

ELASTIC BUCKLING OF UNBALANCED LAMINATED FIBRE REINFORCED COMPOSITE PLATES

(Rectangular plates of AsBtDs type, all edges simply-supported, under biaxial loading)

1. SOLUTIONS, LIMITATIONS AND PLATE BEHAVIOUR

1.1 Introduction

This Data Item together with ESDUpac A9406 provides a Fortran computer program for calculating the buckling load of one of the nine possible classes of unbalanced laminated composite plates. The nine possible classes of unbalanced laminated plates are described in detail in Data Item No. 94003*. The solution and program provided deals with the class coded as AsBtDs in Data Item No. 94003. This class is described in detail in Section 1.3.

This Data Item is provided in two parts. Section 1 describes the theoretical solution to the title problem and its limitations in application to practical plates. Section 2 describes the input and output for the program of ESDUpac A9406 together with example calculations and notes on program testing.

In complex laminated structures it is often not practical to reduce laminate thickness exactly in line with reducing load and maintain a balanced laminate†. Layer continuity over several adjacent panels is often preferred for strength considerations even though it results in unbalanced plates within the structure.

The form of the stiffness matrix for an unbalanced plate can significantly influence the buckling load. Where some choice in the lay-up arrangement is open, selection of a lay-up that produces an AsBtDs type plate is usually preferable since it exhibits bifurcational buckling. The following sections detail the form of plate stiffness sub-matrices covered by this Data Item and show how they influence the buckling load.

1.2 Notation

Attention is drawn to the sign convention used in this Item. The convention is that now in common use and differs from that used in Data Item No. 86020 which this Data Item supersedes. The sign convention for in-plane loads has been changed. It also employs a different sign for the coupling stiffnesses for unbalanced plates (see Data Item No. 94003). The convention used in this Item is consistent with that used in Data Items associated with ESDUview and/or ESDUpac programs for the analysis of composites.

* Data Item No. 94003 "Stiffnesses of laminated flat plates".

† For the complete list of balanced stacking sequences see Data Item No. 82013 "Laminate stacking sequences for special orthotropy. (Application to fibre reinforced composites.)"

Both SI and British units are quoted but any coherent system of units may be used.

1.2.1 Plate overall stiffnesses

$\mathbf{A}$	in-plane stiffness matrix for plate	N/m	lbf/in
A_{ij}	element of $\mathbf{A}$	N/m	lbf/in
$\mathbf{B}$	coupled in-plane and flexural stiffness matrix for plate	N	lbf
B_{ij}	element of $\mathbf{B}$	N	lbf
$\mathbf{D}$	flexural stiffness matrix for plate	N m	lbf in
D_{ij}	element of $\mathbf{D}$	N m	lbf in

1.2.2 Layer properties

C	reference number for layer type (see Section 2.2.3)		
$E_{\alpha k}, E_{\beta k}$	moduli of elasticity of k th layer in α - and β -directions, respectively	N/m ²	lbf/in ²
$G_{\alpha\beta k}$	shear modulus of k th layer in (α, β) directions	N/m ²	lbf/in ²
t_k	thickness of k th layer	m	in
ψ_k	angle of principal axes of orthotropy, α , of k th layer measured from plate reference axis, $0x$, as shown in Sketch 1.1	degree	degree
$\nu_{\alpha\beta k}$	Poisson's ratio for k th layer resulting from load applied in α -direction		

1.2.3 Plate loading and geometry

a	plate length (in x -direction)	m	in
b	plate width (in y -direction)	m	in
m	number of buckle half waves along plate length, a		
N_x, N_y	load intensities applied to plate sides in x - and y -directions, respectively	N/m	lbf/in
n	number of buckle half waves across plate width, b		

1.2.4 Suffixes

b	relates to buckling load
i, j	numbers signifying position in row and column, respectively, of element in matrix
k	layer identification number
$0(x, y)$	plate geometry and material reference axes, see Sketch 1.1
$0(\alpha, \beta)$	layer material reference axes, see Sketch 1.1

