

STIFFNESSES OF LAMINATED FLAT PLATES

1. INTRODUCTION

This Item presents formulae for calculation of the in-plane and flexural stiffnesses and the coupling stiffnesses of laminated flat plates built up from thin orthotropic layers which may have different material properties, thicknesses and orientations of their principal axes of orthotropy. Also provided is a description of how the lay-up arrangement of the orthotropic laminae contributes to the total laminate stiffness. Most laminated plates are designed to be specially orthotropic, that is, direct and flexural stiffnesses are not coupled. Data Item No. 82013* provides the definitive list of all such arrangements of up to 21 layers together with rules for combining these to any number of layers. Nevertheless, it is occasionally necessary to produce a plate that is not specially orthotropic. In such cases an understanding of the stiffnesses associated with the various unbalanced lay-up arrangements is useful since it allows choice of an arrangement with a satisfactory response to the applied loads.

The relationship between the stiffnesses and the complete set of elastic constants for an anisotropic solid is set out in Appendix A.

2. NOTATION

Attention is drawn to the sign convention used in this Item. The convention is that now in common use and differs from that used in Data Item No. 75002 which this Data Item supersedes. The sign convention for moments and curvatures has been changed and, as a consequence, the signs of some of the equations are also changed. It also employs a different sign for the coupling stiffness for unbalanced plates. The convention used in this Item is consistent with that used in Data Items associated with ESDUview and/or ESDUpac programs for the analysis of composites.

Both SI and British units are quoted but any coherent system of units may be used. The signs for distances from the reference plane and loads, moments and axes are positive as shown in Sketch 2.1. Also, tensile stresses are positive and shear stresses are positive when in the same sense as positive shear loads. Units quoted against each matrix defined in this Section are the units of all the elements of that matrix.

2.1 System of Reference Axes

$O(x,y,z)$ plate reference axes

$O(\alpha,\beta)$ principal axes of orthotropy of a layer (used as additional set of reference axes)

* Data Item No. 82013 "Laminate stacking sequences for special orthotropy. (Application to fibre reinforced composites.)".

2.2 Strains

e $\begin{bmatrix} e_x \\ e_y \\ e_{xy} \end{bmatrix}$ column matrix of direct strains and shear strain referred to (x, y) coordinates

$\bar{\mathbf{e}}_k$ $\begin{bmatrix} e_\alpha \\ e_\beta \\ e_{\alpha\beta} \end{bmatrix}_k$ column matrix of direct strains and shear strain of layer k referred to (α, β) coordinates of that layer

T_k matrix relating strain components in (x, y) coordinates to strain components in (α, β) coordinates of k th layer (see Equation (4.2))

ϵ $\begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \epsilon_{xy} \end{bmatrix}$ column matrix of mid-plate direct strains and shear strain referred to (x, y) coordinates

2.3 Stresses

f_k $\begin{bmatrix} f_x \\ f_y \\ f_{xy} \end{bmatrix}_k$ column matrix of direct stresses and shear stress in k th layer referred to (x, y) coordinates N/m² lbf/in²

$\bar{\mathbf{f}}_k$ $\begin{bmatrix} f_\alpha \\ f_\beta \\ f_{\alpha\beta} \end{bmatrix}_k$ column matrix of direct stresses and shear stress in k th layer referred to (α, β) coordinates of that layer N/m² lbf/in²

2.4 Plate Curvatures

w deflection of plate in z -direction m in

K $\begin{bmatrix} -\partial^2 w / \partial x^2 \\ -\partial^2 w / \partial y^2 \\ -2\partial^2 w / \partial x \partial y \end{bmatrix}$ vector of plate middle plane curvatures and twist components m⁻¹ in⁻¹

