

METHODS FOR ANALYSIS OF THE DYNAMIC RESPONSE OF STRUCTURES

1. NOTATION

Vectors are indicated by a bar, *e.g.* $\bar{a}$ and matrices are indicated by a double bar, *e.g.* $\bar{\bar{M}}$.

		<i>Units</i>	
		<i>SI</i>	<i>British</i>
$\bar{\bar{A}}$	BEM matrix		
a_j	displacement of <i>j</i> th trial function or nodal point	m	ft
$\bar{a}$	vector of displacement vector of <i>n</i> th trial function or nodal point	m	ft
$\bar{a}_s, \bar{a}_f$	modal amplitude vectors of structure and fluid		
B'	flexural rigidity per unit length	N/m	lbf/ft
$\bar{\bar{B}}$	damping matrix	Ns/m	lbf s/ft
b	width of structure	m	ft
$\bar{\bar{C}}$	boundary displacement matrix		
c_{LI}	longitudinal wave velocity in plate	m/s	ft/s
$\bar{\bar{D}}$	dynamic stiffness matrix		
d	typical dimension of structure, <i>e.g.</i> dominating length	m	ft
$\bar{d}$	nodal displacement vector	m	ft
E	Young's modulus of elasticity	N/m ²	lbf/ft ²
E_j	total energy of subsystem <i>j</i>	J	lbf ft
E_{kin}	kinetic energy	J	lbf ft
E_{pot}	potential energy	J	lbf ft
F	force amplitude	N	lbf
F_j	generalised force amplitude	N	lbf
$\bar{F}$	force vector	N	lbf
f	frequency	Hz	Hz
$H[\]$	transfer function		

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h	thickness, of <i>e.g.</i> a plate	m	in
I	moment of inertia of area	m ⁴	ft ⁴
$\bar{I}$	unit matrix		
i, j	integers		
j_{jj}	joint acceptance of mode (j, j)		
$\bar{K}$	stiffness matrix	N/m	lbf/ft
k	wave-number	m ⁻¹	m ⁻¹
L	differential operator		
M	modal overlap		
$\bar{M}$	mass matrix	kg	lb
m'	mass per unit length	kg/m	lb/ft
m''	mass per unit area	kg/m ²	lb/ft ²
N	number of modes		
$N_{MAX}, N_{max}, N_{sys}$	integers		
$N_1(x), N_2(x)$	local trial functions		
ΔN	number of modes within the range $\Delta\omega$		
n	modal density ($= \Delta N / \Delta\omega$)	modes/(rad/s)	modes/(rad/s)
$P_{input, j}$	power input to subsystem j	W	ft lbf/s
$P_{diss, j}$	power dissipated by subsystem j	W	ft lbf/s
P_{ij}	power transmitted from subsystem i to subsystem j	W	ft lbf/s
p	pressure	Pa	lbf/ft ²
$\bar{p}$	vector of applied loads, <i>e.g.</i> forces, moments and torsions		
$\bar{Q}$	frequency dependent matrix		
$R[\]$	correlation function		
$\bar{r}$	spatial coordinate vector	m	ft
S	area/surface	m ²	ft ²
$S_{pp}(\omega)$	spectrum of pressure field	Pa ²	(lbf/ft ²) ²
$S[\]$	power spectral density		

$\bar{\bar{T}}$	boundary traction matrix		
t	time	s	s
u	displacement	m	ft
V	volume	m ³	ft ³
$\bar{\bar{W}}$	source function matrix	m	ft
w	displacement	m	ft
$\bar{w}$	displacement vector	m	ft
$\langle w \rangle^2$	mean square displacement	m	ft
x, y, z	coordinates in x-, y-, and z- directions	m	ft
α, β	constants		
δ	Dirac delta function		
η	loss factor of system		
η_{ij}	coupling loss factor from i to j		
λ	bending wavelength in structure	m	ft
ν	Poisson's ratio		
ξ_j	j th eigenvalue		
ρ	density	kg/m ³	*
σ	source strength		
τ	time	s	s
$\bar{\bar{\Phi}}$	matrix involving eigenvalues and eigenvectors		
ϕ_j	j th normal mode shape		
$\bar{\phi}$	normal mode shape vector		
$\bar{\bar{\phi}}$	normal mode shape matrix		
ψ_j	j th trial function		
ω	radian frequency (= $2\pi f$)	rad/s	rad/s
ω_j	j th natural frequency	rad/s	rad/s

* A density value expressed in British units as pounds per cubic foot has to be divided by 32.2 before it can be used in the equations given here.