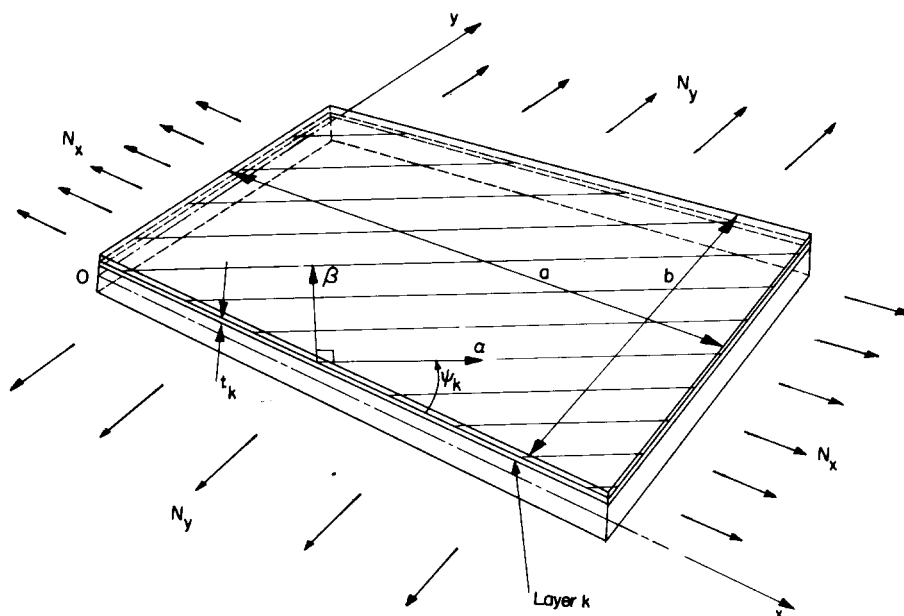
1.2.5 Mathematical notation and sign convention

sym	indicates matrix is symmetric about leading diagonal
	Tensile stress positive, compressive stress negative

1.2.6 Plate stiffness type code

This is as defined in Table 7.1 of Data Item No. 94003. The code consists of the following letters, as appropriate, after each of the symbols for the symmetric stiffness sub-matrices **A**, **B** and **D**, signifying

f	all elements finite
l	only 11, 22, terms non-zero
0	all elements zero
s	specially orthotropic form, 13, 23 terms zero
t	only 13, 23 terms non-zero



Sketch 1.1

1.3 The Class of Unbalanced Plates for Which a Solution Is Given

Data Item No. 94003 shows how the stiffness matrix forms for the various types of unbalanced plate arise. This Data Item deals only with plates in which the **A** and **D** stiffness sub-matrices are of the following forms

$$A = \begin{bmatrix} A_{11} & A_{12} & 0 \\ & A_{22} & 0 \\ (sym) & & A_{33} \end{bmatrix}, \quad D = \begin{bmatrix} D_{11} & D_{12} & 0 \\ & D_{22} & 0 \\ (sym) & & D_{33} \end{bmatrix} \quad (1.1)$$

and the **B** coupling matrix is of the form

$$B = \begin{bmatrix} 0 & 0 & B_{13} \\ & 0 & B_{23} \\ (sym) & & 0 \end{bmatrix}. \quad (1.2)$$

These forms accord with the reference code AsBtDs defined in Data Item No. 94003. Laminates having stiffness sub-matrices of this form most commonly occur when identical pairs of layers, with one layer having its principal axis of orthotropy at $+\psi$ and the other at $-\psi$, are equidistant either side of the central reference plane. For example, the stacks of identical layers arranged at the following values of ψ .

$45^\circ, -45^\circ, 45^\circ, 0^\circ, -45^\circ, 45^\circ, -45^\circ$
 $30^\circ, -30^\circ, 30^\circ, -30^\circ$.

1.4 Statement of Basic Assumptions

Section 1.7 sets out the equations on which the buckling solution provided by the program introduced in Section 2 is based (plates of type AsBtDs). The solution is based upon elastic thin plate small deflection theory in which the plate is assumed to behave as a homogeneous orthotropic material whose axes of orthotropy are aligned with the edges of the plate. The theory only takes account of direct in-plane, flexural and twisting deformations, as governed by the **A**, **B** and **D** stiffness sub-matrices. No account is taken of through-the-thickness shear deformations. The plate is assumed to be initially flat, rectangular and simply-supported along all four sides with the loadings applied in-plane and uniformly along all four edges. The applied loads are assumed to lie in the mid-plane of the plate and this is the plane to which **B** and **D** are related.

