

Quaternion representation of aeroplane attitude and motion characteristics

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QUATERNION REPRESENTATION OF AEROPLANE ATTITUDE AND MOTION CHARACTERISTICS

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QUATERNION REPRESENTATION OF AEROPLANE ATTITUDE AND MOTION CHARACTERISTICS

1. INTRODUCTION

In onboard navigation the motion of an aeroplane relative to the earth is estimated from measurements taken on a moving aeroplane.

In flight simulation both the motion of an aeroplane relative to earth and the aeroplane orientation in space are required in order to provide the appropriate visual presentation relative to the aeroplane motion as produced by pilot actions.

Sensors within an aeroplane, in the form of accelerometers and rate gyros, measure aeroplane motion characteristics relative to axes fixed within the airframe. Three accelerometers, one aligned along each of the three axes, measure the components of absolute linear time-varying acceleration, including the Coriolis acceleration and gravity from a changing aeroplane orientation. Three rate gyros measure the components of aeroplane angular velocities about each of these same fixed axes. The problem is to estimate the aeroplane motion relative to the earth from the measurements of the accelerometers and rate gyros. The two main methods of achieving this conversion are with the use of Euler angles and with quaternions, the first of these being the more traditional approach.

In the Euler method, the orientation of an axis system fixed in the aeroplane relative to another fixed in the earth can be defined from three separate rotations about each of three axes, where the rotations are completed in strict order. The traditional rotational angles are the Euler angles of yaw, pitch and roll (ψ , θ and ϕ , respectively). The Euler angles have a direct physical interpretation and this approach is suitable for all aeroplane motions involving pitch changes of less than $\pm 90^\circ$. When the pitch angle approaches 90° ($\theta \rightarrow 90^\circ$), both ψ and ϕ approach infinity, and so computational difficulties arise.

The quaternion[†] is a four-parameter system that may be physically interpreted as a rotation through some angle about a specific fixed axis system, each axis of which lies at an angle to a fixed orthogonal (x, y, z) axis system. The quaternion is consequently hard to visualise but its advantages outweigh the remoteness of physical interpretation.

This Item first reviews the Euler angle method of axis and motion parameter conversion before outlining the quaternion method. The quaternion method is defined in terms of four parameters that are shown to be equivalent to an alternative set of Euler angle transformations. Expressions are developed for the body angular velocities in terms of the quaternion parameters and then in terms of the rates of change of the quaternion parameters themselves.

Block diagrams illustrate the conversion of the aeroplane motion parameters from body to earth axes for Euler angles and for the quaternion parameters.

[†] The concept of a quaternion as a four-parameter variable was first developed by Hamilton in 1843.

2. NOTATION

Any coherent system of units may be used.

A, B, C, D	substitution quantities (see Equations (A1.2) or (5.25) and (A1.5) to (A1.7))
$\underline{A}$	absolute acceleration vector of origin O relative to O_E
$\{A\}$	components of $\underline{A}$
$\underline{A}_{Acc}$	accelerometer measurement vector
$\underline{e}, \underline{e}^*$	a quaternion and its conjugate, respectively (see Equations (5.1) and (5.3))
e_0, e_1, e_2, e_3	quaternion parameters (see Equations (5.15) to (5.18))
g	acceleration due to gravity
$\{\Delta g\}_B$	gravitational vector (see Equations (4.14) and (5.28))
$[I]$	unit matrix
i, j, k	quaternion operators (see Equation (5.2))
K	constant factor, part of error term $K\sigma$ (see Appendix B)
O	origin of aeroplane axis systems and centre of mass of aeroplane
O_E	origin of fixed earth axis system
$Oxyz$	body-fixed orthogonal axis system with origin O (see Sketch 3.1)
$Ox_0y_0z_0$	body axis system with origin O in which x_0, y_0 and z_0 are at all times parallel to X_0, Y_0 and Z_0 , respectively (see Sketch 3.1)
$Ox_1y_1z_1$	intermediate body axis system with O (see Sketch 3.1)
$Ox_2y_2z_2$	intermediate body axis system with O (see Sketch 3.1)
$O_EX_0Y_0Z_0$	earth-fixed orthogonal axis system with origin O_E (see Sketch 3.1)
$\underline{P}$	spatial vector of origin O relative to O_E
$\{P\}$	components of spatial vector $\underline{P}$
p, q, r	angular velocities of aeroplane about Ox, Oy and Oz, respectively
Q, Q'	quaternions defined by Equations (5.5) and (5.6), respectively
$[R_{\tau\epsilon\delta}]$	quaternion transformation matrix (see Equation (5.24))

