

STATIC AEROELASTICITY: A FORMAL ANALYSIS USING ASSUMED MODES

1. INTRODUCTION

Three earlier Data Items, Nos 94009, 94045 and 96037 (References 2, 3 and 4, respectively), provide a background to the present Item.

The first two Items deal with the loading characteristics of rigid aeroplanes of classical configuration in steady trimmed flight and in quasi-steady symmetric manoeuvres at subsonic speeds. The Items identify the aerodynamic lift and inertial distributions on a wing-fuselage combination and include two wing configurations (one unswept and the other swept). The effects of wing-borne engine nacelles are included. Evaluation of the distribution of shear force, bending moment and torque along the wing span, for each of the wing types, and along the fuselage is also provided. Only overall tailplane loads are given.

Data Item No. 96037 provides a qualitative introduction to static aeroelasticity, discussing some of the fundamental concepts associated with the effects of structural flexibility on aeroplanes of classical configuration undergoing steady, or quasi-steady, longitudinal symmetric manoeuvres at subsonic speeds. The Item concentrates on aspects of controllability, loads and stability, identifying the mutual interactions between aerodynamic lift and the deformation of wing, aft fuselage and tailplane and assesses the effects of deformation on manoeuvrability, both trim and manoeuvre control, and on static stability.

In the present Item the formal equations of a flexible aeroplane of classical configuration in trim and in a quasi-steady symmetric manoeuvre are presented. The aeroplane is assumed to have a flexible swept wing (with wing-borne rigid engine nacelles), a flexible swept tailplane and a rigid fuselage; the wing and tailplane are assumed flexible in both bending and torsion about straight swept flexural (*i.e.* or elastic) axes. Two structural configurations are considered; in the first the wing and tailplane ribs are aligned normal to the swept flexural axis, while in the second the wing and tailplane ribs are aligned in the freestream flow direction.

The bending and twist distributions of the wing and tailplane are each assumed to be the sum of a set of individual assumed modes, each mode consisting of an amplitude multiplied by its spanwise shape.

The assumed mode shapes for bending and twist of the wing are taken to be the set of orthogonal modes that satisfy the boundary conditions for a uniform elastic beam, built-in at an effective wing root assumed normal to the axis of bending and twist.

Expressions for overall static stability, given by the elevator trim margin, for static stability at constant speed, given by the elevator manoeuvre margin, and aeroelastic static stability, are provided.

The coefficients of the set of simultaneous equations that determine the unknown variables of incidence, elevator deflection and the wing and tailplane bending and twist deformations, in terms of forward speed and normal acceleration, for the first configuration only in which the wing and tailplane ribs are aligned normal to the swept flexural axis, are given in an Appendix. It is left to the user to derive the equivalent set of coefficients for the second configuration.

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2. NOTATION

Any coherent system of units may be used.

$B(y_f)$	bending moment distribution (see Equation (6.6))
C_Q	weight coefficient (= $W/S / (\frac{1}{2}\rho V_{trim}^2)$)
E_{man} , E_{trim}	elevator manoeuvre margin and elevator trim margin, respectively
$EI(y_f)$	bending stiffness distribution in which E is Young's modulus of elasticity and I is second moment of area
$GJ(y_f)$	torsional stiffness distribution in which G is shear modulus and J is torsion constant
g	acceleration due to gravity
h_{Nac}	z distance of nacelle/engine below wing
i, j, k	general integers
$L(y_f)$	loading distribution (see Equation (6.5))
$(l_f)_W, (l_f)_T$	distances in x -direction from axis origin O of origins O_W and O_T , respectively (see Sketch 3.1)
$\ell(x_f, y_f), \ell(x)$	lift distributions
$\ell_k(x_f, y_f), \ell_k(x)$	lift distributions of mode k for unit amplitude twist or bending
$\ell_q(x_f, y_f), \ell_q(x)$	rate of change of $\ell(x_f, y_f)$ or $\ell(x)$ with pitch rate (= $\partial \ell(x_f, y_f) / \partial q$ or $\partial \ell(x) / \partial q$)
$\ell_{zk}(x_f, y_f), \ell_{zk}(x)$	lift distributions due to unit amplitude bending of mode k (= $\partial \ell_k(x_f, y_f) / \partial z$ or $\partial \ell_k(x) / \partial z$)
$\ell_0(x_f, y_f), \ell_0(x)$	lift distributions due to camber and, where applicable, washout and setting angle
$\ell_\alpha(x_f, y_f), \ell_\alpha(x)$	rate of change of $\ell(x_f, y_f)$ or $\ell(x)$ with incidence (= $\partial \ell(x_f, y_f) / \partial \alpha$ or $\partial \ell(x) / \partial \alpha$)
$\ell_\eta(x_f, y_f)$	rate of change of $\ell(x_f, y_f)$ or $\ell(x)$ with elevator deflection (= $\partial \ell(x_f, y_f) / \partial \eta$ or $\partial \ell(x) / \partial \eta$)
$\ell_{\theta k}(x_f, y_f), \ell_{\theta k}(x)$	lift distributions due to unit amplitude twist of mode k (= $\partial \ell_k(x_f, y_f) / \partial \theta$ or $\partial \ell_k(x) / \partial \theta$)
$\ell_{\varepsilon zk}(x_f, y_f)$	lift distributions due to unit amplitude bending of mode k resulting from wing downwash
$\ell_{\varepsilon \theta k}(x_f, y_f)$	lift distributions due to unit amplitude twist of mode k resulting from wing downwash

