

Derivation of endurance curves from fatigue test data, including run-outs

**Associated software: ESDUpac A9724
VIEWpac W9724**

Endorsed by
The Royal Aeronautical Society
The Institution of Mechanical Engineers

ESDU DATA ITEMS

Data Items provide validated information in engineering design and analysis for use by, or under the supervision of, professionally qualified engineers. The data are founded on an evaluation of all the relevant information, both published and unpublished, and are invariably supported by original work of ESDU staff engineers or consultants. The whole process is subject to independent review for which crucial support is provided by industrial companies, government research laboratories, universities and others from around the world through the participation of some of their leading experts on ESDU Technical Committees. This process ensures that the results of much valuable work (theoretical, experimental and operational), which may not be widely available or in a readily usable form, can be communicated concisely and accurately to the engineering community.

We are constantly striving to develop new work and review data already issued. Any comments arising out of your use of our data, or any suggestions for new topics or information that might lead to improvements, will help us to provide a better service.

THE PREPARATION OF THIS DATA ITEM

The work on this particular Data Item was monitored and guided by the Fatigue Committee and the Stress Analysis and Strength of Components Committee. The Fatigue Committee first met in 1955 and now has the following membership:

Chairman

Mr A.R. Simpson — British Aerospace Airbus Ltd

Members

Dr A. Burgess	— Messier-Dowty Ltd
Mr K.E. Cheverton	— Independent
Mr D. Crouch	— MATRA BAe Dynamics
Mr R.G. Eastin *	— Federal Aviation Administration, USA
Ms C. Escobedo *	— Airbus, Germany
Dr M. Miller *	— Boeing Commercial Airplane Group, USA
Mr D. Painter	— Westland Helicopters Ltd
Dr P.M. Powell	— QinetiQ
Dr S.H. Spence	— BAE SYSTEMS, Warton
Dr C.H. Wang *	— Aeronautical and Maritime Research Laboratory, Australia
Mr R.T. Watanabe *	— Boeing Commercial Airplane Group, USA
Dr R.N. Wilson †	— Independent.

* Corresponding Member

† Member Emeritus

DERIVATION OF ENDURANCE CURVES FROM FATIGUE TEST DATA, INCLUDING RUN-OUTS

CONTENTS

	Page
1. INTRODUCTION, THEORY AND APPLICATION	1
1.1 INTRODUCTION	1
1.2 NOTATION	1
1.2.1 Suffixes	2
1.3 FATIGUE TEST DESIGN	2
1.3.1 Definition of Population	2
1.3.2 Sample Size and Stress/Strain Levels	2
1.3.3 Censored Data	3
1.4 THEORY AND APPLICATION	3
1.4.1 Form of S-N Curve	3
1.4.2 Parameter Estimation	4
1.5 CURVE-FITTING PROCEDURE	5
1.5.1 Running Strategy	5
1.5.2 User Input	5
1.5.3 Program Output	6
1.5.4 Outliers	7
1.6 DERIVATION AND REFERENCES	8
1.6.1 Derivation	8
1.6.2 References	8
2. ESDUpac A9724 AND VIEWpac W9724	9
2.1 INTRODUCTION TO THE PROGRAMS	9
2.2 SCOPE AND LIMITATIONS OF THE PROGRAMS	9
2.3 DATA INPUT REQUIREMENTS FOR THE FORTRAN PROGRAMS	10
2.3.1 Running the Programs	11
2.3.1.1 Program A9724	11
2.3.1.2 Program W9724	11
2.4 OUTPUT FROM THE PROGRAMS	13
2.4.1 Output of Input Data	13
2.4.2 Results and Output	13
2.4.2.1 Failure to provide a solution owing to errors in the input data	13
2.4.2.2 Failure to provide a solution from valid input data	14