$[R_{\phi\theta\psi}]$	Euler angle transformation matrix (see Equation (4.12))
$[T_{\tau}], [T_{\varepsilon}], [T_{\delta}]$	transformation matrices defined by Equations (5.11), (5.12) and (5.13), respectively
$[T_{\psi}], [T_{\theta}], [T_{\phi}]$	transformation matrices defined by Equations (4.1), (4.2) and (4.3), respectively
$[T_{\tau\varepsilon\delta}]$	‘alternative Euler’ angle transformation matrix (see Equation (5.14)), or quaternion transformation matrix (see Equation (5.20))
$[T_{\phi\theta\psi}]$	Euler angle transformation matrix (see Equation (4.5))
t	time
$\underline{V}$	velocity vector of origin O relative to O_E
$\{V\}$	components of velocity vector $\underline{V}$
X_E, Y_E, Z_E	x, y and z distances from O_E to O (see Sketch 3.1)
σ	error variation in quaternion parameters (see Appendix B)
$\delta, \varepsilon, \tau$	‘alternative Euler’ angles (see Section 5.2)
ψ, θ, ϕ	Euler angles of yaw, pitch and roll, respectively

Subscripts

Acc	denotes accelerometer
B	relates to body
E	relates to earth
$0, 1, 2$	relates to axis systems $Ox_0y_0z_0$, $Ox_1y_1z_1$ and $Ox_2y_2z_2$, respectively

Superscripts

T	denotes matrix transform
$*$	denotes conjugate of quantity

One dot or two dots above a quantity respectively indicate the first or second differential of that quantity with respect to time, t .

3. AXIS SYSTEMS AND EULER ANGLES

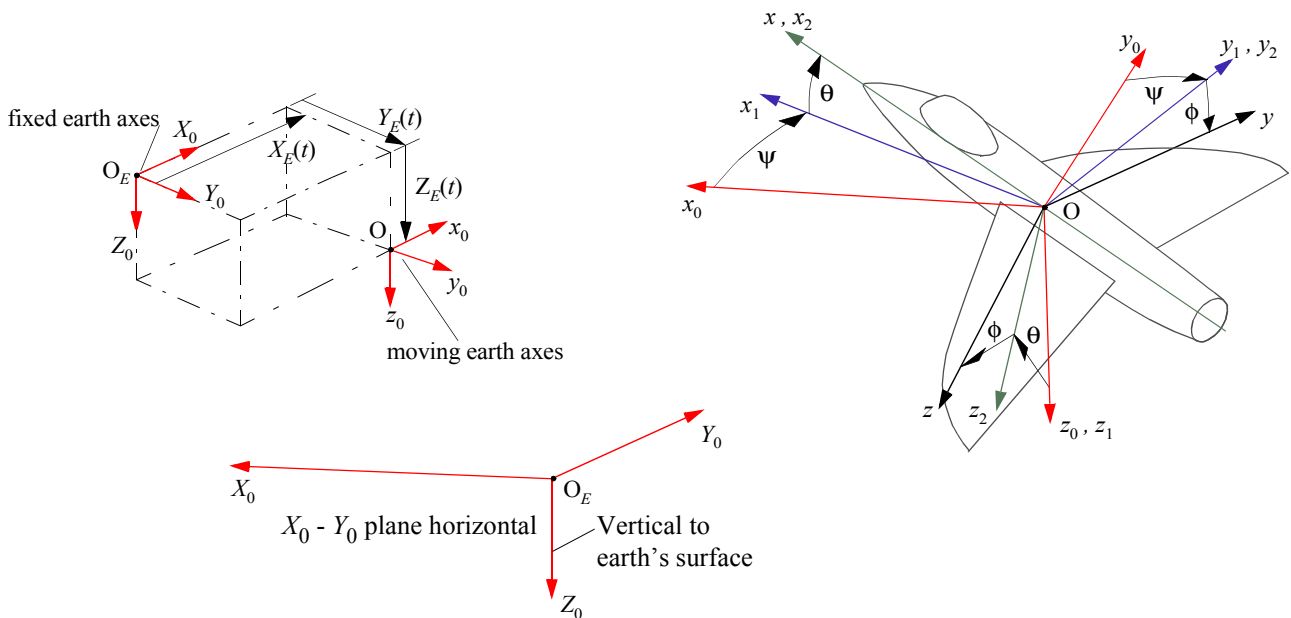
Consider a right-handed orthogonal axis system, $O_E X_0 Y_0 Z_0$, fixed relative to the earth's surface with origin O_E in which the X_0 - Y_0 plane is horizontal and the Z_0 -axis is directed vertically downwards.

