

STATIC AEROELASTICITY: A FORMAL ANALYSIS USING NORMAL MODES

1. INTRODUCTION

Symmetric static aeroelasticity is concerned with the equilibrium equations of a flexible aeroplane in steady trimmed flight or in a quasi-steady symmetric manoeuvre from which the variables of incidence, elevator angle and structural deformation over the entire aeroplane are determined in terms of dynamic pressure, Mach number, altitude and normal acceleration.

Earlier Items in this series on symmetric aeroelasticity consist of the following.

Items 94009^[1] and 94045^[2] identify the aerodynamic lift and inertial distributions on a wing-fuselage combination and evaluate the distributions of shear force, bending moment and torque along the wing span and fuselage.

Item 96037^[3] presents a qualitative introduction to static aeroelasticity, discussing some of the fundamental concepts associated with the effects of structural flexibility. The Item concentrates on aspects of controllability, loads and stability, identifying the mutual interactions between aerodynamic lift and the deformation of the wing, aft fuselage and tailplane, and assesses the effects of deformation on manoeuvrability, both trim and manoeuvre control, and on static stability.

In Item 97032^[4] wing and tailplane flexibility are represented by a finite number of assumed orthogonal modes in bending and torsion. The fuselage is assumed rigid throughout its length, but the extension of the analysis to include a flexible fuselage is straightforward.

In this present Item the aeroplane deformation is represented by a finite number of symmetric structural normal modes. Each normal mode involves wing deformation (a combination of wing bending and torsion), nacelle nodding, pylon torsion and bending, fuselage bending and tailplane deformation (again combining bending and torsion). The formal analysis is similar to that presented in the earlier Item 97032. The Item also presents a discussion of the characteristics of symmetric normal mode shapes and their associated frequencies. For a general comprehensive discussion on normal mode analysis Reference 6, for example, may be consulted.

Normal modes are used extensively in the dynamic analyses of aeroplane responses, including investigations into dynamic instabilities such as flutter. Extensive calculations are carried out to estimate the normal modes and their associated natural frequencies. When aeroplane prototypes are available a thorough testing programme is carried out to measure the normal modes and their natural frequencies. Since these normal modes are available there is benefit to be gained from using them for static problems as well as for dynamic problems and so unifying the approaches to static and dynamic aeroelasticity.

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2. NOTATION

Any coherent system of units may be used.

a_i	amplitude of normal mode i
B	discrete bending moment (see Appendix B)
D_{Fin}	fin drag
$F_{0i}, F_{\alpha i}, F_{qi}, F_{\eta i}, F_{eii}, F_{eji}$	aerodynamic derivatives from structural equation (see Appendix A)
G_i	inertia term (see Appendix A)
g	acceleration due to gravity
i, j	general integers
K_i	stiffness of normal mode i
$L_0, L_\alpha, L_q, L_\eta, L_{ei}$	aerodynamic derivatives from normal force equation (see Appendix A)
$\ell(x, y), \ell(x)$	lift distributions
$\ell_{ei}(x, y), \ell_{ei}(x)$	lift distributions from unit amplitude structural deformation of mode i
$\ell_q(x, y), \ell_q(x)$	rate of change of $\ell(x, y)$ or $\ell(x)$ with pitch rate (= $\partial \ell(x, y) / \partial q$ or $\partial \ell(x) / \partial q$)
$\ell_0(x, y), \ell_0(x)$	lift distribution due to camber and, where applicable, washout and setting angle
$\ell_\alpha(x, y), \ell_\alpha(x)$	rate of change of $\ell(x, y)$ or $\ell(x)$ with incidence (= $\partial \ell(x, y) / \partial \alpha$ or $\partial \ell(x) / \partial \alpha$)
$\ell_\eta(x, y)$	rate of change of $\ell(x, y)$ with elevator deflection (= $\partial \ell(x, y) / \partial \eta$)
M_i	modal mass of normal mode i
$M_0, M_\alpha, M_q, M_\eta, M_{ei}$	aerodynamic derivatives from pitching moment equation (see Appendix A)
$m(x, y), m(x)$	mass distributions
N	number of normal modes of structural deformation