2. INTRODUCTION

The trend in modern design of structures for trains, ships, cars and aircraft has for a long time been towards making them even lighter. This trend is, among other things, driven by a combination of environmental and economic considerations. Lighter constructions imply lower fuel consumption and also reduced expenditure on materials. At the same time it is generally acknowledged that a good design should also be quiet and not exhibit any noticeable vibration levels. Unfortunately, these two objectives of light and quiet, non-vibrating constructions are conflicting. Reducing the material thickness of an element such as a plate will lower the values of its natural frequencies and increase the number of natural frequencies within a given frequency range, thereby increasing the risk of encountering noise and vibration problems.

To reduce noise and vibration levels on existing constructions is often very difficult if not impossible and will often involve high expenditure. In the design process it is therefore desirable to be able to predict the dynamic response of a structure so that excessive vibration levels, resonance problems, failure due to acoustic fatigue and noise problems can be avoided.

The purpose of this Data Item is to provide guidelines on how to choose an appropriate method for prediction of the dynamic response of a structure subjected to some kind of excitation.

3. SCOPE OF THIS DATA ITEM

A large number of methods are available for calculation of the dynamic response of a structure that is subjected to some kind of force excitation. As these methods can be quite complicated, it can be difficult to assess the suitability of a method for a given application. Each method has its specific advantages, disadvantages and limitations. Some methods have found wide use within industry while others are to be considered more as specialist methods generally limited to research applications.

The purpose of this Data Item is to give the engineer faced with a practical problem an introduction to the most commonly used methods and to provide guidance on how to choose a method that is suitable for a particular application. Details of various methods are provided so that the complexity of the mathematical analysis involved can be assessed. On the basis of this information, it should be possible to choose whether an in-house analysis should be performed or the task ought to be assigned to an external consultant. For an in-house analysis, the references provided refer to elementary treatises on the methods described.

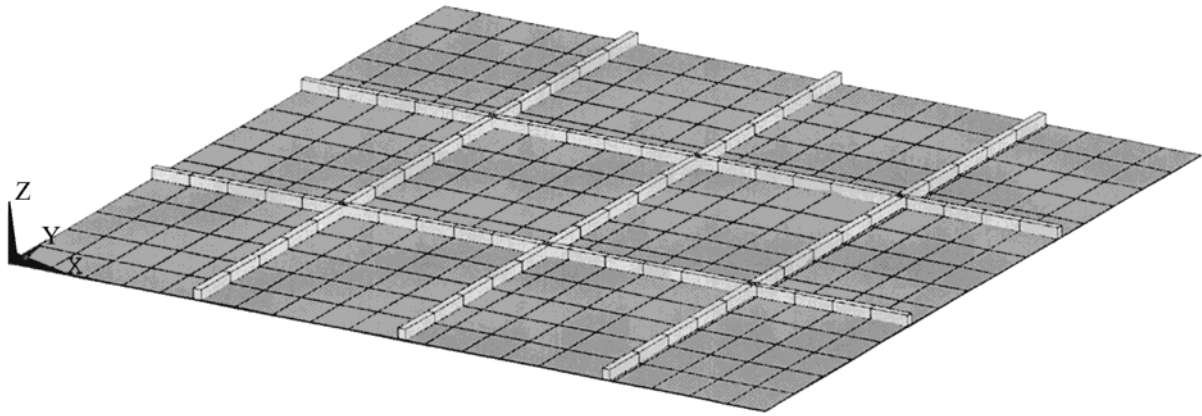
Sections 4 to 7 of this Data Item consider the process of choosing an appropriate method of analysis. In choosing a mathematical model, it is usually required to make an assessment of how the structure is likely to behave when it is vibrating. This is considered in Section 4 which contains a description of some important physical characteristics of structural motion: global response versus localised response, sensitivity to structural design details, *etc.* Section 5 contains a description of various mathematical methods: analytical, numerical and statistical that can be employed to model the structure. In Section 6 a brief description is given of how to model the excitation forces depending on their spatial and temporal characteristics. Section 7 is concerned with methods for solving the equations resulting from Sections 5 and 6 in either frequency or time domains. Finally, Section 8 summarises the content of the previous Sections 4 to 7 in four decision trees. The use of these decision trees is illustrated by three examples in Section 11.

It should be mentioned that extrapolation can sometimes be a useful technique for predicting the response of a new construction. This approach consists of extrapolating measured data pooled from several generally similar structures (termed general extrapolation) or data based on a specific structure (termed specific extrapolation). The rules of extrapolation can either be based on empirical or analytical considerations. Several extrapolation procedures are reviewed in Reference 43.

4. PHYSICAL CHARACTERISTICS OF DYNAMIC RESPONSE

Before considering dynamic analysis techniques in detail, it is helpful to consider three characteristics of the dynamic response that are important for selecting a suitable method for modelling a structure: global modes versus localised modes, sensitivity to structural details and modal overlap.