2.5 Plate Loading Intensities

M	$\begin{bmatrix} M_x \\ M_y \\ M_{xy} \end{bmatrix}$	column matrix of bending moment intensities acting in xz - and yz -planes and twisting moment intensity acting about x - and y -axes respectively	N m/m	lbf in/in
N	$\begin{bmatrix} N_x \\ N_y \\ N_{xy} \end{bmatrix}$	column matrix of direct load intensities and shear load intensity referred to (x, y) coordinates	N m/m	lbf /in

2.6 Plate Stiffnesses

A	in-plane stiffness matrix for plate (see Equations (4.4) and (4.6))	N/m	lbf/in
B	coupled in-plane and flexural stiffness matrix for plate (see Equation (4.4))	N	lbf
b_k	matrix relating stress and strain in (x, y) coordinates for k th layer (see Equation (4.3))	N/m ²	lbf/in ²
C_k	matrix of k th layer relating stress and strain in (α, β) coordinates (see Equation (4.1))	N/m ²	lbf/in ²
D	flexural stiffness matrix for plate (see Equations (4.4) and (4.6))	N m	lbf in

2.7 Layer Properties

E_α, E_β	moduli of elasticity of a layer in α - and β -directions respectively	N/m ²	lbf/in ²
$G_{\alpha\beta}$	shear modulus of a layer in (α, β) directions	N/m ²	lbf/in ²
ρ	$(1 - \nu_{\alpha\beta}\nu_{\beta\alpha})$ for a layer		
$\nu_{\alpha\beta}$	Poisson's ratio for a layer resulting from load applied in α -direction ($\nu_{\alpha\beta} = \nu_{\beta\alpha}E_\alpha/E_\beta$)		
$\nu_{\beta\alpha}$	Poisson's ratio for a layer resulting from load applied in β -direction ($\nu_{\beta\alpha} = \nu_{\alpha\beta}E_\beta/E_\alpha$)		

2.8 Geometry

n	number of layers in plate		
t_k	thickness of k th layer	m	in
z_k	value of z at mid-plane of k th layer	m	in
ψ_k	angle of principal axis of orthotropy, $O\alpha$, of k th layer measured from plate reference axis, Ox , as shown in Sketch 2.1	degree	degree
I_k	second moment of area per unit width of k th layer about its mid-plane ($=t_k^3/12$).	m ³	in ³
$\bar{z}$	distance of a new reference plane from original reference plane of plate positive in z -direction	m	in

2.9 Mathematical Notation

I	unit matrix
sym	indicates matrix is symmetric about leading diagonal
A'	A prime indicates the transpose of a matrix