$n, \Delta n$	load factor and incremental load factor, respectively, measured in terms of g
O	origin of axis system for aeroplane
$p(x, y)$	load distribution (see Appendix B)
q	steady pitch rate ($= \dot{\theta}$)
T	discrete torque (see Appendix B)
Th	thrust of a single engine
t	time
$U, \delta U$	internal strain energy and incremental change in strain energy, respectively (see Appendix B)
V_{trim}	trimmed velocity of aeroplane in flight direction
W	aeroplane weight
δW	virtual work done (see Appendix B)
x, y, z	directions of axes with origin O for aeroplane
$(Y_{ei})_{Nac}$	side force induced by $(\psi_{ei})_{Nac}$
$Y_{Winglet}$	outward side force due to winglet
y_{Nac}	spanwise location of nacelle
$z_e(x, y)$	aeroplane deformation distribution resulting from all modes
$z_{ei}(x, y)$	aeroplane deformation distribution resulting from normal mode i
$\bar{z}_{ei}(x, y)$	shape deformation distribution of normal mode i
z_{Fin}	effective bending moment arm of fin above x -axis
z_{Nac}	distance of line of action of engine thrust below wing
$z_{Winglet}$	effective distance of $Y_{Winglet}$ from wing surface at the wing tip
α	aeroplane incidence
δ	increment
η	elevator deflection

$\theta, \dot{\theta}$	aeroplane pitch angle and rate of change of θ with time, $d\theta/dt$ ($= q$), respectively
Λ	sweep angle of $1/4$ -chord
ρ	air density
$(\psi_{ei})_{Nac}$	angle of twist induced by unit deformation of i^{th} the normal mode
ω_i	frequency of normal mode i

Subscripts

e	relates to structural deformation
F	fuselage
i	normal mode number
le	leading edge
Nac	nacelle
q	relates to aeroplane steady pitch rate
T	tailplane
W	wing
0	denotes quantity independent of aeroplane incidence, elevator angle and structural deformation
$1/4 \bar{c}$	$1/4$ -chord
α	relates to aeroplane incidence
η	relates to elevator

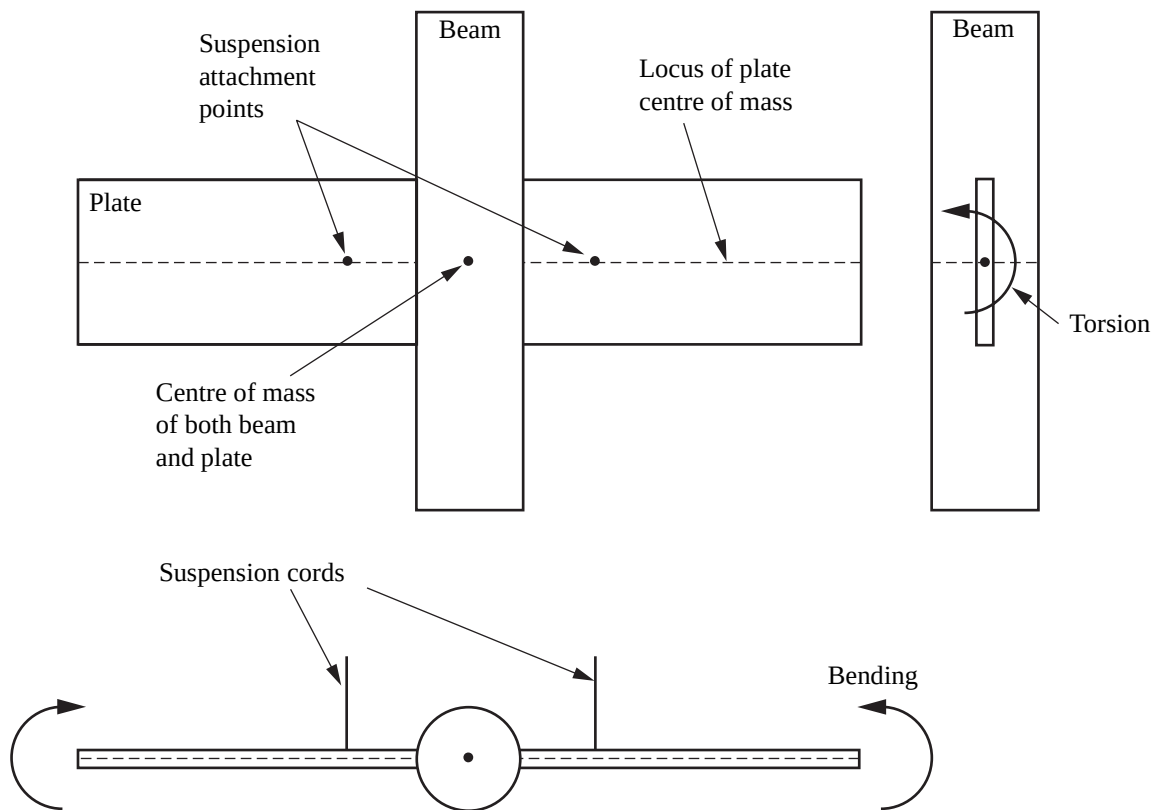
A bar above a quantity indicates a non-dimensional quantity.

