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The statistical analysis of data from normal distributions, with particular reference to small samples

Associated software: ESDUpac A9141

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(Continued on inside back cover)

THE STATISTICAL ANALYSIS OF DATA FROM NORMAL DISTRIBUTIONS, WITH PARTICULAR REFERENCE TO SMALL SAMPLES

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THE STATISTICAL ANALYSIS OF DATA FROM NORMAL DISTRIBUTIONS, WITH PARTICULAR REFERENCE TO SMALL SAMPLES

1. PART I. THEORY AND APPLICATIONS

1.1 Introduction

This Data Item, together with ESDUpac A9141, deals with and gives guidance on the use of the normal, or Gaussian, distribution in the analysis of engineering data. Particular emphasis is placed on the analysis of small samples (typically less than 30 observations).

This Data Item is provided in two Parts. Part I gives guidance on the analysis of one sample and the comparison of two samples. It sets out methods appropriate for hand calculations, and these are accompanied by tables and approximate equations for the requisite statistical functions. Calculations in respect of a single sample may be performed using the computer program of the associated ESDUpac A9141, which is described in Part II and is available both on the Compact Disc and on disc in the Series Software Volume. This utilises more accurate algorithms and calculates more accurate values of statistical functions, thus enabling slightly more precise results to be obtained than with the approximate methods of Part I, particularly for very small samples.

ESDU 92040*, which complements this Data Item, gives a summary of basic statistical terms and principles, and describes the techniques most commonly used in the analysis of engineering data. It includes an introduction to probability, random variables and their distributions, sampling from a population, and the estimation of population characteristics on the basis of a sample.

The aim in analysing data is to estimate the characteristics of the population from which the sample is drawn. The population is the (usually infinite) set of all possible values that the variable under consideration could take. The uncertainty inherent in the estimates of population characteristics is very dependent upon the amount of data available for analysis – the larger the sample, the greater the precision obtainable.

The normal distribution, defined in Section 1.3, is very widely used in engineering. It has been found, experimentally, to be a good model for many quantities that exhibit inherent variability, including the strengths of many engineering materials. It can also be demonstrated theoretically that any quantity that has several additive sources of variability tends to be normally distributed. For the same reason it is often used to describe random measurement errors. Even when there are known to be significant departures from a normal distribution, useful approximate solutions can often still be obtained using normal statistical theory.

It is frequently possible to apply normal statistics to a problem after applying a simple data transformation. For example, it is common practice when analysing fatigue life to assume that the logarithm of the endurance is approximately normally distributed; in this case the methods of the Data Item can be used after first taking logarithms to any convenient base, although care has to be taken when making the inverse transformation.

Section 1.5 describes a test that can be carried out to test that the data are compatible with having arisen from a normally-distributed population. This is useful when there is doubt as to whether the normal distribution is an appropriate model. The test need not be applied to every sample analysed.

* ESDU 92040 "An introduction to the statistical analysis of engineering data".

Assuming the population to have a normal distribution, the sample mean and standard deviation (defined in Section 1.4) are used in Sections 1.6 to 1.8 to estimate the population mean, standard deviation, coefficient of variation and percentiles. In a particular application only those quantities that are of interest need be calculated. The comparison of two samples from normal distributions to test whether they could have come from the same normal population, is covered in Section 1.9. In each case, the calculations can be carried out for a chosen level of confidence (see, for example, Section 1.6).

Worked examples, demonstrating the application of the methods of this Data Item to typical small samples from an approximately normal population are given in Section 1.11. These examples also illustrate the relationship between sample size and the precision of the estimates that can be made. The use of the computer program of ESDUpac A9141, see Part II of this Data Item, is illustrated by its application to the examples in Section 1.11.