$m(x_f, y_f), m(x)$	mass distributions
$(N_z)_W, (N_\theta)_W$	number of modes of wing bending and wing twist, respectively
$(N_z)_T, (N_\theta)_T$	number of modes of tailplane bending and tailplane twist, respectively
$n, \Delta n$	load factor and incremental acceleration in g 's, respectively
O	origin of axis system for overall aeroplane
O_W, O_T	origins of axes systems for wing and tailplane, respectively
q	steady pitch rate ($= \dot{\theta}$)
$r_k(y_f)$	shape function for mode k relating to bending
S	wing area
$(s_f)_W$	distance from O_W to wing tip, measured in y_f -direction
$(s_f)_T$	distance from O_T to tailplane tip, measured in y_f -direction
$s_k(y_f)$	shape function for mode k relating to twist
T	thrust of each nacelle/engine, <i>i.e.</i> engine thrust minus nacelle drag
V_{trim}	trimmed velocity of aeroplane in flight direction
W	aeroplane weight
x, y, z	directions of axes for overall aeroplane with respect to origin O
x_f, y_f, z	directions of axes for wing and tailplane with respect to origins O_W and O_T , respectively (see Sketches 3.1 and 3.2)
x_{fNac}	distance in x_f -direction of nacelle/engine centre of mass (see Sketch 3.2)
y_{fNac}	distance in y_f -direction of nacelle/engine centre of mass (see Sketch 3.2)
$(x_f)_{hinge}$	distance of elevator hinge line from y_f -axis (positive in x_f -direction)
y_f	distance along y_f -axis (for either wing or tailplane)
$z_e(y_f)_W, z_e(y_f)_T$	structural bending distribution of wing and tailplane, respectively, from all modes in z -direction
$z_{ek}(y_f)_W, z_{ek}(y_f)_T$	structural bending distribution of wing and tailplane, respectively, from mode k in z -direction

$(z_{ek})_W, (z_{ek})_T$	bending amplitude in z-direction of mode k for wing and tailplane, respectively
α	aeroplane incidence
$\delta\{y_f - y_{fNac}\}$	denotes impulse function at $y_f = y_{fNac}$
η	elevator deflection
$\theta, \dot{\theta}$	aeroplane pitch angle and rate of change of θ with time, $d\theta/dt (= q)$
$\theta_e(y_f)_W, \theta_e(y_f)_T$	structural twist distribution of wing and tailplane, respectively, from all modes about y_f -axis in x_f -z plane
$\theta_{ek}(y_f)_W, \theta_{ek}(y_f)_T$	structural twist distribution of wing and tailplane, respectively, from mode k about y_f -axis in x_f -z plane
$(\theta_{ek})_W, (\theta_{ek})_T$	twist amplitude of mode k about y_f -axis in x_f -z plane, of wing and tailplane, respectively,
Λ, Λ_f	sweep angle of $1/4$ -chord and y_f -axis, respectively
$\lambda_k, \sigma_k, k = 1, 2, 3, \dots$	variables used in assumed mode shapes
ρ	air density
ϕ_i	constant elements of simultaneous equilibrium equations (see Appendix A)
$(\Psi_A)_i, (\Psi_B)_i$	coefficients of equilibrium equation for α and η , respectively (see Appendix A)
$(\Psi_C)_{ki}, (\Psi_D)_{ki},$ $(\Psi_E)_{ki}, (\Psi_F)_{ki}$	coefficients of equilibrium equation for wing and tailplane bending and twist modes (see Appendix A)
ω_i	variable used in overall static stability (see Equation (9.3))

Subscripts

e	denotes elastic deformation
F	fuselage
f	relates to flexural axis
$front, rear$	refers to extreme front and extreme rear, respectively
le, te	refers to leading edge and trailing edge, respectively, measured in x_f -direction
$man, trim$	denotes manoeuvring flight and trimmed flight, respectively

Nac	nacelle
T	tailplane
W	wing
z	relates to bending deformation
q	relates to aeroplane steady pitch rate
0	denotes quantity independent of aeroplane incidence, elevator angle, twist deformation and bending deformation
α	relates to aeroplane incidence
η	relates to elevator
θ	relates to twist deformation

A bar above a quantity indicates a non-dimensional quantity.

3. BACKGROUND

3.1 Aeroplane Model

The aeroplane configuration and flight conditions in this Item are restricted as follows.