3. SYMMETRIC NORMAL MODE SHAPES AND ASSOCIATED FREQUENCIES

In this section a brief summary of the characteristics of normal modes is presented.

First, consider a structure comprising a thin rectangular plate of uniform thickness built into a central cylindrical beam of finite length as shown in Sketch 3.1, where the centres of mass of the beam and plate coincide. The locus of the sectional centres of mass along the plate 'span' will lie along the centre-line of the plate.

It is assumed that the central cylindrical beam is rigid and the plate is flexible. Since the plate is uniform it will tend to bend, relative to the cylindrical beam, about its attachment to the beam and twist about its centre-line as shown in Sketch 3.1. This configuration, which idealises a wing-fuselage structure, is known as a free-free system.

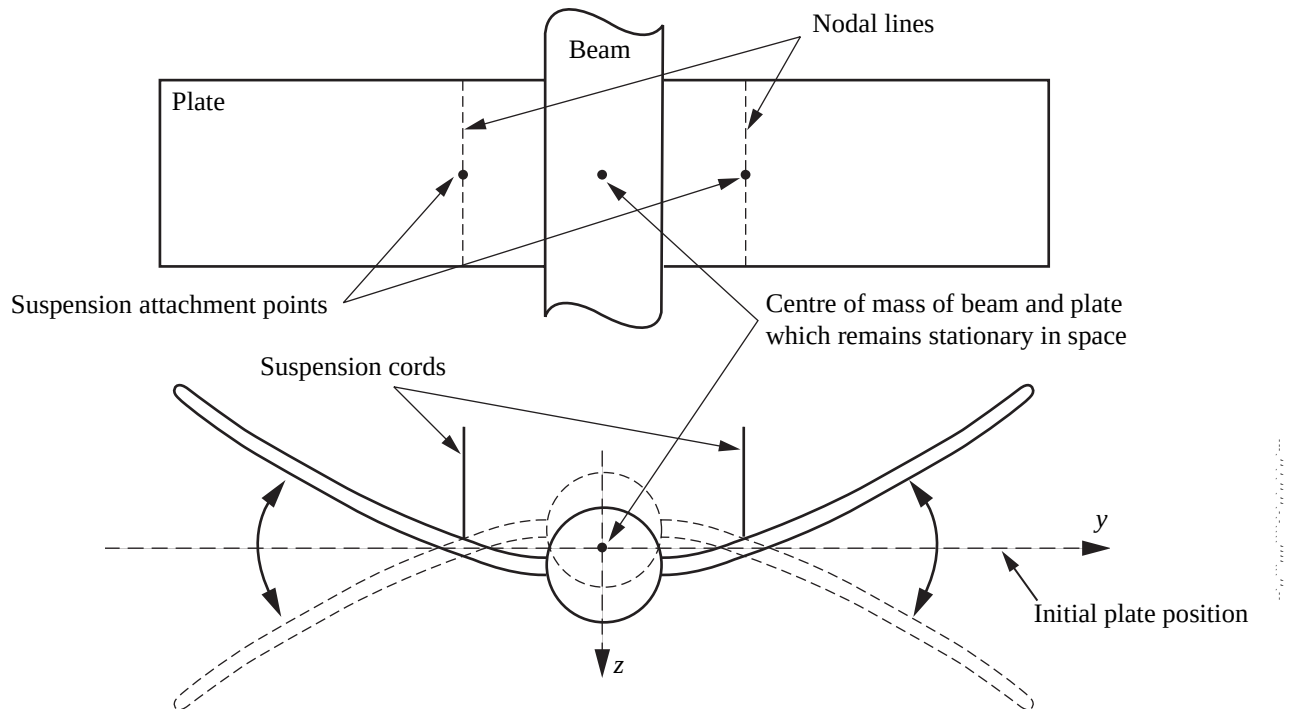


Sketch 3.1 Cylindrical rigid beam and flexible flat plate

Suppose that this structure is suspended and held by two flexible cords connected to points about 10% along the plate centre-line from the root, also as shown in Sketch 3.1.

At a certain frequency the structure can be set vibrating in simple harmonic motion where the vibration of the plate is pure up and down bending, while the beam goes down and up without pitching, as shown in Sketch 3.2. As the plate bends up, the beam moves down; as the plate bends down, the beam moves up. There is a 'nodal line' across the plate, that does not move in space. If there were no structural or aerodynamic damping the structure would continue to vibrate indefinitely with constant amplitude at a constant frequency. Essentially, at each instant of time throughout the dynamic motion, the time-varying dynamic inertial loads exactly balance the time-varying internal strain forces.

This vibration is known as the fundamental normal bending mode and the frequency (ω_f , say) is the fundamental bending natural frequency. The spanwise bending motion of the plate will be of the form, $z(y) \sin(\omega_f t)$, where $z(y)$ are the coordinates of vertical deformation as shown in the sketch. The spanwise 'shape', $z(y)$, of the plate bending deformation together with z_b , the beam displacement, is known as the fundamental bending modal shape in which the amplitude is arbitrary.



Sketch 3.2 Fundamental bending vibration of rigid cylindrical beam and flexible flat plate

If the suspension cords are attached to the points at the intersection of the nodal lines with the plate centre of mass locus, as shown in Sketch 3.2, then the suspension system is not affected by the structural vibration and the overall centre of mass remains stationary in space as the structure vibrates. Also the free-free mode is unaffected by the suspension.