2.10 Suffixes

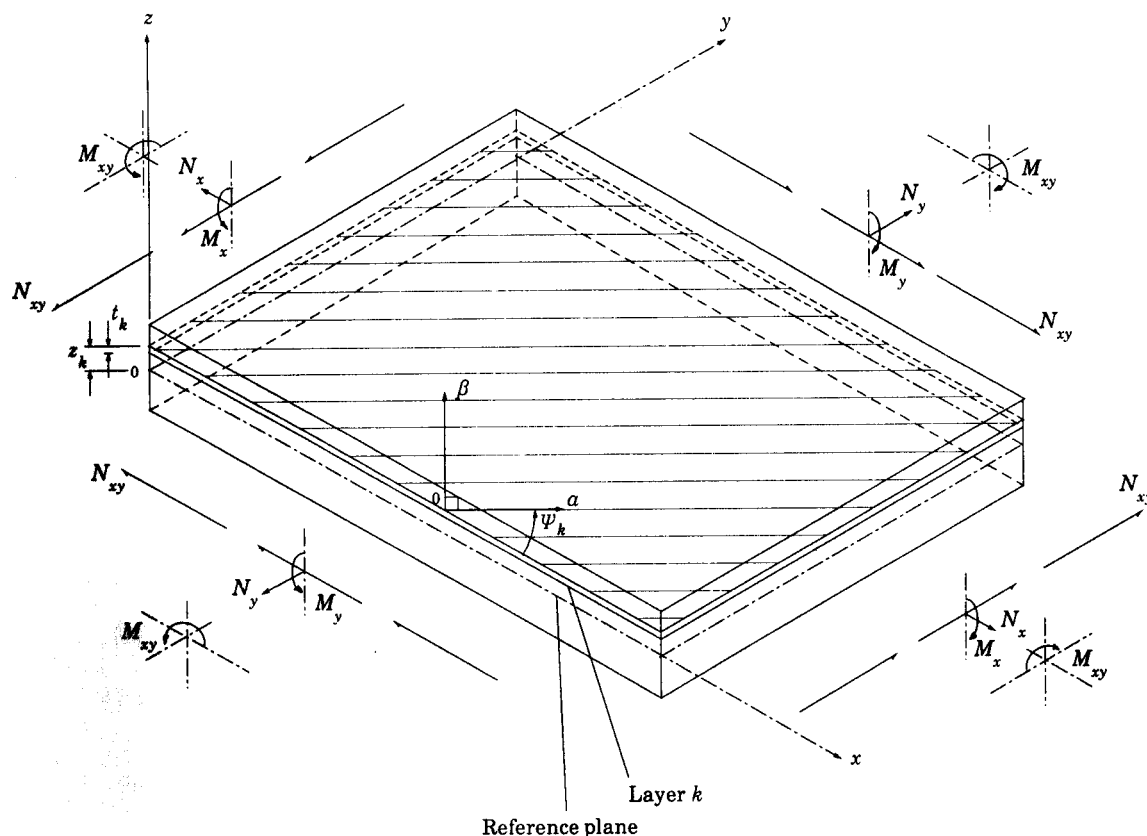
i, j	numbers signifying position in row and column respectively of element in matrix
k	layer identification number
l	identification number for group of layers having identical properties and orientations
m	number of groups having identical properties and orientations
z	suffix added to B and D to indicate that they relate to a plane $\bar{z}$ distant from original reference plane

2.11 Sign Convention

Tensile stresses and loadings are positive, compressive stresses and loadings are negative.

The positive conventions for direct and shear forces, bending and twisting moments are shown in Sketch 2.1

Note: This Data Item supersedes Data Item No. 75002. The sign convention for moments and loadings has been changed to accord with current practice.



Sketch 2.1 Sign convention for forces and moments

3. DESCRIPTION OF LAMINATED PLATES

The general formulae given in the following Sections apply to flat plates built up from a number of thin layers having arbitrary thickness and orientation of their axes of orthotropy. Particular cases of interest are balanced laminates, in which layers of equal thickness, elastic properties and orientation are symmetrically disposed about a central plane, and plates that behave orthotropically under in-plane and flexural loadings. (The latter situation is rarely achieved exactly but can be closely approximated by many multilayered arrangements.)

The properties of a plate are referred to a set of reference axes, $O(x, y, z)$, and the properties of an individual layer, k , to the principal axes of orthotropy of the layer, $O(\alpha, \beta)$. The angle between x and α is termed ψ_k , as shown in Sketch 2.1. In plates that behave orthotropically the axes $O(x, y)$ are chosen as the principal axes of orthotropy. It is convenient to take the central plane as the plane of reference because this minimises the coupling terms between in-plane loading and bending. This is entirely appropriate for balanced arrangements where layers k and $n - (k - 1)$ are identical. For unbalanced arrangements choice of another reference plane may be preferable. Section 6 describes how the properties with respect to any chosen plane may be obtained from those calculated with respect to the original plane.