1.4.1 Implications of lack of balance for buckling

For simply-supported plates having stiffness sub-matrices of the form given by Equations (1.1) and (1.2) (AsBtDs types) the primary response to a direct in-plane load is not one of transverse deflection. Because of this behaviour bifurcational buckling occurs for such plates and the assumption of small deflection theory is realistic. For other unbalanced plates, use of a similar analysis using small deflection theory may not be practical. An example is the AsBsDs type plate, in which the stiffness sub-matrices **A** and **D** are as given in Equation (1.1) but **B** is of the form

$$B = \begin{bmatrix} B_{11} & B_{12} & 0 \\ & B_{22} & 0 \\ (sym) & & B_{33} \end{bmatrix}. \quad (1.3)$$

In Derivation 2 a 'buckling condition' for AsBsDs type laminates is proposed, see equations of Section 1.7.2. In that derivation the equations that result in the buckling relations do not satisfy the requirement of simple-support zero-moment boundary conditions. This has important consequences for the compression behaviour of AsBsDs type plates.

It is a characteristic of AsBsDs type laminates that immediately upon loading in the direction of the mid-plane, bending occurs. This can be seen by considering a two-layered plate, with equal thickness layers, but having different in-plane moduli. Uniform biaxial straining of the layers gives rise to unequal stresses in the layers so that to maintain compatibility bending moments are required. These are directly proportional to the axial load with magnitudes determined by the layer properties. It follows that, for arbitrary stiffness properties, the offset of the neutral plane from the mid-plane differs for loading in the x- and y-directions. It is for this reason that the plate equations are referred to the mid-plane.

The effect of the bending is to produce out-of-plane deflections and, as a consequence, non-uniformity of the direct stress resultants. The latter is the main reason why the out-of-plane equilibrium 'buckling' equation of Derivation 2 is inappropriate for deriving a design reference load for AsBsDs type plates. The equation presumes a uniform prebuckling state, that is, N_x and/or N_y constant over the plate. This implies simple linear shortening in the x- and y-directions, a displacement pattern that does not satisfy the 'classical' simple-support boundary conditions for the unbalanced plate sustaining no edge moments. Strictly, satisfaction of the moment boundary condition requires the simple deformation to be supplemented by a complementary solution of the linearised equations, which implies coupling of the x- and y-direction displacements and the out-of-plane displacements. However, from Derivation 1 the simultaneous occurrence of in-plane compressive loading and bending of a plate requires the strain-displacement equations to be taken from large deflection theory, and that renders the equilibrium equations nonlinear. The nonlinear load-deflection response will initially be comparable to 'beam-column' behaviour and at higher load levels will be influenced by higher order terms. It thus follows that a simple 'buckling criterion' of the type given by Derivation 2 cannot be used to estimate a design load.

To put the above described "buckling condition" in perspective it can be argued that either conservative or unconservative estimates of the design load will be obtained. For a wide plate, the buckling load will correspond to a small overall end-shortening. The actual plate, however, has a deflection pattern at that 'critical' strain with a moment distribution, whose values are related to that small strain. It may thus appear that the actual fibre/matrix stresses at that strain level are perfectly allowable without danger of buckling occurring. Alternatively, a narrow plate, with the same thickness and lay-up, will have a high critical stress. The fibre/matrix strain values that would follow from an 'exact' analysis, which includes the 'beam-column' effect due to the moment distribution in the plate, may be much higher than the allowable value. A strict application of the 'buckling criterion' for design purposes may, in such a case, be very unconservative, because it ignores the prebuckling deflected shape of the plate. An example of the noted non-linear load-deflection response is given in Derivation 4.