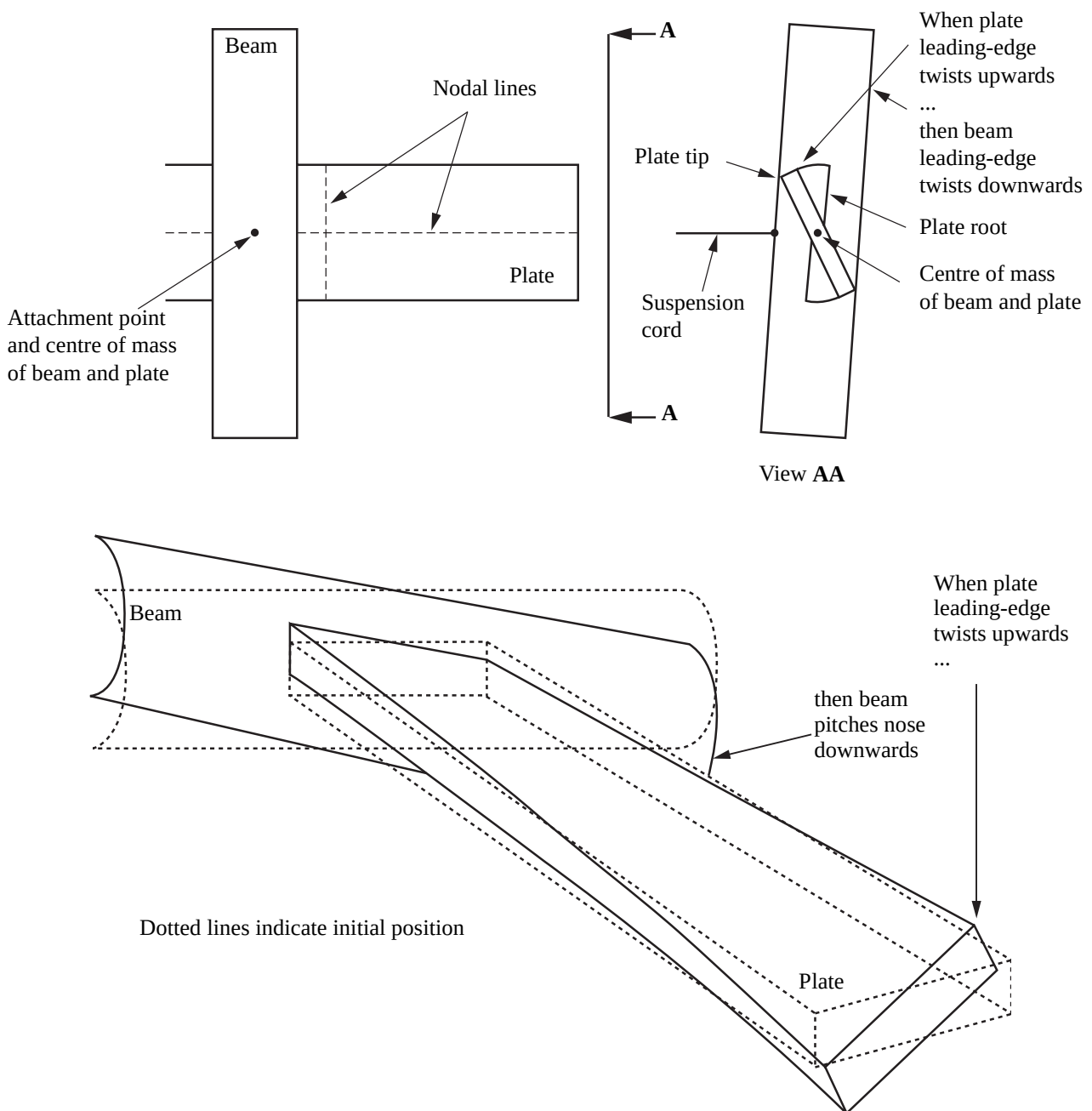
To be more precise, with this flexible suspension system the suspension cords will be elongated in the initial steady-state due to the weight of the structure and there will be a steady structural deformation of the plate due to the weight distribution. The position is the static equilibrium position of the structure. The above vibrating motion will then be superimposed on the initial steady-state deformation and the suspension steady-state elongation will not be affected as the structure vibrates.

Next suppose that the structure is supported by a single suspension chord fixed to the beam above the centres of mass of the beam and plate as shown in Sketch 3.3.

At a higher frequency than the fundamental bending natural frequency the structure can be set vibrating in simple harmonic motion with the plate vibrating in pure torsion about its centre-line with the beam vibrating