4. BASIC FORMULAE AND MATRIX CALCULATION PROCEDURE

4.1 Stress Strain Relationships for a Single Layer

The elastic properties of an orthotropic layer, k , can be defined, using matrix notation, by the equation

$$\bar{\mathbf{f}}_k = \mathbf{C}_k \bar{\mathbf{e}}_k, \quad (4.1)$$

where

$$\mathbf{C}_k = \begin{bmatrix} C_{11} & C_{12} & 0 \\ C_{21} & C_{22} & 0 \\ 0 & 0 & C_{66} \end{bmatrix}_k$$

in which

$$C_{11} = \left(\frac{E_\alpha}{\rho} \right)_k, \quad C_{22} = \left(\frac{E_\beta}{\rho} \right)_k, \quad C_{66} = G_{\alpha\beta k}$$

and

$$C_{12} = \left(\frac{\nu_{\beta\alpha} E_\alpha}{\rho} \right)_k = C_{21} = \left(\frac{\nu_{\alpha\beta} E_\beta}{\rho} \right)_k \quad (\text{see Appendix A}).$$

The stresses and strains referred to the layer coordinate axes $O(\alpha, \beta)$ can be related to those for the plate reference axes $O(x, y)$ by the standard formulae for the transformation of stress and strain components.

Thus,

$$\bar{\mathbf{e}}_k = \mathbf{T}_k \mathbf{e}_k \text{ and } \mathbf{f}_k = \mathbf{T}_k' \bar{\mathbf{f}}_k \quad (4.2)$$

where

$$\mathbf{T}_k = \begin{bmatrix} \cos^2 \psi_k & \sin^2 \psi_k & \frac{1}{2} \sin 2\psi_k \\ \sin^2 \psi_k & \cos^2 \psi_k & -\frac{1}{2} \sin 2\psi_k \\ -\sin 2\psi_k & \sin 2\psi_k & \cos 2\psi_k \end{bmatrix}.$$

For the reference axes the stress-strain relationships of layer k may be deduced from Equations (4.1) and (4.2) as follows.

$$\mathbf{f}_k = \mathbf{T}_k' \bar{\mathbf{f}}_k = \mathbf{T}_k' \mathbf{C}_k \bar{\mathbf{e}}_k = \mathbf{T}_k' \mathbf{C}_k \mathbf{T}_k \mathbf{e}_k.$$

Thus,

$$\mathbf{f}_k = \mathbf{b}_k \mathbf{e}_k \text{ where } \mathbf{b}_k = \mathbf{T}_k' \mathbf{C}_k \mathbf{T}_k. \quad (4.3)$$

4.2 Stiffnesses of an Assembly of Layers

In a single plate consisting of an assembly of n layers the in-plane and flexural load deformation relationships for the plate can be written as follows

$$\begin{bmatrix} \mathbf{N} \\ \mathbf{M} \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B} & \mathbf{D} \end{bmatrix} \begin{bmatrix} \boldsymbol{\epsilon} \\ \mathbf{K} \end{bmatrix} \quad (4.4)$$

The plate stiffnesses **A**, **B** and **D** are given by

$$\mathbf{A} = \sum_{k=1}^{k=n} t_k \mathbf{b}_k, \mathbf{B} = + \sum_{k=1}^{k=n} t_k z_k \mathbf{b}_k \text{ and } \mathbf{D} = \sum_{k=1}^{k=n} (t_k z_k^2 + I_k) \mathbf{b}_k. \quad (4.5)$$

The thicknesses of the individual layers of practical laminates are usually such that I_k is negligibly small in relation to $t_k z_k^2$ and **D** therefore usually reduces to

$$\mathbf{D} = \sum_{k=1}^{k=n} t_k z_k^2 \mathbf{b}_k.$$

From Equation (4.5) it can be seen that for balanced laminates **B** is zero and Equation (4.4) reduces to two uncoupled equations

$$\mathbf{N} = \mathbf{A}\boldsymbol{\varepsilon} \text{ and } \mathbf{M} = \mathbf{D}\mathbf{K}. \quad (4.6)$$

Appendix B sets out an explicit formulation of the terms of **A**, **B** and **D**.

4.3 Calculation Procedure

To determine **A**, **B** and **D** the following procedure is suggested.