1.5 Limitations of Mathematical Model

As stated in Section 1.4 no account is taken of through-the-thickness shear deformations. This is, in effect, an assumption that the through-the-thickness shear modulus is infinite. For specially orthotropic balanced plates finite values of through-the thickness shear stiffness can cause a significant reduction in the buckling load below that calculated if they are assumed infinite. This is shown in Data Item No. 81047 which allows for the effects of through-the-thickness shear stiffness in specially orthotropic balanced plates. It is therefore likely that if the terms of **B** are relatively small, the buckling load will be as much, or more, reduced (below the values given by Equation (1.5)) by the through-the-thickness shear stiffness as by the effects of out-of-balance forces. Alternatively, if the terms of **B** are relatively large, the out-of-balance forces are likely to dominate and the effects of through-the-thickness shear stiffness are likely to be unimportant. In order to assess the influence of through-the-thickness shear stiffness on this type of plate it is suggested that the program of Data Item No. 81047 be run using the **A** and **D** sub-matrices only and comparing the resulting buckling load with that obtained from the program of this Data Item. This will give an indication of the relative influence of the two effects.

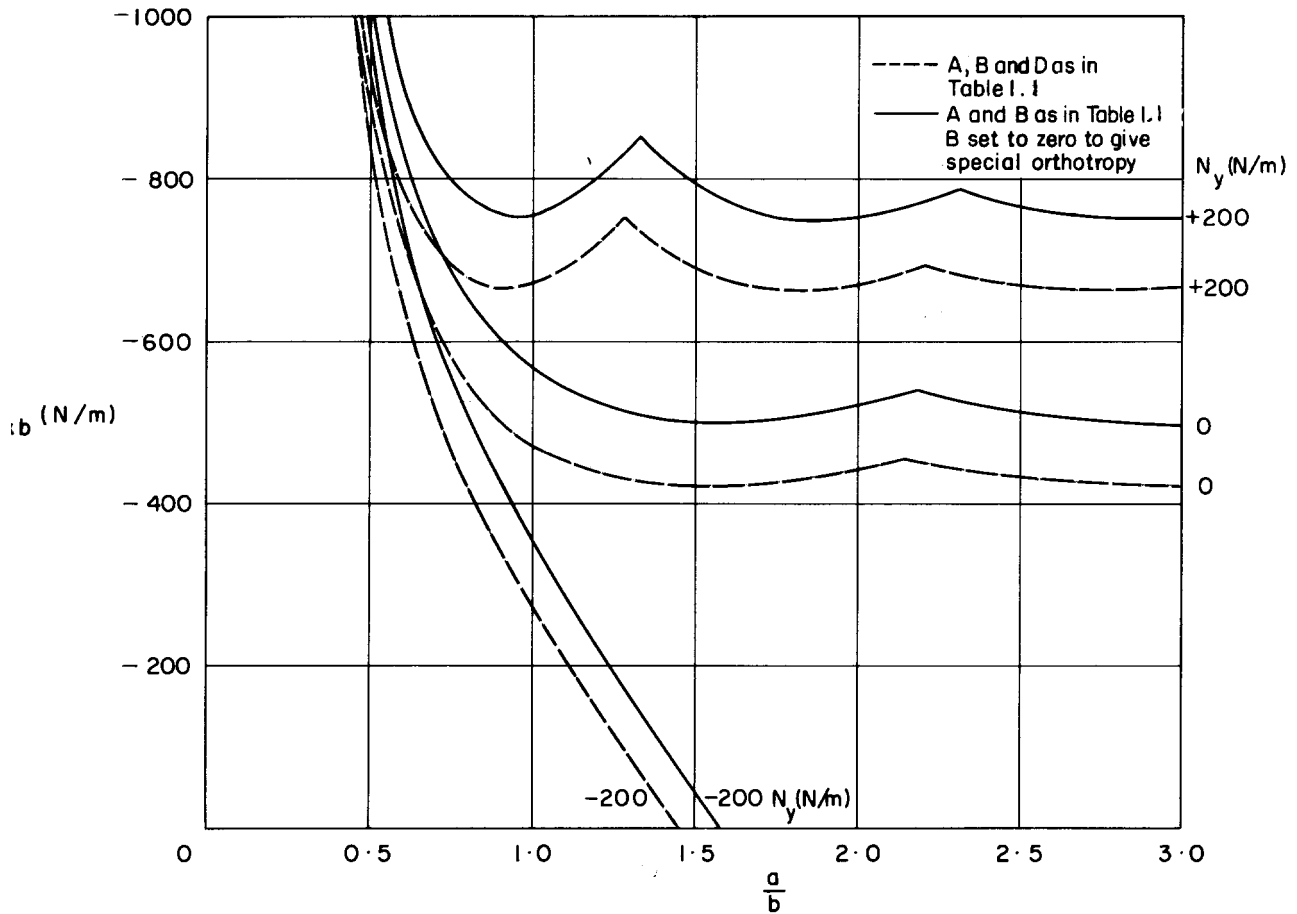
Note that for specially orthotropic balanced plates the program in Section 2 of this Data Item gives buckling loads that agree with Data Item No. 80023. This is because that Data Item also assumes that the through-the-thickness shear modulus is infinite.