1.2 Notation

A^2	Anderson-Darling goodness-of-fit statistic (see Equation (1.9))
A^*	modified Anderson-Darling statistic (see Equation (1.10))
F	cumulative distribution function (see Equation (1.2))
F	F -test statistic (see Equation (1.30))
F_{v_1, v_2}	variable with F distribution, with v_1 and v_2 degrees of freedom (see Section 1.9.2)
F_{p, v_1, v_2}	value of F_{v_1, v_2} below which proportion p of F_{v_1, v_2} distribution lies (see Table 1.10)
f	probability density function (see Equation (1.1))
$k_L(n, p, \gamma)$	lower confidence limit factor for population percentile (see Equation (1.22))
$k_U(n, p, \gamma)$	upper confidence limit factor for population percentile (see Equation (1.23))
n	number of observations
p	probability or proportion (defined in each context)
s	sample standard deviation (see Equation (1.6))
s^2	sample variance (see Equations (1.4) and (1.5))
T	t -test statistic (see Equation (1.31))
t_v	variable with Student's t distribution with v degrees of freedom (see Section 1.6.1)
$t_{p, v}$	value of t below which proportion p of t_v distribution lies (see Table 1.8)
v	population coefficient of variation (see Section 1.8)
$\hat{v}$	point estimate of v (see Equation (1.25))

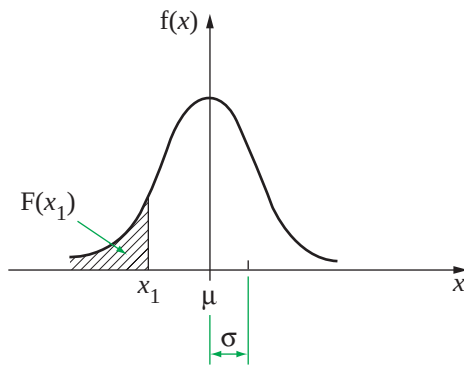
v_L	lower one-sided confidence limit for v (see Equation (1.26))
v_U	upper one-sided confidence limit for v (see Equation (1.27))
X	variable with normal distribution
x	a value of X
x_i	i th observation of X (unordered observations)
$x_{(i)}$	i th smallest observation (ordered observations)
$\bar{x}$	sample mean (see Equation (1.3))
x_p	value of x below which proportion p of normal distribution lies (see Section 1.7)
$\hat{x}_p$	point estimate of x_p
$\hat{x}_p^L$	lower one-sided confidence limit for x_p (see Equation (1.20))
$\hat{x}_p^U$	upper one-sided confidence limit for x_p (see Equation (1.21))
Z	variable with standard normal distribution (see Section 1.3)
z	a value of Z
z_p	value of Z below which proportion p of standard normal distribution lies (see Table 1.7)
α	significance level associated with rejection of hypothesis (see Section 1.5.1)
γ	confidence level associated with an estimate (see Section 1.6)
μ	population mean
$\hat{\mu}$	point estimate of μ (see Equation (1.11))
$\hat{\mu}_L$	lower one-sided confidence limit for μ (see Equation (1.12))
$\hat{\mu}_U$	upper one-sided confidence limit for μ (see Equation (1.13))
v	number of degrees of freedom (see Section 1.6.1)
σ	population standard deviation
$\hat{\sigma}$	point estimate of σ (see Equation (1.15))
$\hat{\sigma}_L$	lower one-sided confidence limit for σ (see Equation (1.16))
$\hat{\sigma}_U$	upper one-sided confidence limit for σ (see Equation (1.17))

Φ	standard normal cumulative distribution function (see Section 1.3)
ϕ	standard normal probability density function (see Section 1.3)
χ^2_v	variable with chi-squared distribution with v degrees of freedom (see Section 1.6.2)
$\chi^2_{p,v}$	value of χ^2_v below which proportion p of χ^2_v distribution lies

1.3 The Normal Distribution

A random variable X is defined as being normally distributed if its probability density is given by:

$$f(x) = \frac{1}{(2\pi)^{1/2}\sigma} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\} \quad \text{where } -\infty < x < \infty, \sigma > 0 \quad (1.1)$$



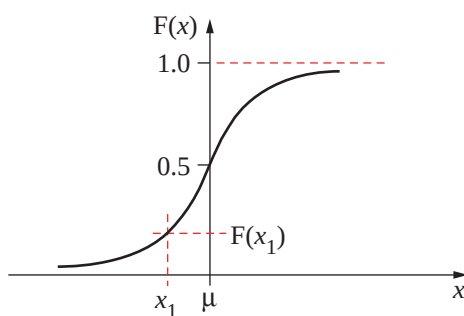
The constants μ and σ are the mean and standard deviation of the distribution, respectively, and are known as the distribution parameters.

The cumulative distribution function of a normal random variable is given by:

$$F(x_1) = \text{Probability}\{X \leq x_1\} = \int_{-\infty}^{x_1} f(x) dx \quad (1.2)$$

(Equation (1.2) is applicable to any continuous random variable, whether normal or not, if the appropriate probability density function is substituted for $f(x)$.)