To illustrate global response and localised response a simply supported orthogonally stiffened plate is shown in Sketch 4.1; this form of stiffening is used widely in the aerospace and marine industries.



Sketch 4.1 Orthogonally Stiffened Panel

The material of the stiffened plate is aluminium alloy and the overall dimensions of the plate are 1 m by 1 m with a thickness of 1 mm. The detailed properties of the plate will be disregarded in this context as they are unimportant for the following considerations. The first mode of free vibration at 36 Hz is shown in Sketch 4.2a, where it can be seen that the motion is distributed throughout the whole structure (global mode) and the half wavelength, $\lambda/2$, of the bending wave in the structure is of the same order of magnitude as the dominating dimensions of the plate, that is, the lengths of the sides. In contrast the 6th mode of vibration at 86 Hz is shown in Sketch 4.2b, where it can be seen that the motion is mainly limited to certain regions of the structure (localised mode) and the half wavelength, $\lambda/2$, of the bending wave in the structure is considerably reduced compared to the lengths of the sides of the plate. This raises two important points regarding the vibration behaviour of a structure with increasing frequency: (i) the modes tend to become localised, that is, the deflections tend to become limited to certain regions of the structure, and (ii) the wavelength of the bending wave decreases. The second point can readily be quantified by considering the analytical expression for the wavelength, λ , of a bending wave of a thin homogeneous plate (Reference 21),

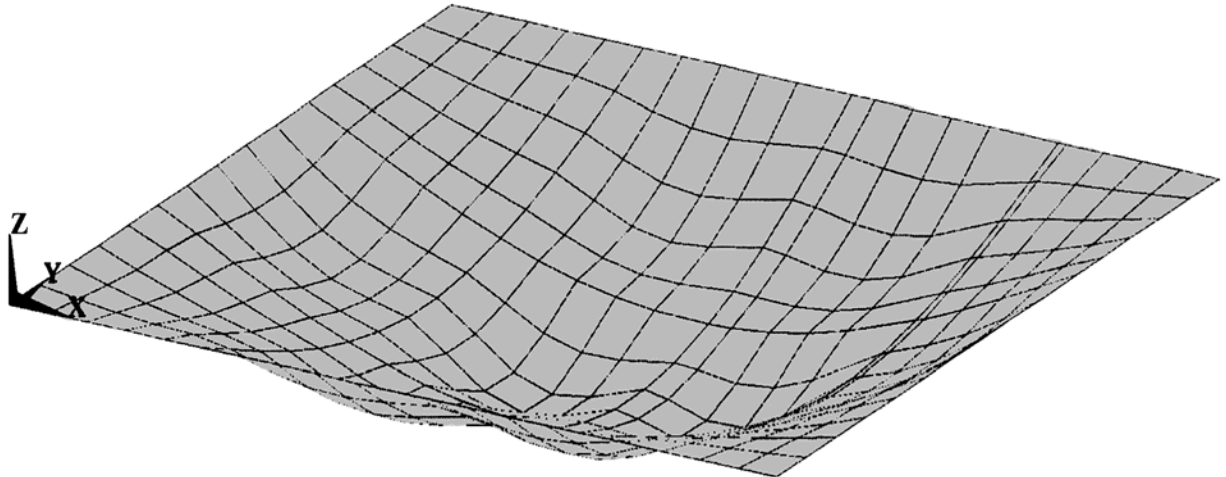
$$\lambda \approx \sqrt{1.8 c_{LI} h / f} \quad , \quad (4.1)$$