2.5	EXAMPLES	15
2.5.1	Example 1, Data for PH13-8Mo Forged Steel Bar taken from MIL-HDBK-5G	15
2.5.1.1	Two parameter curve	15
2.5.1.2	Three parameter curve	19
2.5.2	Example 2, Data from 2025-T6 Forged Aluminium Alloy Slab	21
2.5.2.1	Automatic selection of curve	21
2.5.3	Example 3, Data for a Metal Matrix Composite	25
2.5.3.1	Two parameter curve	25
2.5.3.2	First data point removed	27
2.5.3.3	q predefined	28

APPENDIX A THE METHOD OF MAXIMUM LIKELIHOOD APPLIED TO THE ANALYSIS OF CENSORED FATIGUE DATA

A1.	INTRODUCTION	31
A1.1	NOTATION	31
A1.1.1	Suffixes	31
A1.1.2	Operators	31
A2.	THE METHOD	32
A2.1	Uncensored Observations	32
A2.2	Right-censored Observations (Run-outs)	32
A2.3	Left-censored Observations	32
A2.4	Application of Maximum Likelihood Method	32

DERIVATION OF ENDURANCE CURVES FROM FATIGUE TEST DATA, INCLUDING RUN-OUTS

1. INTRODUCTION, THEORY AND APPLICATION

1.1 INTRODUCTION

Section 1 describes the theory and application of a method for consistent derivation of an $S-N$ curve from a set of fatigue data. The method is applicable to data for both metallic and non-metallic materials as well as to data from tests on coupons or specimens representing structural features. The method encompasses the low-cycle, high-cycle and endurance limit regimes and also takes account of censored data (for example, run-outs) in a rigorous manner.

Data relating to static failures, corresponding to a single load application, are not catered for. In addition to this, it is assumed that the curve of $\log S$ versus $\log N$ is concave upward; this is believed, however, to encompass the vast majority of published $S-N$ data.

S may be in terms of a maximum, an alternating or an equivalent stress, strain or loading with the life, N , expressed in terms of number of cycles, time or some other convenient unit (for example, repeated program blocks). The cycles may be of either constant or variable amplitude. The derived curve is the median curve, that is, it represents a fifty per cent probability of failure.

Ideally, the process of deriving an $S-N$ curve should begin at the test planning stage, the aim being to obtain the maximum amount of useful information from a limited number of tests. Section 1.3 therefore gives guidance on the design of fatigue tests with a view to optimising the testing procedure.

Section 1.4 outlines the theory on which the derivation of the $S-N$ curve is based. Section 1.5 describes the curve-fitting procedure that is implemented by a Fortran computer program, the use of which is described in Section 2.

Definitions of the statistical terms used in this Data Item can be found in Data Item No. 92040^{*}. A definition of the normal distribution is given in Data Item No. 91041[†].

1.2 NOTATION

The units for S and N are totally independent and can, therefore, be chosen as required. However, the values of P_1 , P_2 and P_3 will only apply to the chosen units for S and N (see Section 1.4).

C	number of parameters by which fitted $S-N$ curve is defined (see Table 2.1)
D	number of values of S for which values of N are required to be calculated by the program (see Table 2.1)
l	likelihood (see Section 1.4.2 and Appendix A)
M	number of pairs of $S-N$ values in a particular data set
N	life

^{*} Data Item No. 92040, "An introduction to the statistical analysis of engineering data".

[†] Data Item No. 91041, "The statistical analysis of data from normal distributions, with particular reference to small samples".

P_1, P_2, P_3	constants (see Equation (1.1))
q, r	constants (see Equation (1.1))
S	stress, strain or loading
v	population coefficient of variation of S
ε	random error (see Equation (1.2))

In this Data Item $\ln$ represents the natural logarithm of a number while $\log$ represents the logarithm to any base of a number.

1.2.1 Suffixes

Suffixes 1, 2, . . . D , applied to S , denote specific values of S for which a value of N is required to be calculated by the program.

Suffixes H and L , applied to l , denote values for higher and lower order models respectively (see Section 1.5.3).

Suffixes 1, 2, . . . M , applied to S and N , denote specific pairs of S – N values.