These probability density and cumulative distribution functions are illustrated in Sketch 1.1. Equations (1.1) and (1.2) are given here for reference only.



Sketch 1.1

Throughout this Data Item, the standard normal distribution is used as an aid in analysis. This is the normal distribution that has a mean of zero and standard deviation of one. If a variable X has a normal distribution with mean μ and standard deviation σ , then the variable $Z = (X - \mu)/\sigma$ has a standard normal distribution (see Sketch 1.2). The probability density function of the standard normal distribution is usually denoted by ϕ and its cumulative distribution function by Φ .

For hand calculations, values of the standard normal cumulative distribution function, $\Phi(z)$, are given in Table 1.6, and percentiles (that is, values of z corresponding to a given value of $\Phi(z)$) in Table 1.7. Other distributions used in the analysis of small samples are also tabulated (see Tables 1.8, 1.9 and 1.10). Suitable algorithms and numerical approximations for use in hand calculations are given in Appendix A.

Further information on the properties of the normal distribution, and on the analysis of data arising from it, is given in References 3, 5, 8 and 9.

1.4 Calculation of Sample Mean and Standard Deviation

The **sample mean** is defined as

$$\bar{x} = \frac{1}{n} \sum_{i=1}^n (x_i), \quad (1.3)$$

where $x_i (i = 1, 2, \dots, n)$ are the sampled values, and n the sample size.

The **sample variance** is defined as

$$s^2 = \frac{1}{n-1} \sum_{i=1}^n (x_i - \bar{x})^2. \quad (1.4)$$

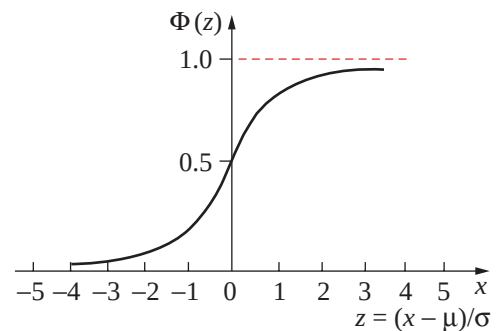
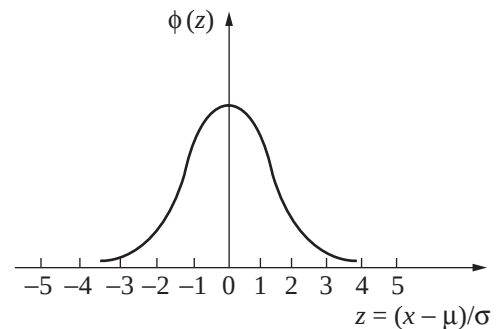
An alternative formula more suited to hand calculation (but more susceptible to rounding errors) is

$$s^2 = \left\{ \frac{1}{n-1} \sum_{i=1}^n (x_i^2) \right\} - \frac{1}{n(n-1)} \left\{ \sum_{i=1}^n (x_i) \right\}^2. \quad (1.5)$$

The **sample standard deviation** is defined as the positive square root of the sample variance

$$s = \left| \sqrt{s^2} \right|. \quad (1.6)$$

The values of $\bar{x}$ and s are used in the following sections of this Data Item to make inferences about the population from which the sample is drawn.



Sketch 1.2

1.5 Testing for Goodness-of-fit

1.5.1 Background

The assumption that a particular sample originates from a normally distributed population will usually be made on the basis of prior knowledge, as described in Section 1.1, because it is not possible to infer a distributional form from a small sample. However, when doubt exists as to the normality of the parent population, and when a sample of at least 8 values is available, a goodness-of-fit test can be carried out to check whether or not the data are consistent with having come from a normal distribution. (It is simply not possible with smaller samples to distinguish between samples originating from a normal distribution and those drawn from an extremely non-normal distribution.) The larger the sample the more sensitive the test will be to departures from normality. It is not necessary to perform this test on every sample analysed.

A large number of goodness-of-fit tests is available. The test given below, a version of the Anderson-Darling test for normality (taken from Reference 7, Section 9.3.2.2), is recommended for the following main reasons.

- (i) It is one of the most powerful tests for normality available, being relatively good at detecting when a sample is from a significantly non-normal population (in which case the methods of analysis given in this Data Item would be inappropriate).
- (ii) It is particularly sensitive to departures from normality in the tails of the distribution. These are regions that are frequently of interest in engineering problems.