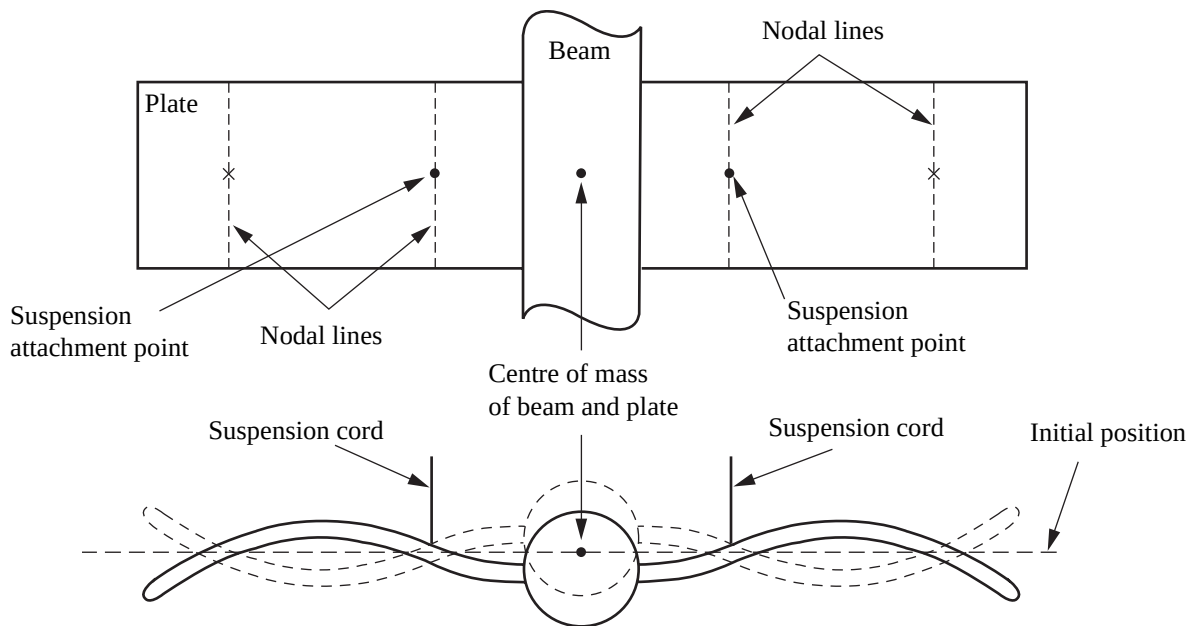
in pitch about the suspension point. As the plate twists nose up the beam pitches nose down, and vice versa, as shown in Sketch 3.3. Again the overall centre of mass remains stationary in space.

The plate torsion, relative to the beam, is zero at the plate root and the plate torsion builds up to a maximum at the plate tip. In this vibration mode there are two nodal lines, which remain stationary in space; one lying parallel to the root close to the central beam and the other lying at right angles along the centre of the plate. Once again if there were no aerodynamic or structural damping then the structure would continue to vibrate indefinitely with constant amplitude at the constant frequency. This vibration is known as the fundamental normal torsion mode and the frequency is the fundamental torsion natural frequency. The spanwise shape of the plate torsional deformation together with the relative pitching motion of the beam is known as the fundamental torsion mode shape in which the amplitude is arbitrary.



Sketch 3.3 Fundamental torsional vibration of rigid cylindrical beam and flexible flat plate

At a higher frequency still, the structure can be induced to vibrate again in pure bending but with a modal shape involving two nodal lines as shown in Sketch 3.4. This normal mode would be known as the first harmonic bending normal mode with its associated natural frequency and modal shape.



Sketch 3.4 First harmonic bending mode of rigid cylindrical beam and flexible flat plate

There is then a succession or sequence of discrete higher frequencies where the structure can be induced to vibrate in simple harmonic motion with more complicated mode shapes:

- a first harmonic torsion normal mode containing three nodal lines,
- a second harmonic bending normal mode containing three nodal lines,
- a second harmonic torsion normal mode containing four nodal lines and so on,

but all of these modes are for the beam/plate structure in which the centres of mass of the beam and plate coincide.

