

EXAMPLES OF FLIGHT PATH OPTIMISATION USING A MULTIVARIATE, GRADIENT-SEARCH METHOD

1. NOTATION*

M	Mach number
V	true airspeed
V_c	calibrated airspeed (CAS)

Subscripts

cl	denotes climb
cr	denotes cruise
des	denotes descent
md	denotes minimum drag

2. INTRODUCTION

The present Data Item deals with one method of solution of the problem of optimisation of a complete sortie. The method is applied to a subsonic transport aircraft and examples are given to demonstrate the penalties incurred by constraints imposed by design and operational considerations.

The method, described in Section 3, requires the use of an appropriate numerical optimisation technique able to solve constrained multivariate problems. That used in the examples of Sections 4 and 5 is RQPMIN which was developed at the Royal Aircraft Establishment (now DRA Farnborough), see Reference 6. A general description of RQPMIN is given in Reference 10, together with a discussion of some of the problems arising in the formulation of this particular application.

Section 4 presents examples of *complete flight path optimisation* for four “objective functions” (minimum fuel; maximum range; minimum time; minimum direct operating cost) – in each of which a common sequence of constraints is applied. Section 5 presents comparisons between results from RQPMIN and from the “Energy Height Method” for the optimum paths for two climbs for a transport aircraft. Data Item No. 94016 (Reference 11) gives a further example showing the effect of varying the range on the optimum values of the flight parameters.

Data Item No. 94012 (Reference 10) deals with the principles and difficulties of formulation of problems such as those addressed here and in Reference 11.

Data Item No. 89015 (Reference 7) gives a general description of the flight path optimisation problem as it may apply to the individual segments of an aircraft sortie or to various combinations of them. Data Item

* This Data Item departs from the usual practice of providing a comprehensive Notation and Units Section. This is done for the following reasons.

- (i) No equations are quoted and many quantities are referred to in words rather than symbols. The exceptions are for the sake of brevity (or space available).
- (ii) Values quoted in Sections 4 and 5, Tables 8.1 to 8.4 and Figures 1 to 8 are expressed in widely-used operational units which do not form a coherent set. The common abbreviations for these units are employed only in those cases where no confusion of meaning is likely.

Nos 90012 and 91016 (References 8 and 9) cover the Energy Height Method as applied to optimisation of climb flight paths.

3. COMPLETE FLIGHT PATH OPTIMISATION

Optimisation of the complete flight path of an aircraft, treating the entire flight as one continuous trajectory with appropriate variations of aircraft and engine controls, is an extremely complex problem (Reference 1), which has rarely, if ever, been attempted outside research establishments, research departments and universities. Generally, when complete flight path optimisation is undertaken, some degree of approximation is incorporated to eliminate some of the variables in order to reduce the complexity of the mathematical problem (see, for example, References 3 – 5).

In many cases (Reference 7), the flight path may be simplified by considering it as consisting of several segments that may include any or all (and repeats, if necessary) of the following,

- climb,
- acceleration,
- cruise,
- descent,
- hold or loiter,
- in-flight refuelling,
- manoeuvre,

joined by negligible transition phases. The methods of analysis to be used are generally those appropriate to quasi-steady flight conditions and the optimisation process is limited to selection of operating conditions for each segment. One such simple case (to be used here to illustrate the process) is that of a subsonic transport aircraft following a flight profile consisting of the following segments (see Sketch 3.1)

- acceleration* from speed at 1 500 ft to climb speed,
- climb at constant calibrated airspeed (CAS)/constant Mach number,
- acceleration* from climb speed to cruise speed,
- cruise at constant height or stepped heights or cruise-climb,
- acceleration* from cruise speed to descent speed,
- descent at constant Mach number/constant CAS to 1 500 ft.

The total fuel required for these segments is referred to as the range (or stage) fuel.