The test involves computing from the sample data a value, known as the Anderson-Darling test statistic, that is a measure of the lack of conformity of the data to a normal distribution. If the calculated test statistic exceeds a predetermined critical value, then the hypothesis that the data originated from a normal population is rejected; if it is less than the critical value the hypothesis is accepted. The critical value is that value (tabulated in Section 1.5.2) that has a probability α of being exceeded when sampling from a perfectly normal distribution, where α is known as the significance level of the test, and can be chosen to suit the circumstances.

The higher the significance level chosen, the less the sample has to deviate from normality to fail the test. Although the power of the test to detect non-normality is enhanced by choosing a larger value of α , it must be borne in mind that a proportion α of samples from perfectly normal populations will also be rejected by the test as non-normal. A compromise therefore needs to be made, and a value of 0.05 is commonly used.

Because the probability of obtaining a given value of the Anderson-Darling statistic, A^2 , when sampling from a normal distribution is dependent upon the sample size, n , a modified statistic, A^* (which is a function of A^2 and n , and whose distribution is insensitive to n), is used; this removes the need for a separate table of points for each n .

Because this goodness-of-fit test does not indicate the type of non-normality exhibited by a sample, it may be useful to supplement its use by a graphical technique. Advice on the use of graphical techniques is given in Reference 7, Chapter 2.

1.5.2 Test procedure

The following procedure is applicable to random samples of at least $n = 8$, when there is no prior knowledge of the population mean and standard deviation.

- (1) Order the sample so that the values are in ascending order of magnitude

$$x_{(1)} \leq x_{(2)} \leq \dots x_{(n)}.$$

- (2) Calculate standardised values $z_{(i)}$, where

$$z_{(i)} = (x_{(i)} - \bar{x})/s \text{ for } i = 1, 2, \dots, n \quad (1.7)$$

and $\bar{x}$ and s are the sample mean and standard deviation, calculated as in Equations (1.3) and (1.6).

- (3) Determine

$$p_i = \Phi(z_{(i)}) \text{ for } i = 1, 2, \dots, n, \quad (1.8)$$

where Φ is the standard normal cumulative distribution function. For hand calculations, values $\Phi(z)$ can be obtained from Table 1.6, or computed using Equations (A1.3) and (A1.4) in Appendix A.

- (4) Calculate the Anderson-Darling statistic from

$$A^2 = \frac{1}{n} \left[\sum_{i=1}^n (2i-1) \left\{ \log_e p_i + \log_e (1 - p_{n+1-i}) \right\} \right] - n. \quad (1.9)$$

- (5) Calculate the modified statistic from

$$A^* = A^2 (1 + 0.75/n + 2.25/n^2). \quad (1.10)$$

- (6) If the calculated value of A^* exceeds the critical value tabulated in Table 1.1 below, for the chosen level of significance α , reject the hypothesis of normality.

TABLE 1.1

Significance level α (per cent)	10	5	2.5	1
Critical value of A^*	0.631	0.752	0.873	1.035

An example of the use of this goodness-of-fit test is given in Section 1.11.1.1. Also, the program of ESDUpac A9141, see Part II, gives the significance level corresponding to the calculated value of A^* .

1.6 Estimation of Population Mean and Standard Deviation

Estimates of the population parameters μ and σ can be given in two basically different ways – either as single values (known as point estimates), or as confidence intervals (defined below).

If there is interest only in a “best estimate” of a parameter, then a point estimate is appropriate. If, on the other hand, a measure of the uncertainty associated with this point estimate is required, which may be considerable when working with small samples, a confidence interval should be calculated. This is an interval that contains the true value of the parameter with a specified degree of confidence, known as the confidence level (or confidence coefficient) and denoted by γ . A confidence level of 0.95 is commonly used, and the corresponding interval is known as a 95 per cent confidence interval.

A 95 per cent confidence interval for a parameter is an interval constructed from a sample in such a way that, if samples were repeatedly drawn and a confidence interval constructed for each sample then, in the long run, 95 per cent of those intervals would contain the true value of the parameter. The higher the confidence level employed, the higher will be the frequency, in the long run, of such intervals encompassing the true parameter but, on the other hand, the wider will be the interval and the less precise the estimate.