- (i) The aeroplane is assumed classically configured and of the transport type with wings of medium to high aspect ratios, typically ≥ 4 , and up to moderate sweep-back angles, typically $\Lambda \leq 30^\circ$.
- (ii) The flight conditions are restricted to symmetric flight in either steady level conditions or in quasi-steady pull-out manoeuvring conditions throughout the subsonic Mach number range up to the drag rise Mach number.
- (iii) Flight angles, which include the angle of incidence, the pitch angle and the flight path angle, are assumed small.

The consequences of (ii) and (iii) are that linearised aerodynamics are valid.

The aeroplane model considered in this Item, as shown in Sketch 3.1, consists of the following elements.

- (a) The model has a flexible swept wing with a sweep angle, Λ_W , defined as the sweep angle of the $\frac{1}{4}$ -chord of the wing. The wing is assumed to possess built-in camber and washout (a built-in nose down twist that increases towards the wing tip) and is set at an angle, the 'wing setting angle', to the fuselage. The wing may also include wing tip winglets.

The flexible wing is assumed to be built-in to an effective root normal to a flexural axis swept at an angle, $(\Lambda_f)_W$, which is not in general equal to the wing sweep. The wing deformation is assumed to comprise bending of, and twist about, the flexural axis. That part of the wing inboard of the effective root is assumed to be rigid.

- (b) A single rigid nacelle, enclosing an engine, is assumed rigidly mounted beneath each half-wing.
- (c) The model has a flexible swept tailplane with a sweep angle, Λ_T , defined as the sweep angle of the $\frac{1}{4}$ -chord of the tailplane.

The tailplane is assumed to be 'conventional' with a full-span rigid elevator activated by the pilot's stick controller and powered by an actuator set within the rear fuselage.

The tailplane is assumed to have zero camber and zero washout, but is assumed set at a fixed angle, the 'tailplane setting angle', to the fuselage.

The flexible tailplane is assumed to be built-in to an effective root normal to a flexural axis swept at an angle, $(\Lambda_f)_T$, which is not in general equal to the tailplane sweep. The tailplane deformation is assumed to comprise bending of, and twist about, the flexural axis. That part of the tailplane inboard of the effective root is assumed to be rigid.

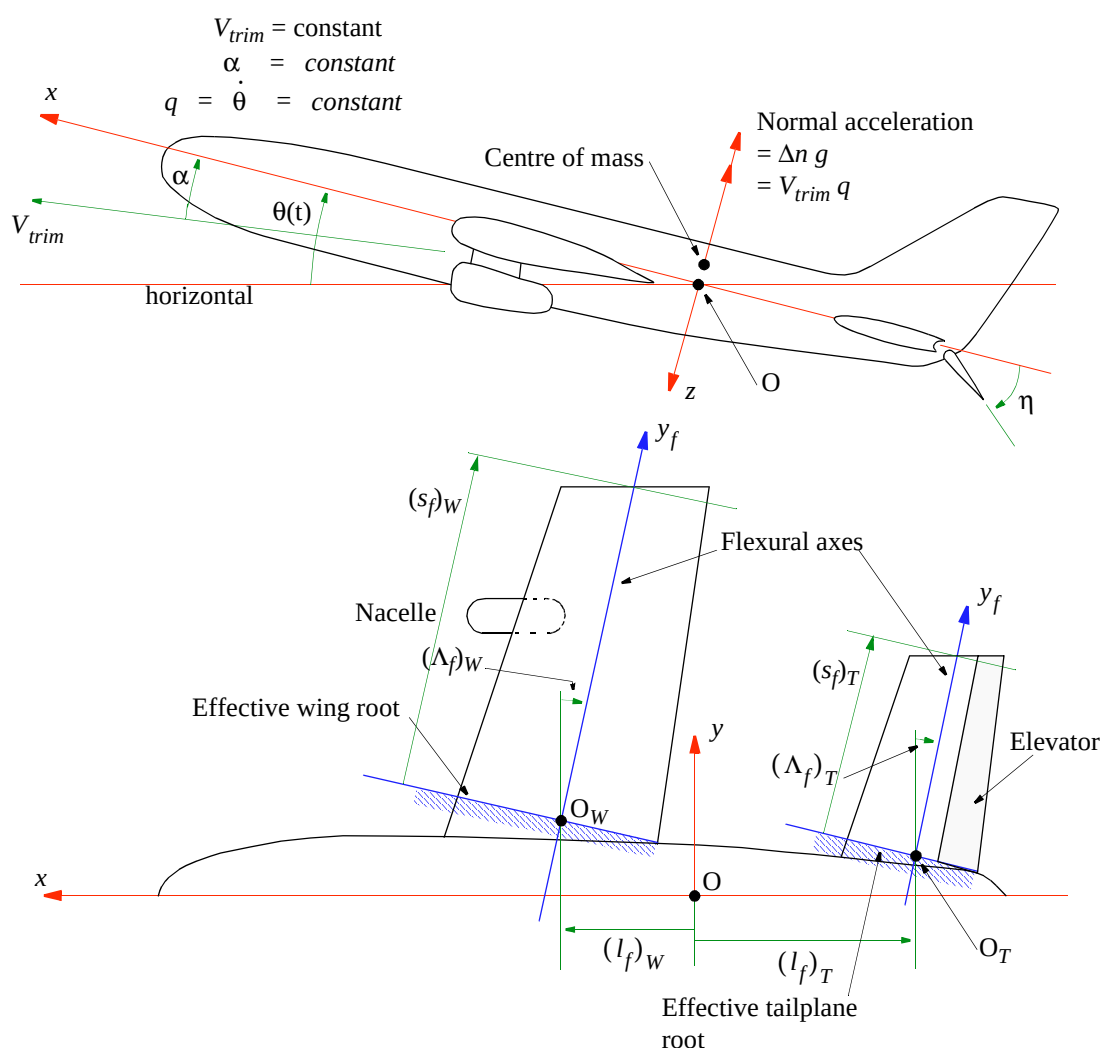
- (d) In this Item the fuselage is assumed rigid throughout its length. This assumption is imposed for simplicity: extension of the analysis to include a flexible fuselage along the lines described in this Item is straightforward.