Block fuel (as used in the minimum fuel and Direct Operating Cost cases in the examples) consists of range fuel plus allowances for

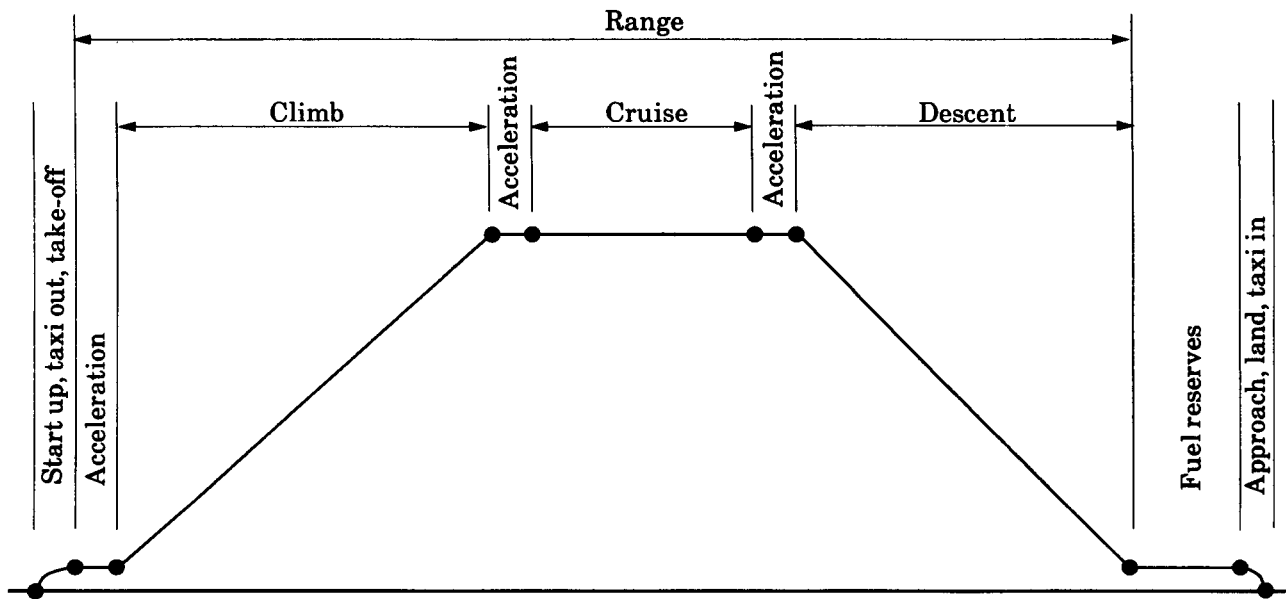
- start-up, taxi out, take-off, initial acceleration and climb to 1 500 ft,
- approach, land and taxi in.

Fuel reserves associated with such a flight typically consist of the elements shown in Sketch 3.2, comprising

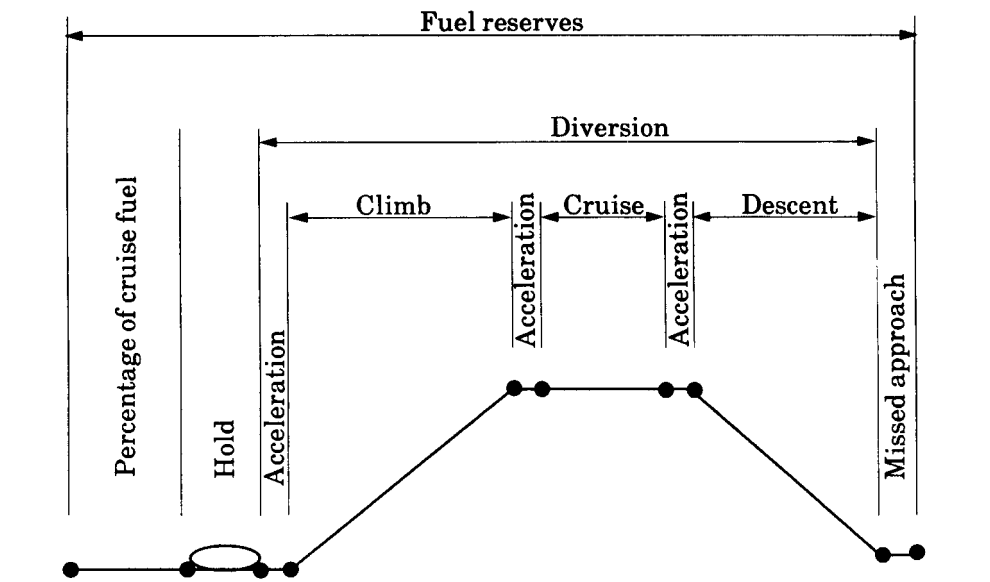
- percentage of range fuel,
- hold,
- diversion (consisting of segments similar to those of cruise),

together with an allowance for a missed approach.