If there is interest only in the minimum, or the maximum, likely value of the parameter, but not both, then a one-sided confidence interval is appropriate. If on the other hand the likely range of values is required, then a two-sided interval should be obtained. One-sided and two-sided confidence intervals are illustrated, for the population mean, in Sketch 1.3.

1.6.1 Population mean

A point estimate of the population mean is given by the sample mean:

$$\hat{\mu} = \bar{x}. \quad (1.11)$$

Confidence intervals for μ are calculated using Student's t distribution (see, for example Reference 8). This is a commonly used distribution, not dissimilar to the standard normal distribution, with a single parameter ν known as the number of degrees of freedom, which is related to the sample size. The appropriate number of degrees of freedom is specified where required in this and later sections.

One-sided 100γ per cent confidence intervals for μ are defined by

$$\mu \geq \hat{\mu}_L = \bar{x} - t_{\gamma, n-1} s/n^{1/2}, \quad (1.12)$$

where $\hat{\mu}_L$ is known as the (one-sided) **lower 100γ per cent confidence limit for μ** , and

$$\mu \leq \hat{\mu}_U = \bar{x} + t_{\gamma, n-1} s/n^{1/2}, \quad (1.13)$$

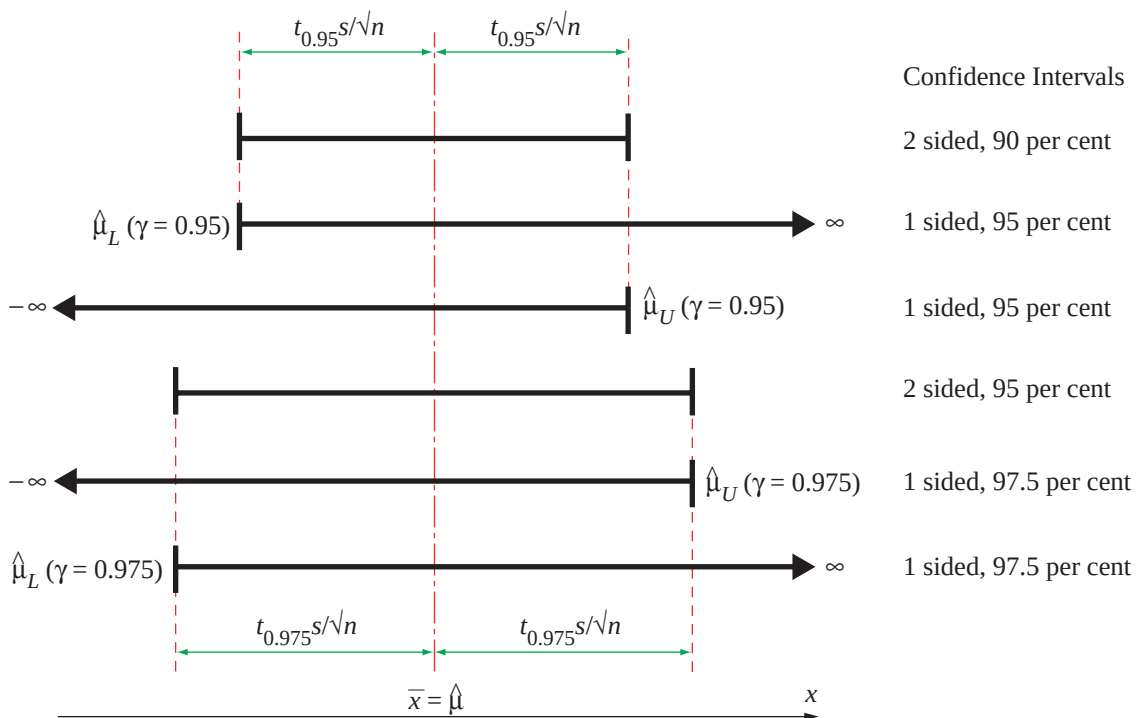
where $\hat{\mu}_U$ is known as the (one-sided) **upper 100γ per cent confidence limit for μ** .

The term $t_{p, n-1}$ is the value of Student's t variable with $n - 1$ degrees of freedom below which a proportion p of the t distribution lies. For hand calculations, an approximate value can be obtained from Table 1.8, or from Equation (A2.1) or (A2.2) in Appendix A.