- (a) Sub-divide the n layers into m groups that have identical values of ψ and elastic properties. For each group, l , calculate the scalar quantities

$$t_l = \sum t_k, (tz)_l = \sum t_k z_k \text{ and } (tz^2 + I)_l = \sum (t_k z_k^2 + I_k)$$

and set up the matrices **T_l** and **C_l**.

- (b) For each group multiply **T_lC_lT_l** to yield **b_l**.
- (c) Multiply **b_l** by the scalar quantities and add as follows

$$\mathbf{A} = \sum_{l=1}^{l=m} t_l \mathbf{b}_l, \mathbf{B} = \sum_{l=1}^{l=m} (tz)_l \mathbf{b}_l \text{ and } \mathbf{D} = \sum_{l=1}^{l=m} (tz^2 + I)_l \mathbf{b}_l$$

5. EFFECT OF LAYER ARRANGEMENTS ON TYPE OF TOTAL PLATE STIFFNESSES

In structural laminates composed of layers of identical properties and thicknesses the layers are usually arranged so as to produce an orthotropic laminate having strength and stiffnesses that match the applied loads. Further, laminate thicknesses are usually reduced in regions where the loading decreases. This is usually done by gradually removing inner layers and requires careful design since the desired degree of orthotropy cannot always be retained as well as maintaining special orthotropy. In some cases the retention of the desired degree of orthotropy may be preferred so that an unbalanced laminate results. In the limit, a specially orthotropic laminate containing angled layers is not possible if less than seven layers are to be used. (See anti-symmetric arrangement A 381 in Table 9.4 of Data Item No. 82013.)

The following Sections describe the various forms of stiffness matrices that result from the possible stacking systems; Table 7.1 summarises these and provides simple example lay-up arrangements. These are based upon the assumptions listed below, all of which commonly apply to laminated fibre reinforced composite plates.

- (1) All layers are specially orthotropic and have the same material properties.
- (2) All layers are of the same thickness.
- (3) The layer principal axes of orthotropy are aligned either with the x - or y -axis or at plus or minus an angle ψ to the x -axis.
- (4) In laminates with an odd number of layers the material of the central layer need not match that of any other layer.

It should be noted, from Equation (4.1), that $\mathbf{C}_k$ is always a symmetric matrix and that, from Equations (4.2) and (4.3), $\mathbf{b}_k$ is also symmetric. In consequence the $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{D}$ matrices are also always symmetric.

Equation (4.5) expresses the stiffness sub-matrices, $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{D}$ as functions of $\mathbf{b}_k$. The value of $\mathbf{b}_k$ is given explicitly in Equation (B1.2) of Appendix B from which it can be seen that $b_{12k} = b_{21k}$, $b_{13k} = b_{31k}$ and $b_{23k} = b_{32k}$. Since most laminated composite plates are composed of layers at angles of $\pm\psi$ to a datum the change in $\mathbf{b}_k$ consequent on changing the sign of ψ has an important influence on the final stiffnesses. From Equation (B1.2) it can be seen that b_{11k} , b_{12k} , b_{21k} , b_{22k} and b_{33k} are all functions of even powers of $\cos \psi$ and/or $\sin \psi$ and so will not be influenced by the sign of ψ . The remaining four terms, b_{13k} , b_{23k} , b_{31k} and b_{32k} , are all functions of odd powers of $\sin \psi$ and $\cos \psi$. Since the range of ψ is from -90° to $+90^\circ$, $\cos \psi$ is always positive. The sign of $\sin \psi$ changes with the sign of ψ . Thus, b_{13k} , b_{23k} , b_{31k} and b_{32k} retain their sign if ψ is positive but change their sign if ψ is negative. Representing the elements of $\mathbf{b}_k$ for a given layer by the symbols a , b , c , d , e and f , all assumed to be positive when ψ is positive, then

$$\mathbf{b}_{k(+\psi)} = \begin{bmatrix} a & b & c \\ & d & e \\ (sym) & & f \end{bmatrix}_k \quad \text{and} \quad \mathbf{b}_{k(-\psi)} = \begin{bmatrix} a & b & -c \\ & d & -e \\ (sym) & & f \end{bmatrix}_k. \quad (5.1)$$

The following Sections describe how these combine using Equations (4.5).