The centre of mass of an aeroplane is denoted as O and the time varying spatial position of O relative to O_E is denoted by the distances $X_E(t)$, $Y_E(t)$ and $Z_E(t)$, as illustrated by Sketch 3.1.

Consider a moving right-handed orthogonal body-fixed axis system, $Oxyz$, fixed in a rigid aeroplane with the origin at O and in which the x - and z -axes are fixed in the aeroplane plane of symmetry, with the x -axis directed towards the nose, the y -axis pointing to starboard and the z -axis directed downwards.

Consider in addition a moving right-handed orthogonal axis system, $Ox_0 y_0 z_0$, with the same origin O but in which the axes x_0 , y_0 and z_0 remain at all times parallel to the fixed earth axes X_0 , Y_0 and Z_0 .

These moving body axes $Oxyz$ and the moving earth axes $Ox_0 y_0 z_0$ are inclined relative to one another by the yaw angle, ψ , the pitch angle, θ , and the roll angle, ϕ . Sketch 3.1 illustrates these axis systems.



Sketch 3.1 Aeroplane body and earth axis system

Sketch 3.2 shows a sequence of plane rotations from the axes, $Ox_0 y_0 z_0$, through two intermediate axes, $Ox_1 y_1 z_1$ and $Ox_2 y_2 z_2$, to the final axes $Oxyz$ defined as follows.

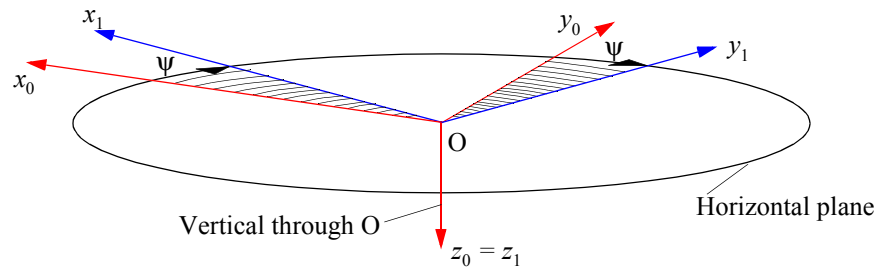
Starting with the $Ox_0 y_0 z_0$ axis system:

first, rotate about the vertical axis Oz_0 through yaw angle, ψ , to the new axis system denoted by $Ox_1 y_1 z_1$, Ox_1 and Oy_1 remain in the horizontal plane and Oz_1 coincides with Oz_0 ,

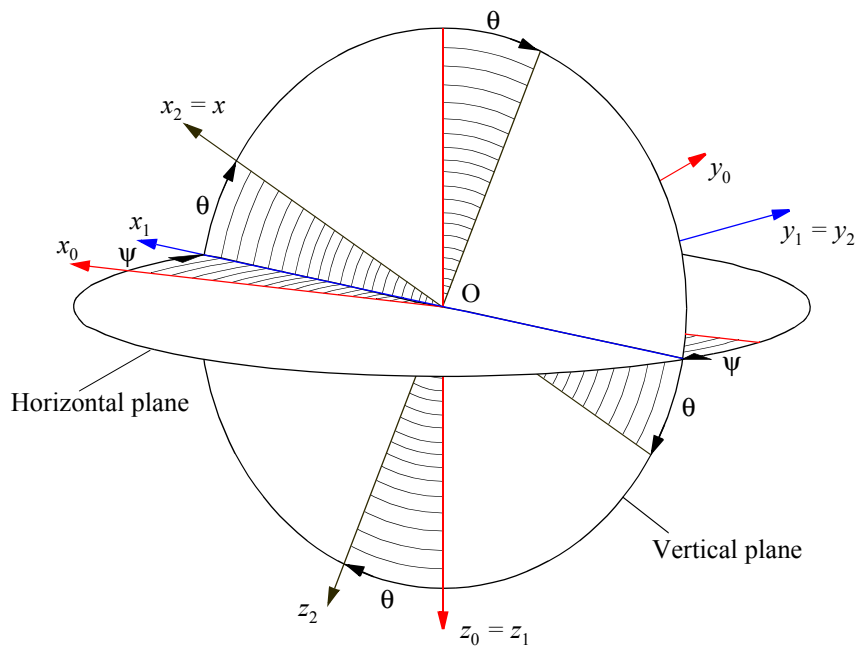
secondly, rotate about the horizontal axis Oy_1 (the displaced Oy_0 axis) through the pitch angle, θ , to the new axis system denoted by $Ox_2 y_2 z_2$

and then finally, rotate about axis Ox_2 (the doubly displaced Ox_0 axis) through the roll angle, ϕ , to the final axis system $Oxyz$.