$$\text{where} \quad c_{LI} = \sqrt{E / \rho (1 - \nu^2)} \quad . \quad (4.2)$$

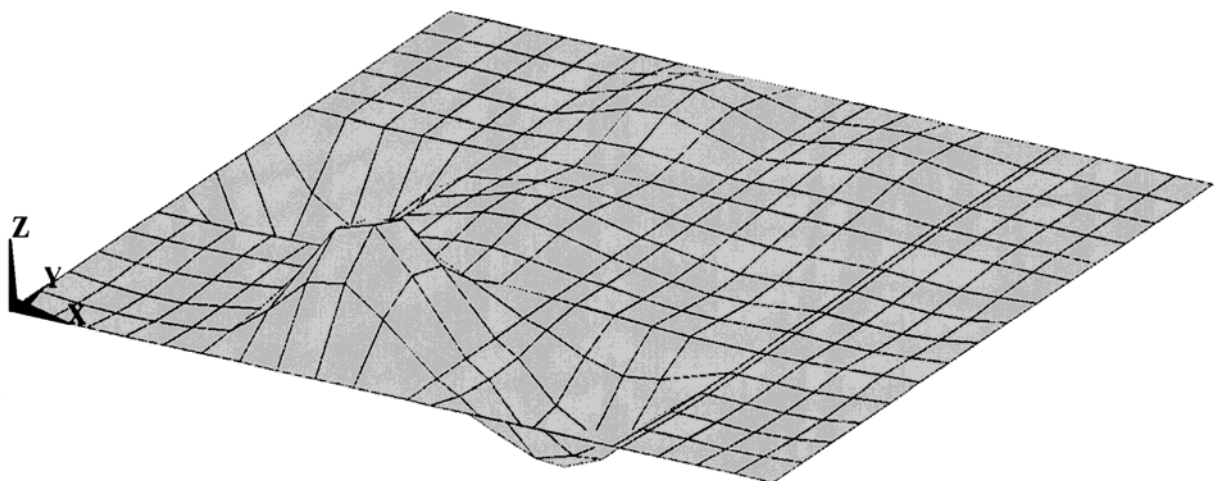
The longitudinal wave speed, c_{LI} , for aluminium alloy is approximately 5410 m/s, so that the bending wavelength in a 1 mm thick plate will be 0.99 m at 10 Hz and 0.10 m at 1 kHz. The implications of points (i) and (ii) are significant. Firstly, if the response at high frequencies is predominantly in localised modes, then it may not be necessary to model the whole structure in order to compute the stresses in a particular region (advantage is normally taken of this fact when predicting acoustic fatigue in aircraft structures). Secondly, the response can be expected to become more and more complicated at higher frequencies due

to the decreasing wavelength of the structural deformations. Modelling difficulties can therefore be expected to increase with frequency because the complexity of the mathematical model generally must increase to describe the more complicated mode shapes of the structure.

(a)



(b)



**Sketch 4.2 (a) First mode of orthogonally stiffened panel at 36 Hz,
(b) sixth mode of orthogonally stiffened panel at 86 Hz**

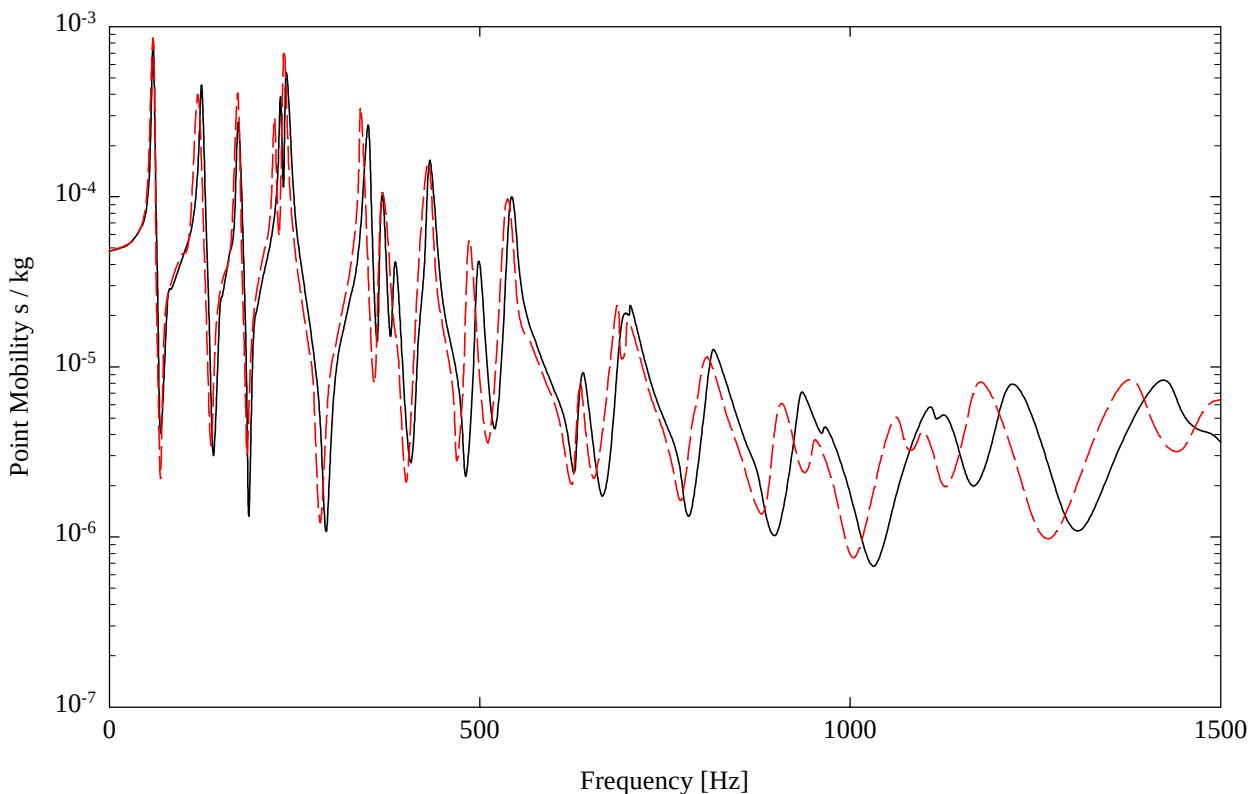
The second general feature of the dynamic response is the increased sensitivity to structural details with increasing frequency. This is illustrated in Sketch 4.3 which shows a comparison between the responses represented by the point mobilities of two almost similar simply-supported rectangular plates. Both of the plates are made of aluminium alloy with thickness 1 mm and width 0.25 m. The only difference between them is the lengths, which are 0.33 m and 0.34 m. As the plates are almost identical, the responses are seen to be almost the same at low frequencies. However, as frequency is increased the difference between the two point mobilities is seen to become larger in spite of the relative small differences in lengths.