* Or deceleration, as appropriate.



Sketch 3.1 Flight profile



Sketch 3.2 Flight profile for fuel reserves

Each of the segments of the flight path may be calculated by simple quasi-steady methods.

Optimisation then consists of determining values of the flight profile parameters, *i.e.*

climb speed,
climb Mach number,
cruise Mach number,
cruise height,*
descent speed,
descent Mach number,

and, possibly (depending on the problem),

take-off mass,

which achieve the best value of some *objective function*. The objective could, for instance, be one of the following,

to minimise block fuel for a given range,
to maximise range for given block fuel,
to minimise time for a given range,
to minimise a combination of block fuel and time, representing a simple estimate of Direct Operating Cost (DOC), for a given range.

Other objectives are possible, including minimising a combination of fuel and non-linear functions of time typical of more complicated DOC formulae.

The solution must also satisfy a number of design and operational constraints. Again using the example of a subsonic transport aircraft, these constraints could include the following,

Design limits

maximum equivalent airspeed (EAS),
maximum Mach number,
maximum cabin differential pressure,

Aerodynamic margins

speed $\geq 1.3 \times$ stalling speed,
speed $\geq 1.05 \times$ minimum drag speed (V_{md}),
0.3 margin in load factor to buffet onset,

Passenger comfort standards

cabin inclination between 10 degrees (in climb) or 5 degrees (in cruise) nose-up and 5 degrees nose-down (in descent),
maximum rate of change of cabin pressure limited to 0.16 (lbf/in²)/minute (1.10 (kN/m²)/minute); (corresponds to 300 ft/minute at sea level) in the descent, see Reference 12.
maximum rate of change of cabin pressure limited to -0.213 (lbf/in²)/minute (-1.47 (kN/m²)/minute); (corresponds to 400 ft/minute at sea level) in the climb, see Reference 12.

Note that this is not a comprehensive list of constraints. Many others are possible.

* Initial height in the case of cruise-climb or stepped climb.

Software is available to solve constrained multivariate problems of this nature; that used in the present examples is RQPMIN (Reference 6); Reference 10 discusses, in detail, the principles and difficulties of formulation of problems for solution by such methods.

