from the evidence of a sample. The data that occur most commonly in the study of Aircraft Performance are either Normally distributed or can be transformed[‡] to a Normal Distribution by a relatively simple modification of the variable. Thus, for the limited set of hypotheses that are considered in the study of Aircraft Performance data there is need only to be familiar with the ways in which the simpler test statistics can be used because these are all based on the assumption that samples are drawn from populations that are (at least approximately) Normally distributed. Specifically, the hypotheses that Performance Engineers test can be regarded as the three questions:

Are the given samples distributed approximately as hypothesised?

Are the means of the given samples significantly different?

Are the variances of the given samples approximately consistent with the hypothesis that they are not significantly different?

The mathematical theory of testing has been established to determine, where possible, the most efficient test for a given purpose. The most efficient test for a particular purpose is defined as that which minimises (or at least optimises) risk.

The most efficient test for the first question has been shown to be equivalent to the t -test which is summarised in Section 4.

The most efficient test for the second question is the χ^2 -test which is considered in the first part of Section 5.

With respect to the third question, the second part of Section 5 is devoted to two special applications of χ^2 – the F -test for determining the consistency of the variances of two samples and a statistic due to Bartlett that can be used to test the consistency of the variances calculated from three or more samples.

[†] See specialist texts on statistics for further information.

[‡] See Reference 7 for a method that can be used to aid in the search for a transformation.

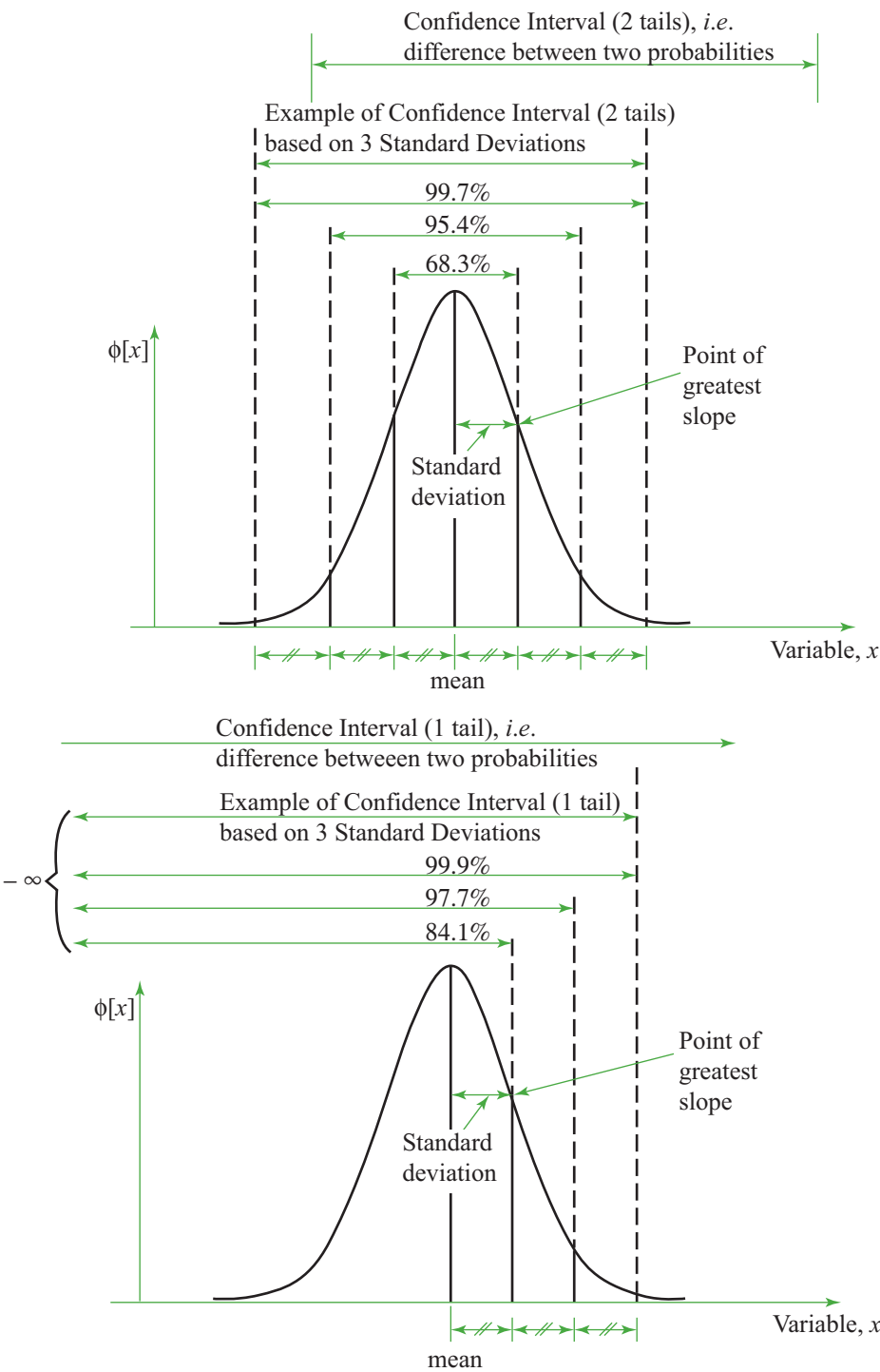
3. THE NORMAL OR GAUSSIAN DISTRIBUTION

The Normal Distribution (attributed to K.F. Gauss) provides a useful numerical description of the distribution of a random variable and, hence, of almost any set of data points that would be expected to cluster symmetrically about a central “mean” value, with a dispersion that is random in nature (for example sets of aircraft climb, cruise or airfield performance test points adjusted to standard conditions). Consequently, before applying this distribution to a set of data points, all sources of systematic variability must first be corrected out to the maximum extent possible. In the context of aircraft performance analysis this usually means that the data must first be corrected to standard conditions or expressed as a difference between measured and calculated (or estimated) values, or between measured values and a regression line (see Sections 7 and 8).

3.1 Main Features of the Distribution

The main features of the Normal Distribution are as follows.

- (i) The distribution is symmetrical about the mean value, and is “bell” shaped, decreasing in height in each direction from a single peak to a value of zero at infinity, see Sketch 3.1.
- (ii) The mean and median values are equal.
- (iii) The horizontal distance between the mean and the points on the curve where the slope is steepest is the standard deviation, see Sketch 3.1.
- (iv) The areas under the curve for intervals ± 1 , ± 2 and ± 3 standard deviations from the mean are 68.3%, 95.4% and 99.7% respectively of the total area under the curve (for two tails of the distribution), see Sketch 3.1.
- (v) From the definition of probability, q , in Section 2.2 these areas represent the probabilities of the sample of the variable being within these intervals (for two tails of the distribution), see Sketches 2.1 and 3.1.



Sketch 3.1 Features of the Normal Distribution