1.3 FATIGUE TEST DESIGN

1.3.1 Definition of Population

The first step in designing a fatigue testing programme is to define clearly the population for which an S – N curve is to be obtained. When referring to fatigue testing, the population may be considered to be the set of all possible test specimens of a particular type and all tested in a particular manner; the specimens actually tested will constitute a sample from that theoretical population. It is important that the sample is as representative as possible of the population and that all relevant details of the specimens, the method of testing and the test conditions are recorded for future reference.

Fatigue data from different sources can be combined, but it should be borne in mind that the population will then usually consist of a combination of two or more individual populations. As a result of this, the scatter may be very large and the derived curve may cease to be representative of any of the individual populations.

1.3.2 Sample Size and Stress/Strain Levels

For a given population, the greater the total number of tests that are performed, the more representative of the population will be the resulting S – N curve. Also, the larger the scatter, the greater the number of test results required to obtain a median curve. Furthermore, the wider the range of endurance over which the S – N curve is to be derived, the larger the number of test results that will be needed to obtain a sufficiently representative median curve.

A lower limit on the number of test results that is required is also imposed by the number of parameters used to define the derived S – N curve (see Section 1.5.2).

The minimum number of specimens to be tested will also depend on the amount of information about the form and parameters of the curve that already exists, since this will determine the number of unknown constants that must be evaluated from the test data.

The optimum allocation of a given total number of tests to different stress or strain levels will also depend on the degree of scatter. The larger the scatter, the more specimens should be tested at each level. Conversely, if the degree of scatter is low it would be advantageous to perform tests at an increased number of stress or strain levels.

In general, it is advisable to choose test levels that result in a reasonably even coverage of the range of endurance of interest. In cases where it is not necessary for all the specimens to be tested simultaneously, good coverage of the range of endurance can sometimes be best achieved by adjusting the stress or strain levels as the test programme proceeds, on the basis of the information gained up to that point. Exceptions to this are cases where the relationship between $\log S$ and $\log N$ is known from experience to be linear over the range of interest. In such cases a good strategy may be to focus testing activity at the lower and upper extremes of the range of endurance of interest.

It is impossible to prescribe the minimum total number of test results that will be needed to obtain a sufficiently representative curve because of the number of separate factors involved. However, two tests in which the specimen fails, performed at each of three different stress or strain levels, is considered an absolute minimum but even this would be unlikely to give a reliable median curve (with the exception of cases where the relationship between $\log S$ and $\log N$ is known to be linear). Section 1.5.2 gives guidance on checking retrospectively whether or not the sample size is adequate.

1.3.3 Censored Data

Although unfailed specimens (run-outs) are taken into account in a mathematically rigorous manner by the computer program that accompanies this Data Item, the aim of a testing programme should be to test to failure as many of the specimens as possible over the range of interest, since more information is obtained from failed specimens than from run-outs. The only exception to this rule is when the user is particularly interested in determining the endurance limit, in which case a number of run-outs is desirable.

Left-censored data, that is data where the number of cycles to failure has not been recorded but is known to be less than a certain value, can also be used in the analysis and should not therefore be discarded.