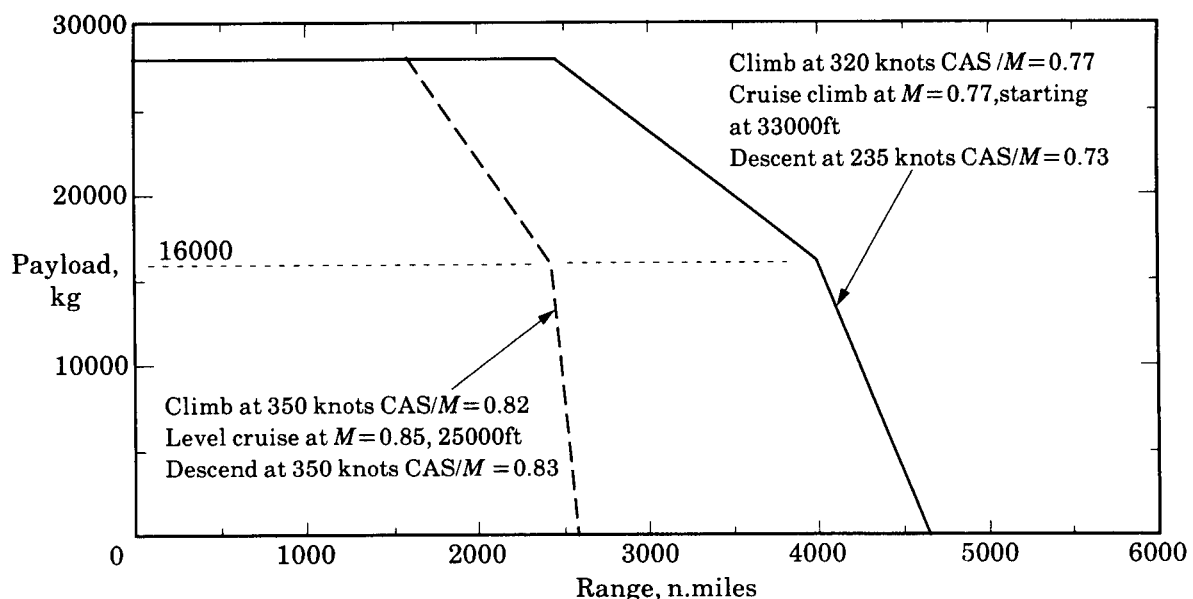
4. APPLICATION TO SPECIFIC EXAMPLES

To illustrate the method, examples are presented of the effects of various constraints on optimum performance in ISA, still-air conditions of a hypothetical subsonic transport aircraft (also used in References 8 and 9) with the following characteristics:

Wing area	200 m ² ,
Aspect ratio	8,
Maximum ramp mass	120 000 kg
Operating mass empty	64 000 kg,
Maximum zero-fuel mass	92 000 kg,
Maximum fuel	40 000 kg
Maximum design EAS	350 knots,
Maximum operating Mach number	0.86,
Engines	2 turbofan,
Bypass ratio	4.4,
Thrust/weight	0.333 (sea level, static, maximum take-off rating; engine derates are not considered in these examples),
Number of flight crew	2 (relevant for DOC study in the examples).

The drag/weight and (maximum climb rating) thrust/weight characteristics in the International Standard Atmosphere are shown in Figure B1 of Reference 8.

For the examples the aircraft has a payload of 16 000 kg (corresponding to a zero-fuel mass of 80 000 kg). A complete analysis of the aircraft's payload-range characteristics results in a payload-range diagram of the form shown in Sketch 4.1.



Sketch 4.1 Payload-range diagram for example aircraft

4.1 Formulation of the Optimisation Problem

RQPMIN^{*} requires the user to provide a sub-routine defining the problem requiring solution. In the present case this “user-written sub-routine” consists of elements defining an aircraft sortie, *i.e.* acceleration, climb, cruise, descent, hold, called in an appropriate sequence, with mass, height, and speed continuous at adjacent ends of contiguous segments. No details of any of these sub-routines are given here since they are made up from conventional elements. Suitable algorithms may be derived from Data Items within the Performance Series. Problems that arise due to the specific needs of RQPMIN^{*} are discussed in Reference 10.

4.1.1 The examples

The examples consist of optimisations for the following four objective functions.

Minimise block fuel for a range of 3 500 n. miles – see Section 4.3.1.

Maximise range for a fixed ramp mass of 120 000 kg – see Section 4.3.2.

Minimise time for a range of 2 000 n. miles – see Section 4.3.3.

Minimise a linear combination[†] of time and block fuel for 2 500 n. miles range – see Section 4.3.4.

The sequence of cases to be investigated for each example is summarised in Table 4.1.

Each example starts with a case in which the only constraints are those necessary to define a flight which is physically possible, *i.e.*

drag $\leq$ thrust available at appropriate engine rating, *i.e.* maximum climb rating in the climb and maximum cruise rating in the cruise,
 contribution to range $\geq$ 0 for each segment,
 speeds $\leq$ maximum equivalent airspeed limit,
etc.,

and uses a constant Mach number cruise-climb technique (except for the minimum time study, which is restricted to constant height cruise, see Section 4.3.3).

Additional constraints are introduced with each subsequent case, as follows

speed $\geq$ 1.00 $\times$ minimum drag speed,
 speed $\geq$ 1.05 $\times$ minimum drag speed,
 0.3 margin in load factor to buffet onset,

rates of climb and descent limited by rate of change of cabin pressure between -0.213 and 0.16 (lbf/in²)/minute (-1469 and 1103 (N/m²)/minute respectively) and cabin differential pressure no greater than 8.5 lbf/in² ($58\,605$ N/m²),

cabin inclination between 10 degrees (climb) or 5 degrees (cruise) nose-up and 5 degrees nose-down.