3.2 Axis System

The axis system chosen for the aeroplane is an Attached or Structural Axis system, in which the axes are fixed in the rigid fuselage with the origin O taken at the mass centre of a 'datum' aeroplane. This 'datum' aeroplane shape is the 'jig' shape of the aeroplane which, in this Item, is taken as the shape when all deformations are zero throughout the complete structure, *i.e.* when all the loads and all the internal stresses are zero. The overall mass distribution is the sum of the structural mass, the equipment mass, the fuel mass and the payload mass distributions.

The body axes are of the right-handed orthogonal type, $Oxyz$, with the x - and z -axes taken in the plane of symmetry of the aeroplane. The x -axis is directed towards the nose, the z -axis is directed downwards and the y -axis points to starboard. The two half-wings are assumed flexible on either side of the rigid fuselage. The tailplane is assumed flexible, mounted on a rigid aft fuselage.

The axis system is illustrated in Sketch 3.1.



Sketch 3.1 Body axis system

The aeroplane shown in Sketch 3.1 is in quasi-steady manoeuvring flight with the centre of mass displaced upwards along the (negative) z -direction from the axis origin. The angles shown are the angle of incidence or angle of attack, α , and the angle of pitch of the aeroplane, θ .

Throughout a quasi-steady manoeuvre the aeroplane trim velocity, V_{trim} , remains constant and the aeroplane has a constant angular velocity, q , which is given by

$$q = \dot{\theta} = \frac{\Delta n g}{V_{trim}}, \quad (3.1)$$

in which the product $(\Delta n g)$ is the constant normal acceleration.

The structural load factor, n is

$$n = 1 + \Delta n.$$

In trimmed steady level flight Δn and q are zero.

In Sketch 3.1, the flexural, or y_f , axes for the wing and tailplane, which lie in planes parallel to the x - y plane of the aeroplane, have been drawn. These axes have origins denoted by O_W and O_T and are located on the effective wing root and effective tailplane root, respectively, taken to be perpendicular to the flexural y_f -axes. The positions of the 'effective root' of the wing and tailplane have each been chosen to coincide with the actual root at the trailing edge of the wing or tailplane. Each flexible half-wing extends over the distance $0 < y_f < (s_f)_W$ and each flexible half-tailplane extends over $0 < y_f < (s_f)_T$.