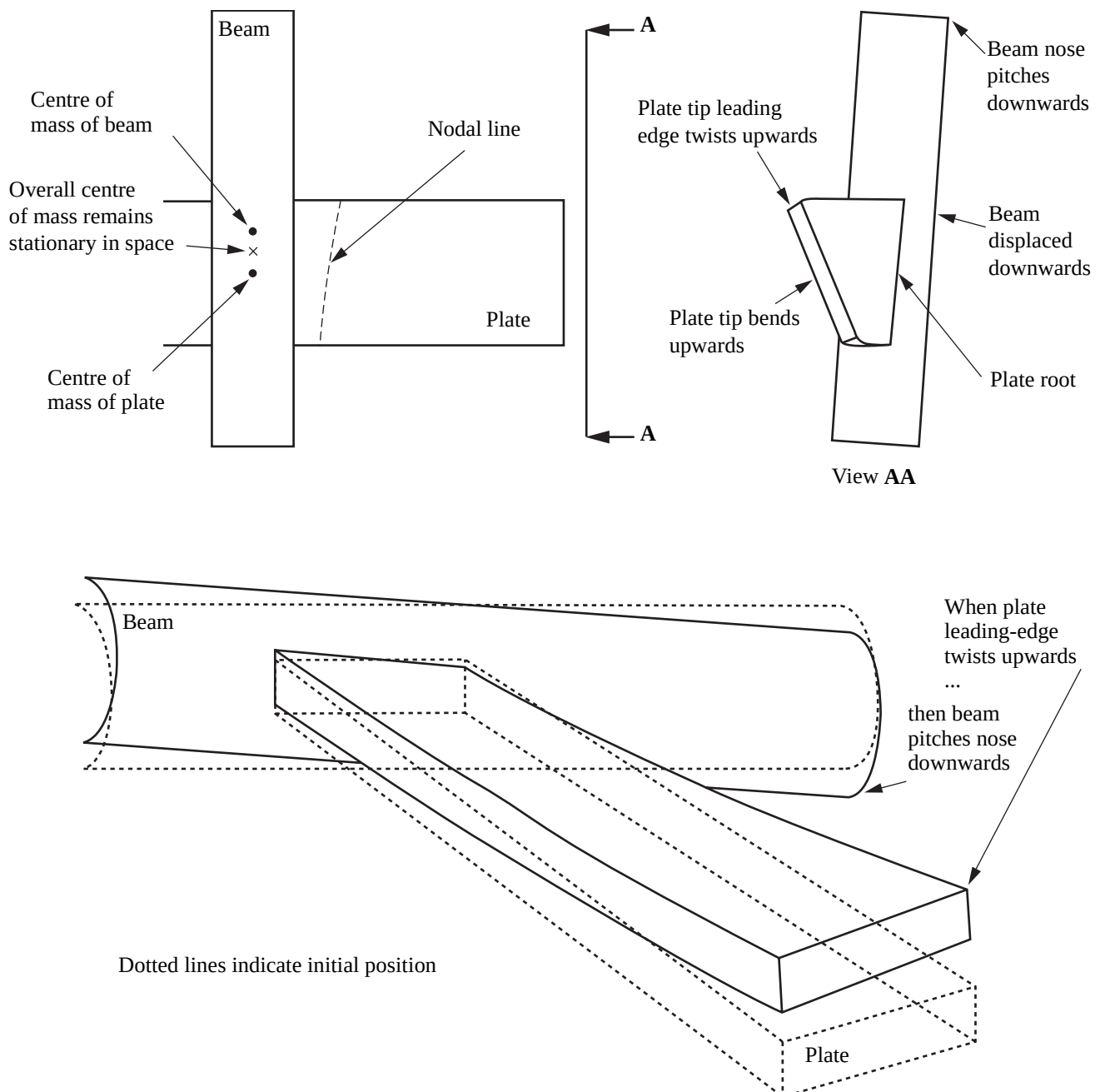
Now suppose that the geometry of the simple beam and plate is changed by displacing the plate aft along the beam so that the centres of mass of the beam and plate no longer coincide, as shown in Sketch 3.5.

The fundamental bending mode will be similar to the earlier case except now, because the beam centre of mass is forward of the plate centre of mass, the dynamic inertial force of the plate bending vibration will induce the beam to vibrate slightly in pitch which will in turn cause a small torsional vibration of the plate. Thus the fundamental normal mode is no longer purely bending but is primarily plate bending with some plate torsion and with beam displacement and some pitch which increases when the beam centre of mass moves forward. The nodal line is now curved as shown in Sketch 3.5.

It is no longer possible to talk in terms of separate bending and torsion modes since all modes will involve plate bending and torsion and higher frequency modes will even involve some plate camber deformation.

However it is customary to refer to the fundamental or first normal mode as primarily a plate bending mode; while the second mode can be either primarily a plate torsion mode or primarily a bending mode with two nodal lines depending on the configuration, and so on.

When the central beam is also flexible the fundamental plate bending mode will be primarily a bending mode of the plate but will include some beam bending. Now there will be some normal modes that are primarily beam bending, with some plate bending.