The terms b_{13k} , b_{23k} , b_{31k} and b_{32k} are all direct functions of $\sin \psi$. Consequently, when $\psi = 0^\circ$ or 90° ,

these terms are also zero and the form of $\mathbf{b}_k$ is

$$\mathbf{b}_k(\psi = 0^\circ \text{ or } 90^\circ) = \begin{bmatrix} b_{11} & b_{12} & 0 \\ & b_{22} & 0 \\ (sym) & & b_{33} \end{bmatrix}_k \quad (5.2)$$

This is also the form of $\mathbf{b}_k$ for isotropic materials with the added simplification then that $b_{11k} = b_{22k}$ and $b_{12k} = \nu_{\alpha\beta} b_{22k}$.

In the special case where $\psi = 45^\circ$ the $\mathbf{b}_k$ matrix is considerably simplified (because $\sin 45^\circ = \cos 45^\circ$) giving, from Equation (B1.2),

$$\left. \begin{aligned} b_{11k} &= b_{22k} = (1/(4\rho_k)) [E_a(1 + 2\nu_{\beta\alpha}) + E_\beta + 4\rho G_{\alpha\beta}]_k \\ b_{12k} &= b_{21k} = (1/(4\rho_k)) [E_a(1 + 2\nu_{\beta\alpha}) + E_\beta - 4\rho G_{\alpha\beta}]_k = b_{11k} - 2G_{\alpha\beta k} \\ b_{13k} &= b_{31k} = b_{23k} = b_{32k} = (1/(4\rho_k)) [E_\alpha - E_\beta]_k \\ b_{33k} &= (1/(4\rho_k)) [E_a(1 - 2\nu_{\alpha\beta}) + E_\beta]_k \end{aligned} \right\} \quad (5.3)$$

5.1 In-Plane Stiffness Matrix A

In general the $\mathbf{A}$ matrix is full. However, for plates composed of stacks of layers of equal thickness and having identical properties, an equal number of which lie at $+\psi$ and at $-\psi$, the A_{13} , A_{23} , A_{31} and A_{32} terms vanish. This is because for pairs of opposed angled layers the 13, 23, 31 and 32 terms of the $\mathbf{b}$ matrices have opposed signs, see Equation (5.1), and therefore they cancel leaving $\mathbf{A}$ in the following form,

$$\mathbf{A} = \begin{bmatrix} A_{11} & A_{12} & 0 \\ & A_{22} & 0 \\ (sym) & & A_{33} \end{bmatrix} \quad (5.4)$$

This form is unaltered by adding any number of layers with ψ at 0° or 90° since, for these, $\mathbf{b}_k$ is of the same form, compare Equations (5.2) and (5.4). For plates composed of layers arranged at plus and minus several values of ψ , the form of $\mathbf{A}$ will be as in Equation (5.4) provided that pairings for each angle are made.

From the foregoing $\mathbf{A}$ cannot be of the form of Equation (5.4) unless there are an even number of layers balanced in sign. For odd numbers of layers the odd layer must be at $\psi = 0^\circ$ or 90° .

From Equation (4.5) it can be seen that $\mathbf{b}$ for a group of n layers of equal thickness may be back calculated from $\mathbf{A}$ thus,

$$\mathbf{b} = \mathbf{A}/(nt_k) \quad (5.5)$$

5.2 Coupled In-Plane Flexural Stiffness Matrix B

Note that the contribution to $\mathbf{B}$ of a layer is proportional to z_k so that a layer whose mid-plane coincides with the central reference plane does not contribute to $\mathbf{B}$.