1.6 Buckling Characteristics of Unbalanced AsBtDs Type Plates

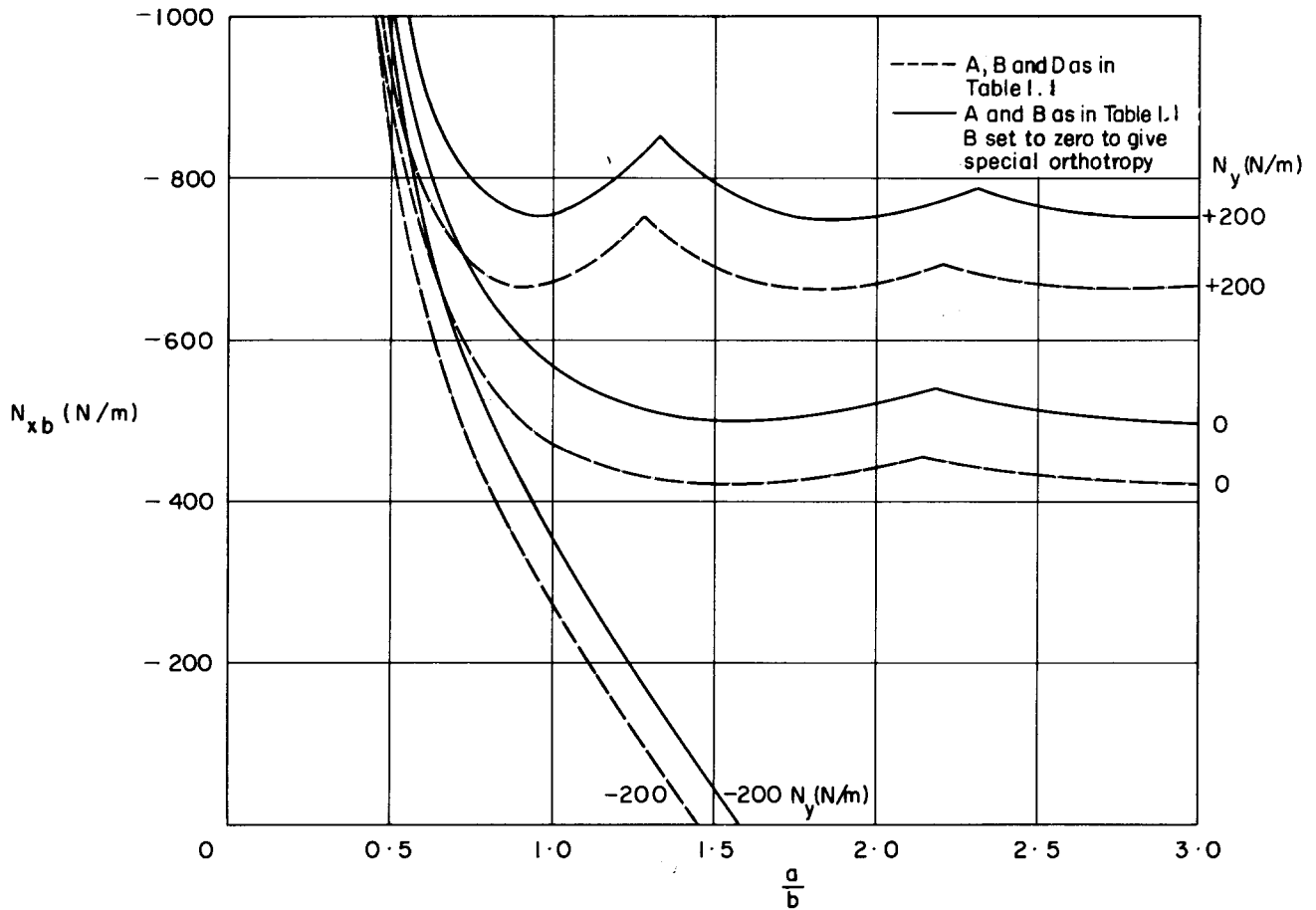
Sketches 1.2, 1.3 and 1.4 illustrate the behaviour of unbalanced AsBtDs type plates with regard to buckling by comparison with that of similar, balanced, specially orthotropic plates. Table 1.1 gives details of the laminates used to produce Sketches 1.2, 1.3 and 1.4. The dotted lines on the figures relate to plates having the stiffness sub-matrices given in Table 1.1. The solid lines on Sketches 1.2 and 1.3 have been obtained by setting the terms of **B** to zero to give specially orthotropic plates with the same **A** and **D** stiffness sub-matrices. (This approach is necessary since with angled layers at least seven layers are required if the plate is to be specially orthotropic, see Data Item No. 82013.) The solid lines on Sketch 1.4 relate to a specially orthotropic plate achieved by reversing the sign of six of the 14 layers, those shown bracketed in the sequence at the bottom of Table 1.1. These changes of sign do not affect **A** or **D** but reduce **B** to zero.