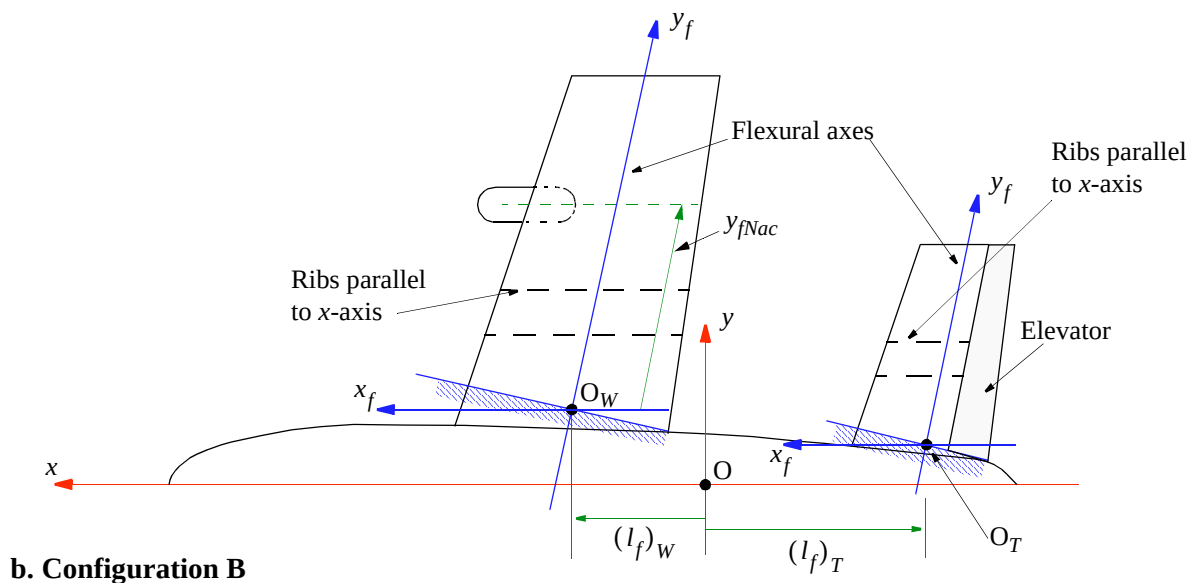
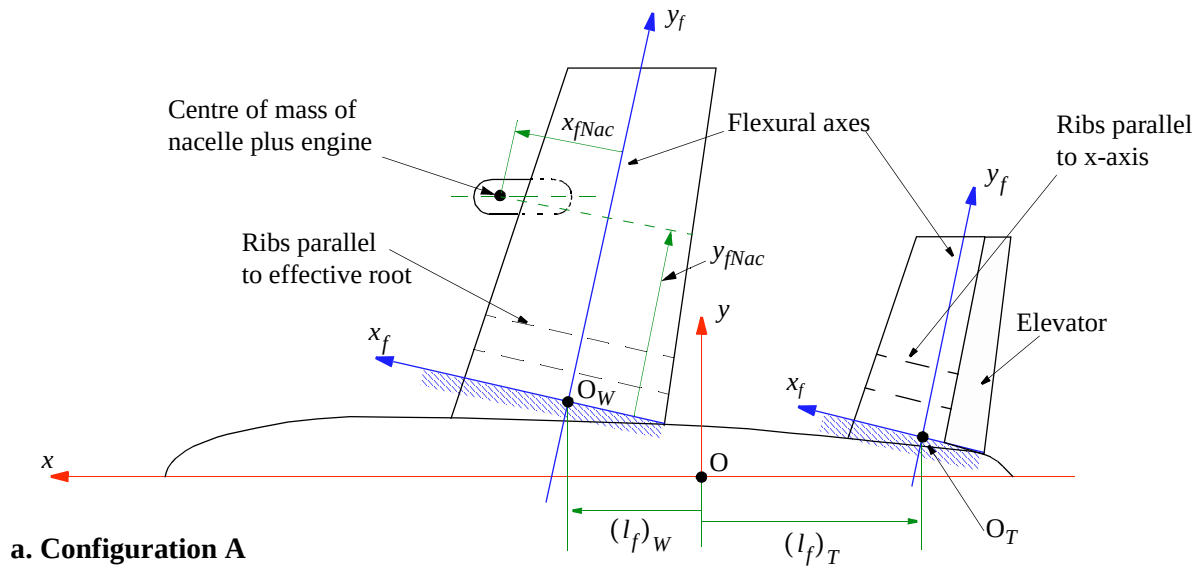
The basis of the assumption of a swept flexural axis normal to an effective built-in root, as indicated in Sketch 3.1, is that the primary structure comprises a twin spar torsion box normal to the built-in root. Now the aerodynamic and inertial wing (or tailplane) loading is transferred into the torsion box via the wing ribs; these wings ribs may be aligned normal to the torsion box (*i.e.* normal to the flexural axis) or may be aligned in the stream direction. These two alternative configurations are shown in Sketch 3.2.

The loading on the torsion box will depend on the orientation of the ribs. It is assumed in this Item that the wing loads and the tailplane loads are transferred into the torsion box in the direction of the ribs.

In **Configuration A**, illustrated in Sketch 3.2a, the ribs of the wing and tailplane are aligned in a direction perpendicular to the flexural axes and parallel with the 'effective root' of the wing and tailplane. An orthogonal axis system $O_W x_f y_f z$ is taken with the x_f -axis normal to the flexural axis; in this case loads are transferred to the flexural axis in the direction x_f normal to the flexural axis. The x_f and y_f distances of the mass centre of the nacelle/engine from the origin, O_W , are denoted by x_{fNac} and y_{fNac} , respectively.

In **Configuration B**, illustrated in Sketch 3.2b, the ribs of the wing and tailplane are aligned in the freestream direction. An oblique axis system $O_W x_f y_f z$ is taken with the x_f -axis parallel to the stream direction; in this case loads are transferred to the flexural axis in the direction of x_f in the freestream direction. The y_f distance of the nacelle/engine from the origin, O_W , is denoted by y_{fNac} , now relative to the oblique axis system.

Although the same (x_f, y_f) notation is used for both configurations (see Sketches 3.1 and 3.2) it should be obvious in the text which set is being used.



Sketch 3.2 Structural configurations

3.3 Aeroplane Degrees-of-freedom

In this Item the dependent variables of the aeroplane are
the aeroplane incidence, α ,
the elevator deflection, η , and
the wing and tailplane deformation modes in both bending and twist,
that each need to be obtained in terms of the independent variables
the trim speed, V_{trim} , and
the normal acceleration, $\Delta n g$.