* And, by implication, other optimising procedures of a similar nature.

† For this example, the combination used was
 $(0.267 \times \text{block fuel}) + (13.48 \times \text{time})$

(fuel in kilograms and time in minutes) which corresponds to the linear fuel and time terms in one particular formula for Direct Operating Cost per flight. Although other terms are necessary to complete the formula, these are all that are necessary for the present purpose. The coefficients relate to a particular set of economic conditions and are typical of the *relative* costs of fuel and time. Other conditions, altering the relative cost of fuel and time, could be expected to produce different optimum flight paths.

Finally, the operating constraints of stepped-climb cruise and cruise at a single altitude (“level cruise”) are introduced, separately. In both of these cases initial altitude is fixed (*i.e.* it is not an independent variable in the optimisation process. In keeping with normal air traffic control procedures, the steps in altitude are limited to 2 000 ft below 29 000 ft, and to 4 000 ft above 29 000 ft. The particular altitudes used here are from the sequence 23 000, 25 000, 27 000, 29 000, 33 000, 37 000, 41 000 ft. Note that the level-cruise cases are all at constant Mach number. It may be possible in these cases to improve the optimisation of fuel or distance by allowing Mach number to vary along the flight path (see Reference 2).

TABLE 4.1 Progressive Introduction of Constraints

Case	Cruise Type	Minimum V/V_{md}	Minimum buffet margin	Cabin pressure limits	Maximum cabin inclination	Height
0	Cruise-climb	No additional constraints				
1	Cruise-climb	1.00				
2	Cruise-climb	1.05				
3	Cruise-climb	1.05	I			
4	Cruise-climb	1.05	I	I		
5	Cruise-climb	1.05	I	I	I	
6	Stepped cruise	1.05	I	I	I	Next below Case 5 optimum
7	Stepped cruise	1.05	I	I	I	Next above Case 5 optimum
8	Level cruise	1.05	I	I	I	Taken as better of Cases 6 & 7

Cases 6 and 7, immediately below the broken line, represent the same constraint (stepped-climb) at two different initial altitudes and, as such, do not fit the principle of progressive introduction of constraints. Case 8 (level cruise) uses the same height as the better of Cases 6 and 7.

In the minimum time example it was found that shorter flight times were obtained from constant height cruise than from a cruise-climb technique. Consequently, each case uses level cruise and Case 8 is irrelevant. The reason is discussed in Section 4.3.

4.1.2 Fuel reserves

In the minimum fuel and maximum range examples, it would be possible to include fuel reserves in the optimisation process, selecting diversion climb and descent calibrated airspeeds and Mach numbers, together with diversion cruise Mach number and height. In the minimum fuel example, the fuel required for range increases with the mass at the start of the diversion and this should lead, in principle, to minimisation of the fuel reserves. However, the dependence of block fuel on the size of the fuel reserves is weak, with the result that minimum reserves will not necessarily be achieved in this way. (Similar considerations apply to the maximum range example.) Furthermore, it would not be possible to include calculation of the fuel reserves in either the minimum time or minimum DOC examples since two separate objective functions would be involved, *i.e.* minimising fuel for the reserves and minimising time or DOC for range. Consequently, optimisation of the fuel reserves was undertaken separately prior to the main calculations. The reserves policy assumed was as follows

30 minutes hold at 1 500 ft altitude, plus

200 n. miles diversion.

A 5% margin on range fuel was also imposed but, since this is dependent on ramp mass (which is a variable in three of the four examples), its calculation is included in all four examples for consistency.