1.4 THEORY AND APPLICATION

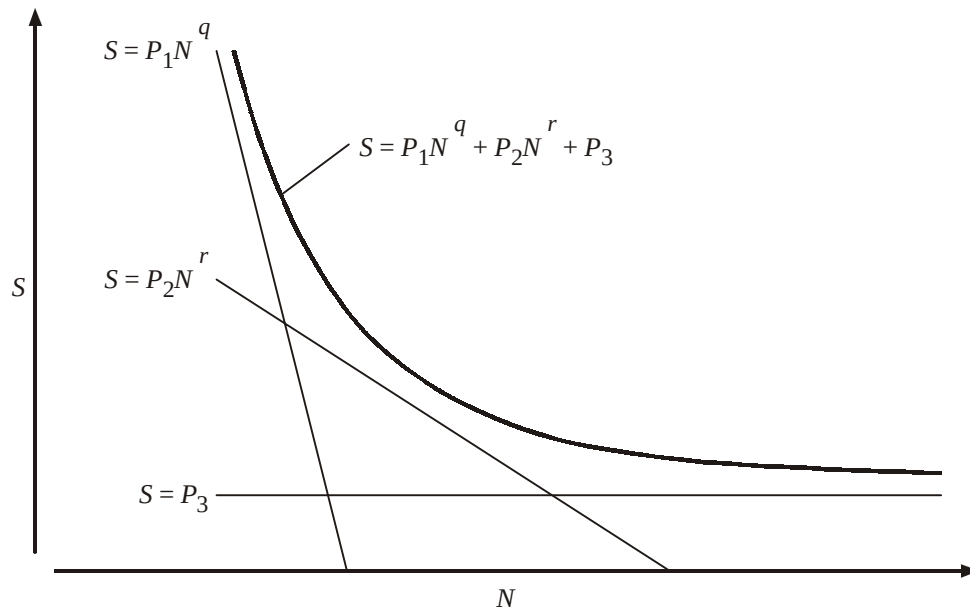
1.4.1 Form of $S-N$ Curve

The program that accompanies this Data Item derives from the fatigue data a curve of the form:

$$S = P_1 N^q + P_2 N^r + P_3, \quad (1.1)$$

where P_1 , P_2 , P_3 , q and r are constants with P_1 , P_2 , $P_3 \geq 0$ and $q < r \leq 0$.

This general equation is capable of representing a wide range of shapes of $S-N$ curve where S may be in terms of a maximum, an alternating or an equivalent stress, strain or loading with the life, N , expressed in terms of number of cycles, time or some other convenient unit. The equation has a sound theoretical basis that is applicable to many materials.



Sketch 1.1 Components of S - N curve plotted on a log-log scale

The first term on the right-hand side of Equation (1.1), when taken in isolation, represents a straight line on a log-log scale, the slope of the line being q . Similarly, the second term on the right-hand side represents a straight line with shallower slope r on a log-log scale. The third term represents a horizontal straight line. Since different terms are dominant over different ranges of N , the curve tends to be asymptotic to each straight line in turn, as illustrated in Sketch 1.1.

The model described by Equation (1.1) is consistent with the results of research on metallic materials which have shown that the low-cycle fatigue region, which is characterised by predominantly plastic strain, can be described on a log-log scale by a straight line. These results have also shown that the high-cycle fatigue region, which is characterised by predominantly elastic strain, can be described by a straight line with a fairly shallow slope, with a gradual transition taking place between the low-cycle and high-cycle regions. Further information on the origins of Equation (1.1) and details of other S - N relationships that are commonly used elsewhere are given in Derivation 3. The third term on the right-hand side of Equation (1.1) enables an endurance limit to be modelled, since the first two terms tend to zero for high values of N . The value of the constant term fitted by the program should not, however, be taken as an estimate of the endurance limit unless there is a significant number of data points in that region.

In practice, an appropriate model is often described by just one or two terms on the right-hand side, especially when the available data may cover only a restricted range of endurance.

1.4.2 Parameter Estimation

The Fortran program performs a non-linear constrained optimisation procedure using a maximum likelihood criterion. *This means that the values chosen for the unknown parameters in Equation (1.1) are those that maximise the probability of the observed data being obtained.* The method of maximum likelihood is covered in most text books on statistics. The simplified treatment given in Reference 7 and the more rigorous treatment given in Reference 6 are recommended. A brief description of the use of the method in the derivation of an S - N curve from fatigue test data is given in Appendix A. The method is particularly appropriate for application to a mixture of censored and uncensored data. In practice, it is convenient to maximise the natural logarithm of the likelihood, $\ln l$, rather than the likelihood, l , itself, and

the value of $\ln l$ corresponding to the fitted relationship provides a useful measure of how good a fit has been obtained. (The same optimal solution is obtained whether it is l or $\ln l$ that is maximised.)

The regression relationship that is fitted is:

$$\log S = \log(P_1 N^q + P_2 N^r + P_3) + \epsilon, \quad (1.2)$$

where ϵ is a normally distributed error with a mean of zero and a constant standard deviation, independent of N .