TABLE 1.1 DETAILS OF PLATE CONSTRUCTION FOR SKETCHES 1.2, 1.3 AND 1.4.

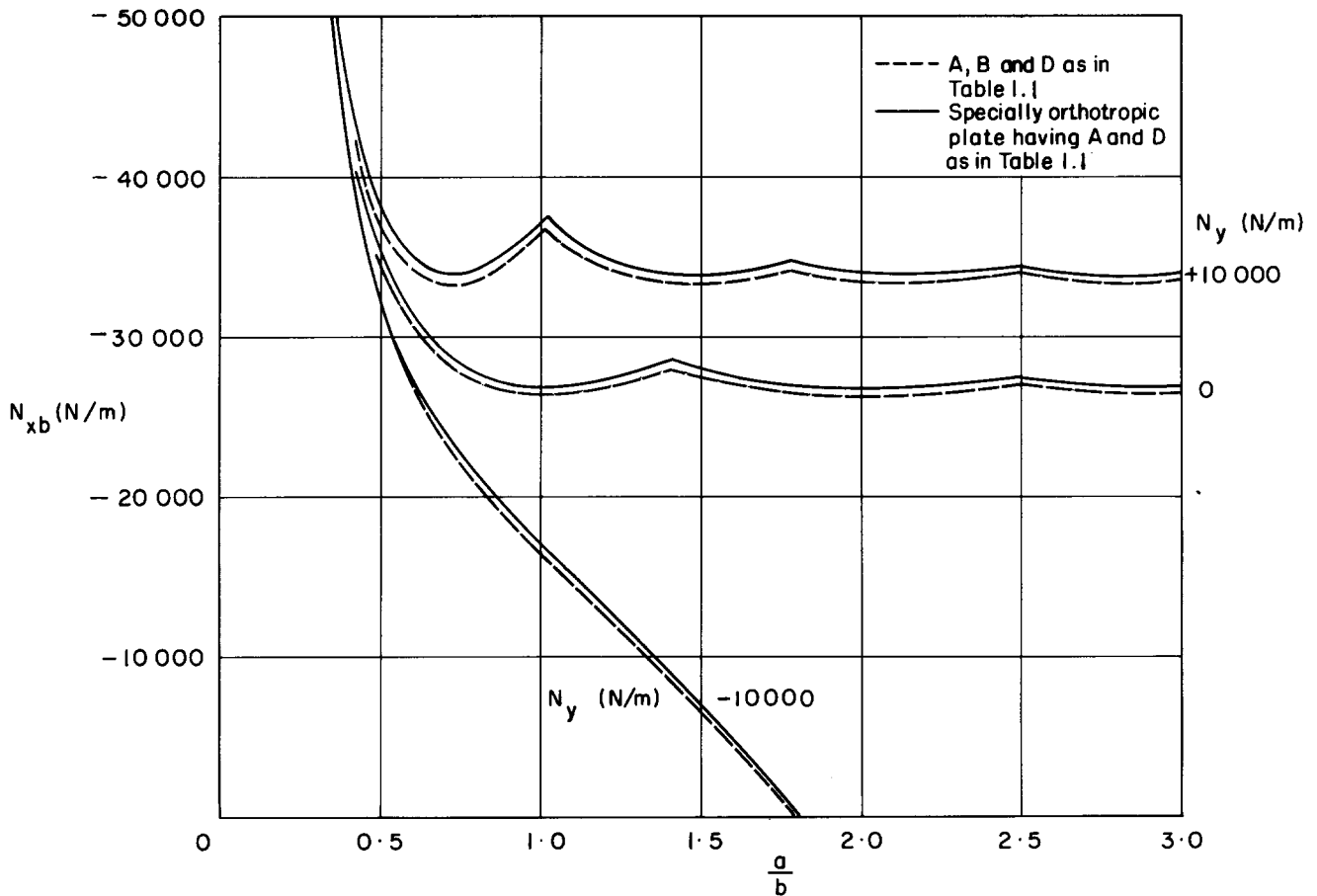
Layer properties and thickness and plate geometry		
$E_{\alpha k} = 207 \times 10^9 \text{ N/m}^2$ $G_{\alpha\beta} = 5.0 \times 10^9 \text{ N/m}^2$ $t_k = 0.12 \text{ mm}$ $E_{\beta k} = 7.6 \times 10^9 \text{ N/m}^2$ $\nu_{\alpha\beta k} = 0.30$ $b = 250 \text{ mm}$		
Sketch number	Number of layers	Stacking sequence and laminate stiffnesses
1.2	2	$45^\circ, -45^\circ$ $\mathbf{A} = \begin{bmatrix} 1.44 & 1.20 & 0 \\ & 1.44 & 0 \\ (sym) & & 1.26 \end{bmatrix} \times 10^7$ $\mathbf{B} = \begin{bmatrix} 0 & 0 & 7.20 \\ & 0 & 7.20 \\ (sym) & 0 & \end{bmatrix} \times 10^2$ $\mathbf{D} = \begin{bmatrix} 0.0691 & 0.0576 & 0 \\ & 0.0691 & 0 \\ (sym) & & 0.0607 \end{bmatrix}$