4.2 Optimisation Runs

The results of the various optimisation runs are shown in Tables 8.1 to 8.4 The explanation of the column headings (where necessary) is as follows

Minimum V/V_{md}	Minimum value for V/V_{md} . In the output values, it refers to the lowest value occurring in the specific phase of the flight. Values above about 1.0 ensure a degree of speed stability which increases with V/V_{md} .
Ramp mass	Mass of aircraft before engine start-up, kg,
Block fuel	Fuel used for the flight (excluding reserves), kg,
Time	Duration of the flight (corresponding to "Range" in Sketch 3.1), minutes,
Range	Distance flown (see Sketch 3.1), n. miles,
Cost function	Linear combination of block fuel and time (\$), the objective function in the minimum DOC study,
V_{cl}	Climb speed, knots, CAS,
M_{cl}	Climb Mach number,
M_{cr}	Cruise Mach number,
V_{des}	Descent speed, knots, CAS,
M_{des}	Descent Mach number,
Buffet margin	The margin in load factor to the onset of buffet. The figures quoted represent the lowest value of the margin at any point in the flight.
Cabin inclination	The maximum inclination (nose-up or nose-down, as appropriate), in degrees, of the cabin during the specific phase of the flight.

Pressure differential	The maximum difference, in lbf/in^2 , between cabin and ambient atmospheric pressures during the specific phase of the flight.
Descent time	Minutes. This must be large enough to allow the cabin pressure to return to sea level value at no more than $0.16 (\text{lbf/in}^2)/\text{minute}$ ($1103 (\text{N/m}^2)/\text{minute}$).

The results for the four examples are shown in Figures 1 to 4 in bar charts of the particular parameters being optimised (the objective functions), and the respective optimum values of the flight parameters (see Section 4.1.1). Scales used in these charts have been selected for ease of comparison between the various examples, being identical for the flight parameters and covering approximately the same relative range of values for the objective functions.

4.3 Discussion of Results

The sequential application of the constraints is shown in Tables 8.1 to 8.4 by the stepped lines running through the columns labelled “minimum V/V_{md} ”, “buffet margin”, “pressure differential” and “cabin inclination”. Above each line the constraint is not applied, below the line the constraint is included in the optimisation process.

The first thing to note is that, in most cases, the required values of the constraint parameters are already met before the constraint is imposed. Nevertheless, because of the way in which the optimiser program works, introduction of a constraint affects the way in which a search is made for the optimum. In general, this should not have a significant effect on the result (within the limitations of the accuracy to which the optimiser program accepts that it has reached an optimum, see Reference 10).

4.3.1 Minimum fuel for 3 500 n. miles range

After a minor effect on minimum fuel with the introduction of the more severe constraint on minimum speed (Case 2), there is no further significant effect until the change to stepped cruise (Cases 6 and 7). The better of the two stepped cruise cases (Case 6, starting at 33 000 ft) uses 3.2% more fuel than the equivalent cruise-climb case. The magnitude (3.2%) of this penalty is directly related to the constraint of 4 000 ft steps in altitude above 29 000 ft. The larger the step, the longer the time spent at each level, where, as time progresses, cruise efficiency becomes progressively less economic. If the allowable steps in altitude were smaller, the resulting flight profile would more nearly approximate to a cruise-climb, with consequent reduction in the penalty of stepped cruise relative to cruise-climb. In particular, subsidiary studies (not included in Tables 8.1 to 8.4) show that with 2 000 ft steps the fuel penalty would be only 2.1%, and with 1 000 ft steps it would be down to 1.3%. Level cruise (Case 8) is an even greater departure from the efficiency of the cruise-climb and, as a result, requires 4.2% more fuel than the equivalent cruise-climb case.

Flight parameters follow a similar pattern, *i.e.* initially some changes with the introduction of minimum speed requirements, then little change until the flight technique is varied. Mach numbers for climb and cruise are identical in most cases, supporting a simple practice in long range cruise of climbing at constant calibrated airspeed until cruise Mach number is reached.

4.3.2 Maximum range

Results for maximum range follow a similar pattern to those for minimum fuel, which is to be expected since both call for maximum fuel efficiency. The differences stem simply from the ramp masses at which the aircraft is operated. The higher ramp mass in the maximum range example results in the initial cruise altitude being slightly lower than in the equivalent minimum fuel example. In particular, Case 5 was at 32 759 ft (compared with 33 672 ft for the minimum fuel example). Since this is just below 33 000 ft, Cases 6 and 7 were then run at the fixed initial heights of 29 000 ft and 33 000 ft compared with 33 000 ft and