This approach was chosen, in preference to the derivation of a regression curve of N on S , because the standard deviation of $\log N$ is not constant but increases with increasing N and tends to infinity as the endurance limit is approached. This makes it impossible to fit a regression curve of N on S that extends to high endurances where the slope of the curve levels out. Analysis of typical fatigue data from a variety of materials has shown that the standard deviation of $\log S$ is, on the other hand, approximately constant; this is equivalent to stating that S has a constant coefficient of variation. The regression relationship for S on N can therefore be fitted over the whole applicable range of endurances.

The optimisation procedure is a globally-iterated version of a simplex algorithm constructed by Nelder and Mead (Derivations 1 and 5). Constraints in the algorithm mean that negative values of the parameters P_1 , P_2 and P_3 , and positive values of the parameters q and r are not permitted. This ensures that, even for very small samples where the true shape of the curve tends to be obscured by scatter, the resulting S - N curve will have an appropriate shape.

1.5 CURVE-FITTING PROCEDURE

1.5.1 Running Strategy

When using the program the limitations on its use that are set out in Section 2.2 should be given careful consideration. It is envisaged that the program will probably be used in the following way. Initially a plot of the data will be obtained that will enable a preliminary assessment of the data to be made. Depending upon this and the amount of information about the population already available, the number of parameters used to define the curve can be decided upon and, if desired, values assigned to some or all of them. A plot showing the data and the derived curve can then be viewed together with the corresponding value of the natural logarithm of the likelihood, $\ln l$. A satisfactory curve having been derived, values of N for specific values of S can be obtained.

1.5.2 User Input

The basic input to the program consists of pairs of S and N values, plus an indication of which values of N are censored; that is, cases where the value of N is known to be either greater or less than the value input. The program provides a facility for the user to display a scatter plot of the input data at this stage which might assist in deciding what order of model is appropriate.

Equation (1.1), with all the constant parameters being given values other than zero, represents the highest order (that is the most complex) model that can be fitted. The options available to the user are as follows.

1 parameter model	$S = P_1,$
2 parameter model	$S = P_1 N^q,$
3 parameter model	$S = P_1 N^q + P_2,$

4 parameter model

$$S = P_1 N^q + P_2 N^r$$

and

5 parameter model

$$S = P_1 N^q + P_2 N^r + P_3$$

No general rules on choosing a model can be given but, as well as examining a scatter plot of the data, it might be useful, especially when analysing very small samples, to draw upon experience with similar but larger data sets. For small data sets the absolute maximum number of unknown parameters that can be evaluated by the program is equal to one less than the number of pairs of S – N values corresponding to failed specimens but, because of the high degree of scatter inherent in fatigue data, a lower-order model will usually be found to be more satisfactory. When doubt exists as to whether or not there are sufficient data, it is recommended that each data point be removed in turn and the analysis repeated; if the fitted model varies greatly between the analyses then a lower order model should be tried, or more data collected. If, however, the inclusion of a particular data point affects the model significantly, it is possible that this point is an outlier that is not representative of the population (see Section 1.5.4).

The full 5 parameter model is rarely needed, and a satisfactory fit to the vast majority of fatigue data can be obtained using 2, 3 or 4 parameters. The simplest model that gives an adequate fit is to be preferred, since this is likely to give a better representation of the population S – N curve than a higher order model that gives a slightly better fit. In the absence of any prior knowledge, all five orders of model can be fitted automatically and the most appropriate model selected; this facility is described in Section 1.5.3.

A facility is also provided for the user to specify the values of any of the non-zero parameters that can be estimated from experience; in practice this is likely to apply only to the indices q and r . Again this might be of value when dealing with very small data sets, for which the shape of the population S – N curve might be completely obscured by scatter. It might be useful to analyse old data sets using this Data Item in order to obtain this information. This facility is not available in conjunction with automatic selection of the most appropriate model (see preceding paragraph).

An additional option, which is only available when censored data are involved, is that of specifying the value of v , the coefficient of variation of S , if it is known.