The downwards spanwise bending deformation of the wing flexural axis, $z_e(y_f)_W$, is taken as the sum of a set of assumed individual deformation distributions, $z_{ek}(y_f)_W$, $k = 1$ to $(N_z)_W$, in which $(N_z)_W$ is the number of assumed modes. The deformation is defined as

$$z_e(y_f)_W = \sum_{k=1}^{(N_z)_W} z_{ek}(y_f)_W = \sum_{k=1}^{(N_z)_W} (z_{ek})_W r_k(y_f), \quad (3.2)$$

where $(z_{ek})_W$ is the amplitude of the k^{th} deformation mode and $r_k(y_f)$ is the spanwise shape of the k^{th} mode.

Each bending mode should satisfy the following boundary conditions,

$$\left(r_k(y_f) \right)_{@y_f=0} = 0 \quad \text{and} \quad \left(\frac{dr_k(y_f)}{dy_f} \right)_{@y_f=0} = 0$$

$$\text{and} \quad \left(\frac{d^2 r_k(y_f)}{dy_f^2} \right)_{@y_f=(s_f)_W} = 0 \quad \text{and} \quad \frac{d}{dy_f} \left(EI(y_f)_W \frac{d^2 r_k(y_f)}{dy_f^2} \right)_{@y_f=(s_f)_W} = 0,$$

where $(s_f)_W$ is the 'effective' half-wing length taken in the y_f -direction from O_W to the wing tip.

These boundary conditions are those for pure bending of a cantilever beam built-in at the effective wing root in which the dimensions of the beam cross-section are small compared with the beam length.

Similarly, for the tailplane the downwards bending deformation, $z_e(y_f)_T$, is taken as the sum of a set of assumed individual deformation distributions, $z_{ek}(y_f)_T$, $k = 1$ to $(N_z)_T$, in which $(N_z)_T$ is the number of assumed modes. The deformation is then defined by

$$z_e(y_f)_T = \sum_{k=1}^{(N_z)_T} z_{ek}(y_f)_T = \sum_{k=1}^{(N_z)_T} (z_{ek})_T r_k(y_f), \quad (3.3)$$

where $(z_{ek})_T$ is the amplitude of the k^{th} deformation mode and $r_k(y_f)$ is the shape of the k^{th} mode. The mode shapes assumed for the tailplane are taken to be the same as those for the wing. Boundary conditions for each bending mode for the tailplane are the same as those for the wing.

The nose-up twist distribution of the wing about the flexural axis, $\theta_e(y_f)_W$, is taken as the sum of a set of assumed individual twist distributions, $\theta_{ek}(y_f)_W$, $k = 1$ to $(N_\theta)_W$, where $(N_\theta)_W$ is the number of modes. The wing twist deformation is then given by the following expression

$$\theta_e(y_f)_W = \sum_{k=1}^{(N_\theta)_W} \theta_{ek}(y_f)_W = \sum_{k=1}^{(N_\theta)_W} (\theta_{ek})_W s_k(y_f), \quad (3.4)$$

where $(\theta_{ek})_W$ is the amplitude of the k^{th} deformation mode and $s_k(y_f)$ is the spanwise shape of the k^{th} mode.

Each twist distribution must satisfy the following boundary conditions,

$$\left(s_k(y_f) \right)_{@y_f = 0} = 0 \quad \text{and} \quad \left(\frac{ds_k(y_f)}{dy_f} \right)_{@y_f = (s_f)_w} = 0 .$$

These boundary conditions are those for pure torsion of a beam in which, as for the bending case, the dimensions of the beam cross-section are small compared with the beam length.

For the tailplane the nose-up twist deformation, $\theta_e(y_f)_T$, is similarly taken as the sum of a set of assumed individual deformation distributions, $\theta_{ek}(y_f)_T$, $k = 1$ to $(N_\theta)_T$, where $(N_\theta)_T$ is the number of modes. The tailplane deformation is defined as

$$\theta_e(y_f)_T = \sum_{k=1}^{(N_\theta)_T} \theta_{ek}(y_f)_T = \sum_{k=1}^{(N_\theta)_T} (\theta_{ek})_T s_k(y_f) , \quad (3.5)$$

where $(\theta_{ek})_T$ is the amplitude of the k^{th} deformation mode and $s_k(y_f)$ is the shape of the k^{th} mode, again taken to be the same as the wing twist modal deformations. Boundary conditions for each twist mode are identical with those for the wing.

4. AEROPLANE LOADING CHARACTERISTICS

The formulae for the loading characteristics of elements of the aeroplane, wing, tailplane, fuselage and nacelles, given in this section, apply to both structural Configurations A and B. In the case of Configuration A the (x_f, y_f) axis system is that described by Sketch 3.2a and in the case of Configuration B the (x_f, y_f) axis system is that described by Sketch 3.2b.

All of the loading descriptions following can include the effects of compressibility at subsonic Mach numbers below the drag rise Mach number.