1.3	4	$30^\circ, -30^\circ, 30^\circ, -30^\circ$ $\mathbf{A} = \begin{bmatrix} 5.85 & 1.83 & 0 \\ & 1.05 & 0 \\ (sym) & & 1.96 \end{bmatrix} \times 10^7$ $\mathbf{B} = \begin{bmatrix} 0 & 0 & 18.4 \\ & 0 & 6.53 \\ (sym) & 0 & \end{bmatrix} \times 10^2$ $\mathbf{D} = \begin{bmatrix} 11.2 & 0.351 & 0 \\ & 0.202 & 0 \\ (sym) & & 0.376 \end{bmatrix}$
1.4	14	$45^\circ, -45^\circ, 45^\circ, -45^\circ, 45^\circ, -45^\circ, 0^\circ, 0^\circ, 45^\circ, -45^\circ, 45^\circ, -45^\circ, 45^\circ, -45^\circ$ $\mathbf{A} = \begin{bmatrix} 13.6 & 7.25 & 0 \\ & 8.82 & 0 \\ (sym) & & 7.71 \end{bmatrix} \times 10^7$ $\mathbf{B} = \begin{bmatrix} 0 & 0 & 43.2 \\ & 0 & 43.2 \\ (sym) & 0 & \end{bmatrix} \times 10^2$ $\mathbf{D} = \begin{bmatrix} 23.9 & 19.70 & 0 \\ & 23.60 & 0 \\ (sym) & & 20.8 \end{bmatrix}$ $45^\circ, -45^\circ, (-45^\circ), (45^\circ), (-45^\circ), (45^\circ), 0^\circ, 0^\circ, 45^\circ, -45^\circ, 45^\circ, -45^\circ, (-45^\circ), 45^\circ$ $\mathbf{A}$ as above $\mathbf{B} = [0]$ $\mathbf{D}$ as above