The third general feature of the dynamic response is increased modal overlap, M , with increasing frequency. This feature can also be observed from any of the two responses shown in Sketch 4.3. At low frequencies the individual natural frequencies of the system are clearly visible as peaks in the point mobility. As the frequency increases, the peaks tend to merge into one another and the response curve becomes smoother. This is a feature of increasing modal overlap, M , in the sense that the peaks become broad in comparison to the average spacing in frequency between the natural frequencies. The modal overlap, M , is defined (Reference 42) as

$$M = \omega \eta n(\omega) , \quad (4.3)$$

where ω is the radian frequency, η is the loss factor and $n(\omega)$ is the modal density (the average number of natural frequencies per unit radian frequency). The modal density is on average constant for a plate. The modal overlap for a plate with a constant loss factor will therefore increase with frequency.

These three features of the dynamic response must be borne in mind when considering the selection of a dynamic analysis method.



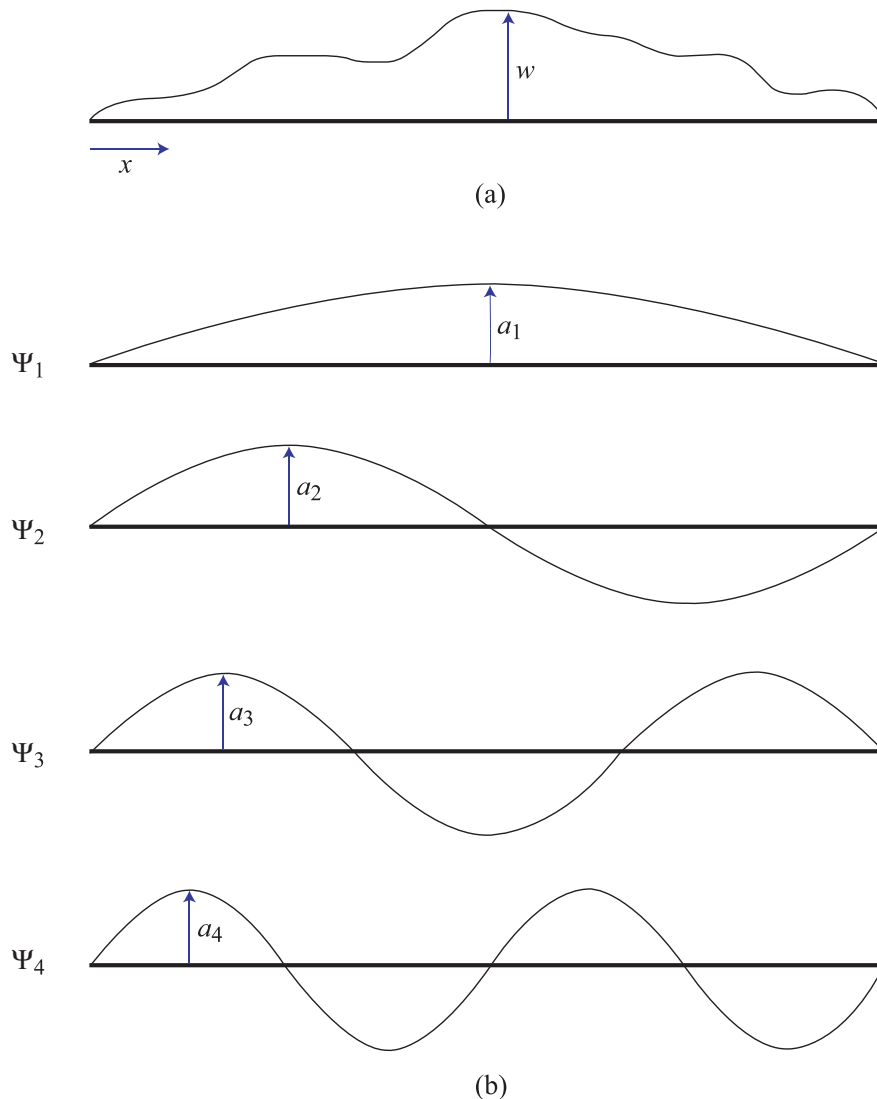
Sketch 4.3 Point mobilities of simply-supported aluminium alloy panels both with width 0.25 m and thickness 0.001 m: — length 0.33 m; - - - length 0.34 m