4.1 Wing

The lift distribution on a wing including winglets can be denoted by $\ell(x_f, y_f)_W$, which extends over the entire wing including the wing region between the effective wing root and the fuselage. This wing lift distribution is made up from,

a lift, $\ell_0(x_f, y_f)_W$, due to wing setting angle, camber and washout (a built-in nose down twist which increases towards the wing tip),

a lift, $\ell_\alpha(x_f, y_f)_W \alpha$, due to aeroplane incidence,

a lift, $\ell_q(x_f, y_f)_W q$, due to pitch rate from the pull-out

and a lift due to the structural deformations caused by bending and twist of the wing.

The wing lift distribution can be expressed as, assuming linearity,

$$\begin{aligned} \ell(x_f, y_f)_W &= \ell_0(x_f, y_f)_W + \ell_\alpha(x_f, y_f)_W \alpha + \ell_q(x_f, y_f)_W q \\ &+ \sum_{k=1}^{(N_z)_W} \ell_{zk}(x_f, y_f)_W (z_{ek})_W + \sum_{k=1}^{(N_\theta)_W} \ell_{\theta k}(x_f, y_f)_W (\theta_{ek})_W, \end{aligned} \quad (4.1)$$

noting that $q = (\Delta n \, g/V_{trim})$ from Equation (3.1). The terms $\ell_{zk}(x_f, y_f)_W$ and $\ell_{\theta k}(x_f, y_f)_W$ are the wing lift distributions due to unit amplitude bending and twist, respectively, of the k^{th} deformation modes of the wing.

The mass distribution of the wing is denoted by

$$m(x_f, y_f)_W.$$

4.2 Tailplane

The lift distribution on a tailplane, including the elevator, can be denoted by $\ell(x_f, y_f)_T$, which extends over the entire tailplane including the region inboard of the effective tailplane root and the fuselage. This tailplane distribution is made up from,

a lift, $\ell_0(x_f, y_f)_T$, due to tailplane setting angle (the tailplane is assumed to possess zero camber and washout) plus downwash effects due to the constant lift on the wing,

a lift, $\ell_\alpha(x_f, y_f)_T \alpha$, due to aeroplane incidence which includes the downwash effects at the

tailplane induced by the wing lift due to incidence,

a lift, $\ell_{\eta}(x_f, y_f)_T \eta$, due to elevator deflection,

a lift, $\ell_q(x_f, y_f)_T q$, due to pitch rate which includes the downwash due to the wing lift due to q ,

a lift from the structural deformations caused by bending and twist of the tailplane,

plus lifts due to the downwash effects from wing bending and twist deformations.

The tailplane lift distribution can be expressed as, assuming linearity

$$\begin{aligned} \ell(x_f, y_f)_T &= \ell_0(x_f, y_f)_T + \ell_{\alpha}(x_f, y_f)_T \alpha \\ &+ \ell_{\eta}(x_f, y_f)_T \eta + \ell_q(x_f, y_f)_T q \\ &+ \sum_{k=1}^{(N_z)_T} \ell_{zk}(x_f, y_f)_T (z_{ek})_T + \sum_{k=1}^{(N_{\theta})_T} \ell_{\theta k}(x_f, y_f)_T (\theta_{ek})_T \\ &+ \sum_{k=1}^{(N_z)_W} \ell_{\varepsilon zk}(x_f, y_f)_T (z_{ek})_W + \sum_{k=1}^{(N_{\theta})_W} \ell_{\varepsilon \theta k}(x_f, y_f)_T (\theta_{ek})_W . \end{aligned} \quad (4.2)$$

The terms $\ell_{zk}(x_f, y_f)_T$ and $\ell_{\theta k}(x_f, y_f)_T$ are the tailplane lift distributions due to unit amplitude bending and twist, respectively, of the k^{th} deformation modes of the tailplane.

The terms $\ell_{\varepsilon zk}(x_f, y_f)_T$ and $\ell_{\varepsilon \theta k}(x_f, y_f)_T$ are the tailplane lift distributions due to wing downwash from unit amplitude bending and twist, respectively, of the k^{th} deformation modes of the wing.

The mass distribution of the tailplane is denoted by

$$m(x_f, y_f)_T .$$

4.3 Rigid Fuselage

The lift distribution along the fuselage, $\ell(x)_F$, includes the nose lift, the lift of the carry-over section of the wing, *i.e.* the fuselage section between the two halves of the net wing, and the lift on the aft fuselage. It is assumed that the fuselage lift effects at the tailplane position, *i.e.* the carry-over section of the tailplane, can be neglected and so the tailplane deformation does not induce fuselage lift.

This fuselage lift distribution is made up from,

an incidence-independent lift, $\ell_0(x)_F$, from fuselage camber and wing induced effects on the fuselage at zero incidence but excluding wing bending and twist deformation,

a lift, $\ell_{\alpha}(x)_F \alpha$, due to aeroplane incidence,

a lift, $\ell_q(x)_F q$, due to pitch rate, q ,

and a carry-over lift due to the structural deformations caused by the bending and twist of the wing.