Sketch 1.2 See Table 1.1 for Details of Laminate



Sketch 1.3 See Table 1.1 for Details of Laminate



Sketch 1.4 See Table 1.1 for Details of Laminate

Sketches 1.2, 1.3 and 1.4 show that, as a general rule, unbalanced plates buckle at a lower load than balanced plates in which all other factors are the same. The differences in reduction of N_{xb} for unbalanced plates below that for specially orthotropic plates shown by the sketches is primarily due to differences in the ratios of the terms of **D** and of **B**. For example, D_{11} increases by a factor of nearly 350 between Sketches 1.2 and 1.4 whereas the terms of **B** only rise by a factor of 6. Thus, in this case, D_{11} has risen by a factor of 60 over the rise in the terms of **B**, so reducing the influence of **B** to a few per cent (see Sketch 1.4). Inspection of Equation (1.5) shows that, strictly, the reduction is also dependent on **A** and is negligible if

$$\bar{B}^2/(\bar{D}\bar{A}) \ll 1. \quad (1.4)$$

Sketch 1.2 shows that a compressive N_y can in some cases cause buckling before N_x is applied but it also shows that this may not occur in a comparable specially orthotropic plate (see for example, $N_y = 10 \text{ N/m}$ and $a/b = 2.0$ on Sketch 1.2). It is therefore recommended that in all cases a full buckling analysis is conducted.

1.7 Basic Equations

1.7.1 AsBtDs plates (Derivation 3)

$$N_{xb} = \lambda^2 N_y - \frac{n^2 \pi^2}{b^2} (\bar{D} + \bar{B}^2 / \bar{A}) \quad (1.5)$$

in which

$$\left. \begin{aligned} \bar{A} &= \bar{a}_{22} \lambda^{-2} + (2\bar{a}_{12} + \bar{a}_{33}) + \bar{a}_{11} \lambda^2 \\ \bar{B} &= (2\bar{b}_{23} - \bar{b}_{31}) \lambda^{-1} + (2\bar{b}_{13} - \bar{b}_{32}) \lambda \\ \bar{D} &= \bar{d}_{11} \lambda^{-2} + 2(\bar{d}_{12} + 2\bar{d}_{33}) + \bar{d}_{22} \lambda^2 \\ \lambda &= na/(mb) \end{aligned} \right\} \quad (1.6)$$

and

$$\begin{bmatrix} \bar{a}_{11} & \bar{a}_{12} & 0 \\ \bar{a}_{21} & \bar{a}_{22} & 0 \\ 0 & 0 & \bar{a}_{33} \end{bmatrix} = A^{-1} \quad (1.7)$$

$$\begin{bmatrix} 0 & 0 & \bar{b}_{13} \\ 0 & 0 & \bar{b}_{23} \\ \bar{b}_{31} & \bar{b}_{32} & 0 \end{bmatrix} = -A^{-1} B \quad (1.8)$$

$$\begin{bmatrix} \bar{d}_{11} & \bar{d}_{12} & 0 \\ \bar{d}_{21} & \bar{d}_{22} & 0 \\ 0 & 0 & \bar{d}_{33} \end{bmatrix} = D - BA^{-1} B \quad (1.9)$$

1.7.2 AsBsDs and AsB1Ds plates (Derivation 2)

(Refer to Section 1.4.1 for notes on this solution.)

$$N_{xb} = \lambda^2 N_y - T_{33} - \frac{2T_{12}T_{13}T_{23} - T_{11}T_{23}^2 - T_{22}T_{13}^2}{T_{11}T_{22} - T_{12}^2} \quad (1.10)$$