5. MODELLING OF THE STRUCTURE

The basic issue to be addressed in formulating a theoretical model of a vibrating structure is how to represent the deformed shape in mathematical terms. In theory, an infinite quantity of data is needed to describe fully the motion of a continuous system, since each 'point' on the system can be displaced and there are an infinite number of points. Clearly some degree of simplification is required if a useful mathematical model is to be established; specifically, the structural motion must be expressed in terms of a finite number of variables known as the degrees of freedom of the system. The most common structural modelling techniques are described in the following Sections, and a summary of their main advantages and disadvantages is given in Table 9.1.

5.1 Global Trial Functions

With this approach the displacement of the system is modelled in terms of a set of known global trial functions, ψ_j , where the subscript j refers to global trial function number j . This is illustrated in Sketch 5.1 for the case of a string.



Sketch 5.1 (a) Displacement of a tensioned string, (b) displacements of trial functions $\psi_j(x)$

The displacement $w(x)$ to be modelled is shown in Sketch 5.1a and the first four global trial functions, ψ_1 to ψ_4 , are shown in Sketch 5.1 as sine terms. The displacement $w(x)$ is expressed as the sum of each trial function ψ_j times an unknown amplitude a_j , i.e.,

$$w(x) \approx \sum_{j=1}^{N_{MAX}} a_j \psi_j(x) \quad , \quad (5.1)$$

where N_{MAX} is the total number of global trial functions. In theory any motion of a system can be modelled exactly by using an infinite number of trial functions; in practice a sufficiently accurate representation is obtained by using a finite number of terms, N_{MAX} , corresponding to the degrees of freedom of the model. The aim of the analysis is then to formulate the equations of motion that govern the system and from them determine the unknown amplitudes a_j .

The trial functions shown in Sketch 5.1b are referred to as global trial functions, in the sense that each function extends over the whole structure. The choice of global trial functions is largely a matter of judgement and the most suitable set of functions will depend to a large extent on the structure under consideration. Having selected the global trial functions, the equations of motion that govern the amplitudes, a_j , can, in principle, be formulated by using Lagrange's equation or some other approach such as Galerkin's method (Reference 19). In practice, this procedure requires specialist skills and the process may be unwieldy in an engineering environment. One notable exception to this arises when the structure under consideration is a simple component such as a beam, plate, or shell, with known natural frequencies and normal mode shapes. In this case the global trial functions can be selected to be the normal modes, and the equations of motion will then have a simple uncoupled form that can be written down immediately. This approach is sometimes employed when considering the response of structural components to acoustic excitation (References 43 and 44). The sine terms shown in Sketch 5.1b are in fact the first four normal modes of the string.

Individual segments of aircraft and spacecraft structures can often exhibit quite self contained local response to the noise environment, particularly plate-like items such as aircraft panels and large dish antennas. The local modes of such components most susceptible to noise induced damage are those where the trace wave length of the incident sound field is longer than the bending wave length of the structural mode in question (References 5 and 27). In these circumstances, use can be made of the single mode response to estimate the dynamic stress and by that the risk of acoustic fatigue (Reference 44).

Although the main concern of the present Data Item is with mathematical modelling, it can be noted that experimentally determined mode shapes can be used as global trial functions for a complicated structure. This leads to a set of modal equations of motion that can be updated to investigate the effect of structural modifications.