This fuselage lift distribution can be expressed as, assuming linearity,

$$\begin{aligned} \ell(x)_F &= \ell_0(x)_F + \ell_\alpha(x)_F \alpha + \ell_q(x)_F q \\ &+ \sum_{k=1}^{(N_z)_W} \ell_{zk}(x)_F (z_{ek})_W + \sum_{k=1}^{(N_\theta)_W} \ell_{\theta k}(x)_F (\theta_{ek})_W, \end{aligned} \quad (4.3)$$

in which $\ell_{zk}(x)_F$ and $\ell_{\theta k}(x)_F$ are the fuselage lift distributions from carry-over due to unit amplitude bending and twist, respectively, of the k^{th} deformation mode of the wing.

The lengthwise mass distribution of the fuselage is denoted by

$$m(x)_F.$$

4.4 Rigid Nacelle

A single nacelle/engine is assumed hung beneath each half-wing.

The lift distribution along a nacelle, $\ell(x)_{Nac}$, is made up from,

a lift, $\ell_0(x)_{Nac}$, due to wing setting angle, wing camber and wing washout and any nacelle camber,

a lift, $\ell_\alpha(x)_{Nac} \alpha$, due to incidence of the aeroplane,

a lift, $\ell_q(x)_{Nac} q$, due to pitch rate, q ,

and a lift from the structural deformations caused by the bending and twist of the wing at the position of the nacelle.

This nacelle lift distribution can be expressed as, assuming linearity,

$$\begin{aligned} \ell(x)_{Nac} &= \ell_0(x)_{Nac} + \ell_\alpha(x)_{Nac} \alpha + \ell_q(x)_{Nac} q \\ &+ \sum_{k=1}^{(N_z)_W} \ell_{zk}(x)_{Nac} (z_{ek})_W + \sum_{k=1}^{(N_\theta)_W} \ell_{\theta k}(x)_{Nac} (\theta_{ek})_W, \end{aligned} \quad (4.4)$$

in which the terms $\ell_{zk}(x)_{Nac}$ and $\ell_{\theta k}(x)_{Nac}$ are the lifts on the nacelle due to unit amplitude bending and twist deflection, respectively, of the k^{th} modes of the wing.

A net thrust, T , defined as the engine thrust minus the nacelle drag, acts on each nacelle/engine. This net thrust acts in the stream direction. The distance of the nacelle/engine below the wing in the z -direction is denoted by h_{Nac} .

The mass distribution of a single engine, nacelle and pylon is denoted by

$$m(x)_{Nac}.$$

5. EQUILIBRIUM EQUATIONS IN QUASI-STEADY SYMMETRIC MANOEUVRES FOR AEROPLANES WITH STRUCTURAL CONFIGURATION A (RIBS NORMAL TO FLEXURAL AXIS)

The equilibrium equations for an aeroplane with a flexible wing and tailplane in quasi-steady symmetric manoeuvres (*i.e.* trimmed flight and pull-out manoeuvres) are satisfied by the following conditions:

- (i) zero resultant aeroplane normal force,
 - (ii) zero resultant aeroplane pitching moment,
 - (iii) structural equilibrium bending of the wing,
 - (iv) structural equilibrium bending of the tailplane,
 - (v) structural equilibrium twist of the wing
- and (vi) structural equilibrium twist of the tailplane.

It is assumed that the additional force equilibrium equation in the flight direction involving thrust and drag is decoupled from the equations expressing the above conditions.

In this section (x_f, y_f) refer to the orthogonal coordinate system shown in Sketch 3.2a.

5.1 Overall Aeroplane Normal Force and Moment Equilibrium

Equilibrium of forces in the normal direction, *i.e.* the negative z -direction, gives the following equation

$$\text{Total Aeroplane Lift} - \text{Total Aeroplane Mass} \times (1 + \Delta n) g = 0, \quad (5.1)$$

where the component of thrust in the normal direction has been neglected. With the use of the lift and mass characteristics of Section 4, Equation (5.1) becomes

$$\begin{aligned} & 2 \iint_{\text{wing}} \left(\ell(x_f, y_f)_W - m(x_f, y_f)_W (1 + \Delta n) g \right) dx_f dy_f \\ & + 2 \iint_{\text{tailplane}} \left(\ell(x_f, y_f)_T - m(x_f, y_f)_T (1 + \Delta n) g \right) dx_f dy_f \\ & + \int_{\text{fuselage}} \left(\ell(x)_F - m(x)_F (1 + \Delta n) g \right) dx \\ & + 2 \int_{\text{nacelle}} \left(\ell(x)_{Nac} - m(x)_{Nac} (1 + \Delta n) g \right) dx = 0, \quad (5.2) \end{aligned}$$

where *wing* and *tailplane* refer to the net half-wing and net half-tailplane, *i.e.* a single exposed surface outside the fuselage, including the region between the effective roots and the fuselage.