Equation (1.1) gives a convenient means of estimating S from N , but not vice versa. The program will, however, if required, calculate values of N corresponding to values of S specified by the user; these values of S form an additional input into the program if it is decided to make use of this facility.

1.5.3 Program Output

The program outputs values of the parameters that define the derived S – N curve, a maximum likelihood estimate of v , the population coefficient of variation of S , a list of any outliers identified, and values of N for any values of S that were specified in the input. The program also outputs the value of the natural logarithm of the likelihood corresponding to the fitted curve. If program W9724 is used, a plot of the data and the fitted S – N curve is output.

The curve obtained is that which gives the best fit to the data within the constraints imposed by the basic model of Equation (1.1) and by the user. The purpose of these constraints is to ensure that all available information, including that obtained from relevant experience, is taken into account to obtain the best possible estimate of the population median S – N curve.

The graphical output covers the same range of endurances as the input data. Extrapolation beyond this range is not recommended and a facility to do this is not provided in the program. Similarly, if the program is used to calculate values of N corresponding to given values of S , results will only be provided within the range of the fitted curve.

It should be emphasised that the maximum likelihood estimate of the coefficient of variation of S given by the program should not be taken to give an accurate indication of the population value. This is because there will be some error in the estimated median curve and also the degree of scatter will vary from sample to sample. The user should therefore exercise great caution when using this value for design purposes.

The value of the natural logarithm of the likelihood corresponding to the fitted curve provides a useful means of comparing and choosing between different orders of model, using Akaike's criterion (Derivations 2 and 4). Akaike's criterion states that, when comparing the natural logarithm of the likelihood of a lower order model, $\ln l_L$, with that of a higher order model, $\ln l_H$, involving an extra unknown parameter, the higher order model should only be preferred if $\ln l_H$ is greater than or equal to $\ln l_L + 1$. This comparison can be carried out by the user, or, if automatic selection has been requested, the most appropriate order of model will have been selected according to Akaike's criterion. However, if knowledge of the expected form of the curve is available, this knowledge may be used to override Akaike's criterion.

1.5.4 Outliers

Outliers are points that appear to be inconsistent with the majority of the data. They may, of course, be genuinely extreme values, but, on the other hand, they may result from defective specimens or improper test procedures, or they may simply be recording errors. Another possibility is that an inappropriate $S-N$ model has been fitted. The program identifies all points corresponding to failures that lie at a distance of more than two standard deviations of $\log S$ from the fitted curve. The purpose of this is to alert the user who can then investigate the possible causes of the discrepancy and, if necessary, take appropriate action. Outliers should only be removed from the data set if there is positive evidence that they do not represent genuine population values.

The fact that no outliers have been identified does not necessarily mean that all of the data points are genuine, particularly in small samples where a single erratic value could modify the fitted curve to such an extent that it does not appear to be an outlier.

1.6 DERIVATION AND REFERENCES

1.6.1 Derivation

This section lists selected sources that have assisted in the preparation of this Data Item.

1. NELDER, J.A. MEAD, R. A simplex method for function minimization. *Computer Journal*, Vol. 7, pp. 308-313, 1965.
2. AKAIKE, H. A new look at the statistical model identification. *IEEE Transactions on Automatic Control*, Vol. AC-19, No. 6, pp. 716-723, 1974.
3. MANSON, S.S. Predictive analysis of metal fatigue in the high cyclic life range. *Methods for predicting material life in fatigue*. Proceedings of the winter annual meeting, American Society of Mechanical Engineers, pp.145-183, December 1979.
4. PRIESTLY, M.B. *Spectral analysis and time series*. Academic Press, London, 1981.
5. PRESS, W.H. *Numerical recipes in FORTRAN*. Second Edition, Cambridge University Press, Cambridge, 1992.

1.6.2 References

The references given are recommended sources of information supplementary to that in this Data Item.

6. SILVEY, S.D. *Statistical inference*. Chapman and Hall, London, 1975.
7. CHATFIELD, C. *Statistics for technology*. Third edition, Chapman and Hall, London, 1984.