5.2 Local Trial Functions: The Finite Element Method

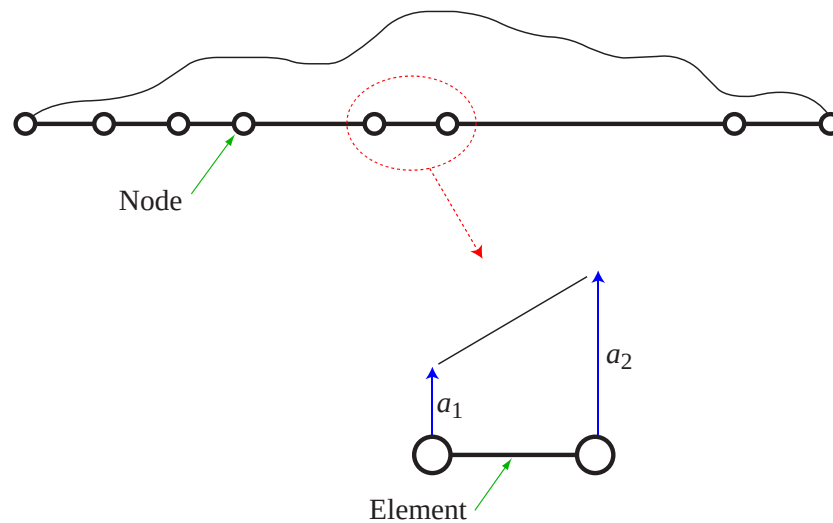
The Finite Element Method (FEM) (Reference 9) is without doubt the most widely used structural modelling technique in industry. This method is again based on the use of trial functions to model the system displacements, but in this case each trial function is defined over a limited region of the structure. This is shown in Sketch 5.2 for the case of a tensioned string.

The structure is divided into a number of 'elements' and each element has a number of 'nodes' - in this case two. The number of nodes form the degrees of freedom of the model, and the displacement $w(x)$ of

some point within an element is interpolated from the nodal displacements a_1 and a_2 as

$$w(x) \approx N_1(x) a_1 + N_2(x) a_2, \quad (5.2)$$

where $N_1(x)$ and $N_2(x)$ are local trial 'shape' functions. A complicated structure can be represented by an assembly of beam, plate, shell and solid elements, and this versatility, together with the availability of a wide range of commercial software, has led to the popularity of FEM.



Sketch 5.2 Finite element model of tensioned string with modes and element with trial function

The accuracy afforded by a finite element model can be improved by increasing the number of nodes and hence increasing the number of degrees of freedom. The number of degrees of freedom that are required will depend to a large extent on the properties of the applied load and in particular on the upper frequency limit of the model. As discussed in Section 4, the wavelength of a bending wave, apart from some very special cases, decreases with increasing frequency, which implies that more degrees of freedom are needed to capture the details of the response at higher frequencies. As a very rough guideline, approximately four elements are needed per half wavelength, so that the fourth mode of a simply-supported beam would require 16 elements. Similarly, the (4,4) mode of a simply-supported rectangular plate would require 256 rectangular elements. Clearly, the number of degrees of freedom involved in a finite element model can become very large, and models having hundreds of thousands of degrees of freedom are not uncommon. For example, around 550 000 degrees of freedom are required to capture the modes of vibration below 225 Hz in a 2 m section of the Fokker F27 commuter aircraft fuselage (Reference 35). Various methods are available for reducing the size of a dynamic model, including Guyan reduction, substructuring, h - p technology, and modal synthesis. Full details of these methods can be found in specialist texts (References 9, 12 and 13), although modal synthesis is briefly discussed in what follows.

The most direct method of computing the response of a system by using the finite element method is to formulate the equations of motion that govern the nodal displacements and then solve them to yield the nodal displacements for a given excitation. For large finite element models this procedure can require excessive computer resources, particularly for the case of broad-banded loading where the response must be computed at numerous excitation frequencies. Another approach is to use modal synthesis in which the finite element model with N_{MAX} degrees of freedom is used to calculate the natural frequencies, ω_j , and

normal mode vectors $\bar{\phi}_j$ of the model, *i.e.*, solving the eigenvalue problem described by

$$\left[-\omega_j^2 \bar{M} + \bar{K} \right] \bar{\phi}_j = 0 . \quad (5.3)$$

The system response is then represented in terms of a limited number, N_{max} , of these mode shapes, *i.e.* by

$$\bar{w} \approx \sum_{j=1}^{N_{max}} a_j \bar{\phi}_j . \quad (5.4)$$

In this way, the number of degrees of freedom might be reduced from hundreds of thousands of nodal degrees of freedom to tens of modal amplitudes. The number of modes to be included is decided on the basis of the frequency range covered by the excitation. Normally only those modes with natural frequencies that fall within the frequency band of the excitation are included, and the quasi-static response arising from the higher modes is recovered by using the mode acceleration technique (References 12 and 13). The modal synthesis method can be further extended by the inclusion of static modes (Reference 12).

If the structural damping matrix can be written as a linear combination of the mass and stiffness matrices (Rayleigh damping) then the modal equations of motion are uncoupled and the system reduces to a series solution of single degree of freedom responses. If the damping is non-proportional then the modal equations of motion are coupled through the effects of damping. Although this coupling effect is analytically straightforward, most commercial finite element codes do not include this capability. Often it is stated that non-proportional damping requires the use of the 'direct' solution procedure. However, this will depend on the details of the structure under investigation and the software employed for the analysis. It should be noted that possible modal coupling can be ignored if the modal overlap is low, *i.e.* the modal bandwidth is less than the modal spacing. Another approach to modal analysis in the presence of non-proportional damping is to employ uncoupled complex modes (Reference 25), though this facility is normally not available in commercial software.