Sketch 3.5 Fundamental normal mode exhibiting primarily plate bending but with some plate torsion and pitching of beam (non-coincident centre of mass of beam and plate)

The structure shown in Sketch 3.5 idealises a wing-fuselage combination. In a complete configuration there will also be a flexible tailplane on the rear fuselage. The first normal mode of primarily wing bending will induce, through the small vibrations of the fuselage, vibrations of the tailplane. So all the normal modes will involve wing bending and torsion, fuselage pitch and bending deformation and tailplane bending and torsion. Some of the higher frequency normal modes will be described as primarily tailplane bending modes or as primarily tailplane torsion modes.

The mass of an engine and nacelle attached to the leading edge of a wing will further complicate the coupling between wing bending and wing torsion.

When an engine nacelle is underslung beneath a wing the connecting pylon is usually flexible in pitch, bending and torsion. There will be a symmetric normal mode primarily involving pylon yawing vibrations (the two pylons on either side of the fuselage vibrating in anti-phase) with some associated wing bending and torsion vibration.

To repeat a point made earlier, all normal mode vibrations are superimposed on the steady-state or quasi-static structural deformation.

3.1 Aeroplane Symmetric Normal Modes

Table 3.1 lists a selection of typical symmetric normal modes of a twin-engine transport aeroplane configuration. Sketches 3.6 to 3.11 illustrate six of the lower frequency normal modes. (The deflections shown in the sketches have been exaggerated for the purposes of visual presentation.)

TABLE 3.1 Typical flexible symmetric normal modes of response and associated frequencies for a large transport aeroplane

<i>Flexible Normal Mode Number</i>	<i>Frequency (Hz)</i>	<i>Primary Description</i>	<i>Sketch Number</i>
1	1.5	Wing bending (fundamental) with one nodal line	3.6
2	2.25	Nacelle torsion mode	3.7
3	2.75	Fuselage bending with one nodal line	3.8
4	3.25	Nacelle pitch	3.9
5	4.0	Wing bending (first harmonic) with two nodal lines	3.10
7	5.0	Tailplane bending (fundamental) with one nodal line on the tailplane	3.11
15	11.0	Wing torsion mode (fundamental)	—
26	20.0	Wing torsion mode (first harmonic)	—
28	22.0	Tailplane torsion mode (fundamental)	—

Sketch 3.6 illustrates Mode 1 that primarily consists of wing bending in which an upwards elastic deformation of each half-wing occurs about a nodal line, shown dotted in the plan view. The nodal line is a gently curved line that crosses the wing at about 0.25 of the span, extremely close to the position of the engine nacelle. The views also show a small upward deflection of the rear fuselage and a small upward deflection of each half-tailplane.

Sketch 3.7 illustrates Mode 2 that is a pylon bending and torsion mode in which the engine nacelle twists and moves laterally towards the wing tip. Two nodal lines are seen on the half-wing, one close to the engine nacelle and parallel to the wing chord and the other that crosses the wing trailing edge at approximately 0.25 of the span and cuts the wing leading edge at about 0.6 of the span. A small amount of downward wing bending also occurs.

Sketch 3.8 illustrates Mode 3 that is primarily a fuselage bending mode where the fuselage nose and rear sections are deflected upwards. The outer sections of the wing are also seen to be deflected in bending upwards about a nodal line located at about 0.7 of the span.

Sketch 3.9 illustrates Mode 4 that is essentially an engine nacelle pitch mode but also incorporates outer wing bending that occurs at about 0.7 of the span.

Sketch 3.10 illustrates Mode 5 which is the second wing bending mode and has two nodal lines on each half-wing. One nodal line is located at roughly the position of the engine nacelle and the other is at about 0.8 of the span. The wing bends upwards and the tailplane is seen to be deflected downwards in bending.

Mode 6 which is not included in Table 3.1 is a wing fore and aft mode, with a vibration frequency of approximately 4.5 cycles/s, in which the wing primarily bends backwards and forwards in the x-y plane. Some wing bending and torsion also occur about two nodal lines on each half-wing. Also some tailplane bending occurs.

Sketch 3.11 illustrates Mode 7 that is primarily a tailplane bending mode in which the tailplane bends upwards. Little wing bending or torsion occurs and little fuselage bending is seen.