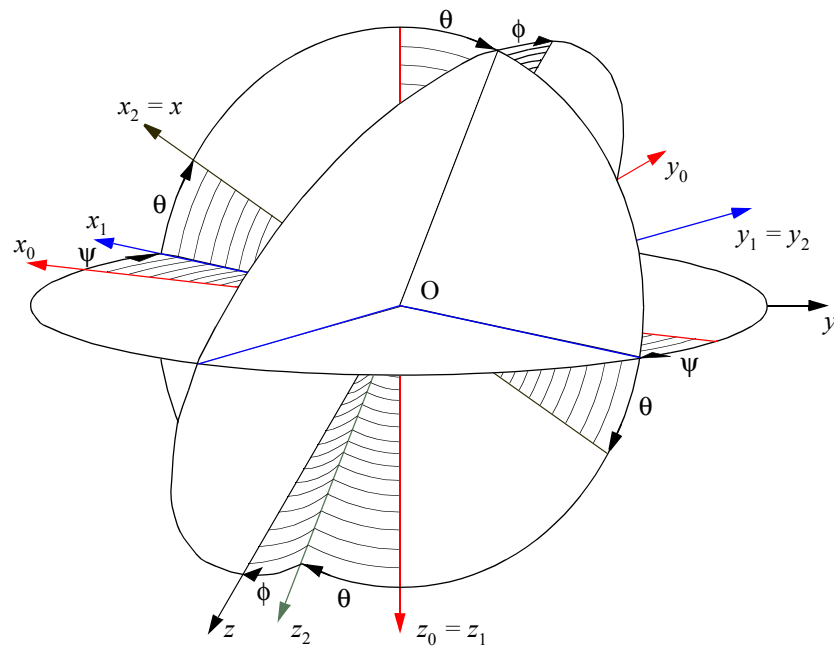
It is important to stress that the sequence of rotations (ψ , θ , ϕ) cannot be interchanged but it is also important to appreciate that other sequences of rotations are possible.



(a) First rotation through ψ about Oz_0 ($\equiv Oz_1$)



(b) Second rotation through θ about Oy_1 ($\equiv Oy_2$)



(c) Third rotation through ϕ about Ox_2 ($\equiv Ox$)

Sketch 3.2 A sequence of rotations leading to a given aeroplane attitude

4. VECTOR TRANSFORMATION IN TERMS OF EULER ANGLES

Consider the spatial vector $\underline{P}$ which has components resolved along the body-axis system, $Oxyz$, expressed by

$$\{P\}_B = \begin{Bmatrix} x \\ y \\ z \end{Bmatrix}.$$

The components of $\underline{P}$ resolved along the earth axis system $Ox_0y_0z_0$, are similarly expressed by

$$\{P\}_0 = \begin{Bmatrix} x_0 \\ y_0 \\ z_0 \end{Bmatrix}.$$

There is a transformation relationship between these two sets of components.

This transformation can be achieved by applying the rotations, yaw angle, ψ , pitch angle, θ , and roll angle, ϕ , as described in Section 3 and illustrated by Sketches 3.1 and 3.2.

The components of $\underline{P}$ relative to the intermediate moving axis systems, $Ox_1y_1z_1$ and $Ox_2y_2z_2$, are defined as

$$\{P\}_1 = \begin{Bmatrix} x_1 \\ y_1 \\ z_1 \end{Bmatrix} \quad \text{and} \quad \{P\}_2 = \begin{Bmatrix} x_2 \\ y_2 \\ z_2 \end{Bmatrix}.$$

The transformation from axis system $Ox_0y_0z_0$ to axis system $Ox_1y_1z_1$ may be written as

$$\{P\}_1 = [T_\psi] \{P\}_0,$$

where the transformation matrix, $[T_\psi]$, obtained from simple geometry and illustrated by Sketch 3.2(a), is

$$[T_\psi] = \begin{bmatrix} \cos \psi & \sin \psi & 0 \\ -\sin \psi & \cos \psi & 0 \\ 0 & 0 & 1 \end{bmatrix}. \quad (4.1)$$

It is noted that the matrix of Equation (4.1) is skew symmetric, *i.e.* $(T_\psi)_{ij} = -(T_\psi)_{ji}$ for $i \neq j$, and so a rotation through the angle $-\psi$ yields the matrix transpose.

Similar transformations of $\underline{P}$ from systems $Ox_1y_1z_1$ to $Ox_2y_2z_2$ and from $Ox_2y_2z_2$ to $Oxyz$, respectively, are obtained as follows

$$\{P\}_2 = [T_\theta] \{P\}_1 \quad \text{and} \quad \{P\}_B = [T_\phi] \{P\}_2,$$