Modal synthesis will enable efficient analysis of systems in which a limited number of modes are being excited. In certain vibro-acoustic problems, thousands of resonant modes may be excited, and the finite element method can become computationally unfeasible. Furthermore, allowance should be made for the fact that at higher frequencies the response of a structure can become very sensitive to structural details, and the sensitivity of the finite element model to small structural changes should be investigated.

One aspect of the finite element model not mentioned thus far is that most commercial packages allow the study of non-linear problems. Non-linearities can be important if large deformations occur, or the material properties are significantly non-linear such as is the case with rubber and plastics.

5.3 The Boundary Element Method

The Boundary Element Method (BEM) (Reference 11) differs in concept from the trial function methods outlined in the previous two sections. The trial function approach in the FEM is based on assuming a particular form of structural displacement and then employing Lagrange's equation or a similar approach to yield the equations of motion of the system. This approach leads to a system mass matrix and stiffness matrix from which the natural frequencies and/or the forced response can be computed. The resulting solution satisfies the governing differential equations approximately and the displacement type boundary conditions exactly. In contrast, a solution yielded by the BEM satisfies the governing differential equations exactly and the boundary conditions approximately. The method is based on Green's Formula, which represents the displacement at some point within a structural component in terms of an integral of certain

terms around the structural boundary. For example, in the indirect version of the BEM, the displacement of a membrane would be represented as a distribution of point sources around the membrane boundary. The degrees of freedom in the formulation would consist of the source strengths at N_{MAX} locations and the governing equations would be developed by applying the boundary conditions at N_{MAX} boundary locations. The direct version of the method differs slightly in that the degrees of freedom would consist of the physical displacements and/or slopes on the boundary.

The main advantage of the BEM is that only the boundary of a component needs to be discretised. In general, this leads to fewer degrees of freedom than are needed by the finite element method, particular for two- and three-dimensional components. If the BEM is to be applied to structures then there must exist a Green's function for the type of structural element, that is, a solution to differential equation for a point force. In general, the BEM will only be advantageous to use as compared to the FEM if some of the elements extend to infinity. The main disadvantages of the BEM are that the matrices tend to be fully populated rather than banded as in the FEM and the computational effort to assemble the equations can be significant. Furthermore as the BEM yields a frequency dependent stiffness matrix, the natural frequencies will normally have to be found by a determinant search rather than by solving an eigenvalue problem as in the FEM.

The main use of the BEM in industry has been in vibro-acoustics where it has been employed to model either radiation from a structure into a semi-infinite acoustic space or free-field incidence on a structure. In this regard, a boundary element model of the acoustic volume can be coupled to a finite element model to produce an efficient hybrid analysis scheme (Reference 33).

5.4 The Dynamic Stiffness Method

The FEM approach applied to a continuous structure results in a lumped parameter model where the degree of discretisation determines the upper frequency limit of the model. For certain continuous elements it is possible to derive the dynamic properties of a structural component exactly. This can always be done for a homogeneous one-dimensional system. The string in Sketch 5.2 would in the Dynamic Stiffness Method (DSM) be represented by a single element and further subdivision will not improve the accuracy. This approach leads to an exact description of the structure of the form (Reference 15)

$$\bar{\bar{D}}\bar{a} = \bar{F}, \quad (5.5)$$

where $\bar{\bar{D}}$ is an exact dynamic stiffness matrix, $\bar{a}$ is a vector with the nodal displacements (angular, radial and axial) and $\bar{F}$ is a vector containing the external forces and moments at the nodes. The presence of distributed loading can be accounted for through the use of particular integrals, and this leads to an effective nodal load vector. The elements in the dynamic stiffness matrix are frequency dependent and contain both mass and stiffness information. At any selected frequency, the equations of motion can be solved to yield the response of the structure. A typical application of the direct-dynamic stiffness method is in the assembly of one-dimensional elements assembled in two- and three-dimensional frameworks (Reference 28). The method is ideally suited to study high frequency response both to harmonic and random excitation. The method has also been applied to satellite dynamics (Reference 7) and via integral transforms to impulsive loading (Reference 15).

The dynamic stiffness method is also applicable to a limited class of two-dimensional structural elements such as plates and shells. The main restriction for this application is that the each element must be simply supported on two opposite edges to allow the displacement to be expanded as a Fourier series in one of the spatial coordinates. For a rectangular plate with simple supports at $y = 0$ and $y = b$, the displacement can