A two-sided 100γ per cent confidence interval for μ is defined by

$$\mu = \bar{x} \pm (t_{[(1+\gamma)/2], (n-1)} s/n^{1/2}). \quad (1.14)$$

These confidence intervals are illustrated in Sketch 1.3 for some common values of γ . See also the worked example in Section 1.11.1.2.



Sketch 1.3 Confidence intervals for population mean

1.6.2 Population standard deviation

A point estimate of the population standard deviation is given by the sample standard deviation:

$$\hat{\sigma} = s \quad (1.15)$$

The chi-squared (χ^2) distribution (see, for example, Reference 3 or Reference 8) is used in the calculation of confidence intervals for σ . This is, like Student's t distribution, a single-parameter distribution, the parameter ν being known as the number of degrees of freedom.

One-sided 100 γ per cent confidence intervals for σ are defined by

$$\sigma \geq \hat{\sigma}_L = [(n-1)/\chi_{\gamma, n-1}^2]^{1/2} s, \quad (1.16)$$

where $\hat{\sigma}_L$ is known as the (one-sided) **lower 100 γ per cent confidence limit for σ** , and

$$\sigma \leq \hat{\sigma}_U = [(n-1)/\chi_{1-\gamma, n-1}^2]^{1/2} s, \quad (1.17)$$

where $\hat{\sigma}_U$ is known as the (one-sided) **upper 100 γ per cent confidence limit for σ** and $\chi_{p, n-1}^2$ is the value of the chi-squared variable with $n-1$ degrees of freedom below which a proportion p of the χ_{n-1}^2 distribution lies. For hand calculations, percentiles of the chi-squared distribution are given in Table 1.9 (applicable to sample sizes up to 31) and approximate values can also be calculated using Equation (A3.1) or (A3.2) in Appendix A.

A two-sided 100 γ per cent confidence interval for σ is defined by

$$\left\{ \frac{(n-1)}{\chi^2_{[(1+\gamma)/2], (n-1)}} \right\}^{1/2} s \leq \sigma \leq \left\{ \frac{(n-1)}{\chi^2_{[(1-\gamma)/2], (n-1)}} \right\}^{1/2} s. \quad (1.18)$$

Note that this confidence interval, unlike that for μ , is not symmetrical about the point estimate.

Examples of the calculation of point estimates and confidence intervals for the mean and standard deviation of a normal population are given in Sections 1.11.1.2 and 1.11.2.2. Note that, whilst a fairly accurate assessment of μ can be made on the basis of a small sample, that is, the confidence interval is relatively narrow, a much larger sample is required to obtain a useful estimate of σ .

1.7 Estimation of Population Percentiles

It will often be necessary to estimate, on the basis of sample data, the value x_p of a normally distributed variable below which a given proportion p of the population lies. This is known as a population percentile.

The method that follows should not be used to estimate $x_{0.5}$ which, for a normally distributed variable, is equal to the mean, μ , which can be more readily, and more accurately, estimated using Section 1.6.1.

It should be noted that, in practice, most populations deviate appreciably from normality in the extreme tails. Any calculations involving these regions must therefore be regarded as approximate.

As with the population parameters μ and σ either point estimates or confidence intervals may be obtained (see Section 1.6).

A point estimate of x_p is given by:

$$\hat{x}_p = \bar{x} + z_p s, \quad (1.19)$$

where z_p is the value of the standard normal variable below which a proportion p of the standard normal population lies, and can be obtained from Table 1.7. Note that, for $p < 0.5$, z_p is negative.

For small samples, the uncertainty associated with the estimate will be appreciable, and it is usually necessary to quantify this uncertainty in the form of a confidence interval (defined in Section 1.6). Either one-sided or two-sided 100 γ per cent confidence intervals, which include the true value x_p with confidence γ , can be obtained as follows, a confidence level of 0.95 being commonly chosen.

One-sided 100 γ per cent confidence intervals for x_p are defined by:

$$x_p \geq \hat{x}_p^L = \bar{x} + k_L(n, p, \gamma)s, \quad (1.20)$$

where $\hat{x}_p^L$ is known as the (one-sided) **lower 100 γ per cent confidence limit for x_p** , and

$$x_p \leq \hat{x}_p^U = \bar{x} + k_U(n, p, \gamma)s, \quad (1.21)$$

where $\hat{x}_p^U$ is known as the (one-sided) **upper 100 γ per cent confidence limit for x_p** .