

INTRODUCTORY SHEET ON ISENTROPIC FLOW AND SHOCK WAVES

1. NOTATION AND UNITS

		<i>SI</i>	<i>British</i>
a	speed of sound	m/s	ft/s
A	cross-sectional area of duct	m ²	ft ²
c_p	specific heat capacity at constant pressure	J/kg K	ft lbf/slug K
c_v	specific heat capacity at constant volume	J/kg K	ft lbf/slug K
E	internal energy per unit mass	J/kg	ft lbf/slug
i	enthalpy per unit mass	J/kg	ft lbf/slug
M	Mach number		
p	pressure	N/m ²	lbf/ft ²
Q	heat per unit mass	J/kg	ft lbf/slug
R	gas constant for a particular gas	J/kg K	ft lbf/slug K
s	entropy per unit mass	J/kg K	ft lbf/slug K
T	temperature	K	K
u	velocity of flow in one dimension	m/s	ft/s
V	velocity of flow	m/s	ft/s
γ	ratio of specific heat capacities, c_p/c_v		
δ	wall deflection angle	rad	rad
μ	Mach angle, $\sin^{-1}(1/M)$	rad	rad
ρ	density	kg/m ³	slug/ft ³
ω	Prandtl-Meyer angle	rad	rad
$\delta\omega$	infinitesimal flow deflection angle	rad	rad

Subscripts

H	denotes stagnation values
n	denotes normal component
0	denotes reference values

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2. GENERAL

This Item defines some of the expressions and quantities that occur frequently in calculations related to high-speed flow, summarises the basic thermodynamic concepts, and gives notes on the use of the Data Items dealing with isentropic flow and shock waves.

2.1 Fundamental Considerations

It is usually permissible to assume that the air behaves as a *perfect gas* for which the *equation of state* is

$$\frac{p}{\rho} = RT, \quad (2.1)$$

where R is a constant for a given gas. Its value for air is 287.1 J/kg K (3090 ft lbf/slug K). The temperature, T , is measured in Kelvin.

The *first law of thermodynamics* states that the heat content increase of a gas is equal to the sum of the external work done by the gas and the increase of its internal energy. For unit mass this may be written

$$dQ = dQ_1 + dQ_2 = dE + pd\left(\frac{1}{\rho}\right), \quad (2.2)$$

where dE is the change of *internal energy* and dQ_1 and dQ_2 are respectively the quantity of heat added from an external agency and that due to the internal dissipation of mechanical energy set up during the process. If the process is always in equilibrium with its surroundings no internal mechanical energy is dissipated so that dQ_2 is zero. The process is then spoken of as *reversible*, since it could be reversed with the result that the gas would pass back through all the stages through which it had originally passed and would be at the same conditions at each corresponding stage in the process. All natural processes are in some measure irreversible and particularly those involving heat conduction and viscosity. If dQ_1 is zero, the process is said to be *adiabatic* or, in the case of flow through channels, *adiathermal* since the term adiabatic is sometimes used to describe a flow in which there is no exchange of heat between any element of the gas and surrounding gas elements. The term adiabatic has sometimes been used loosely to describe isentropic processes (see below); the latter must be reversible.

Assuming constant specific heat capacities, the *internal energy* E , for unit mass, is given by

$$dE = c_v dT \quad (2.3)$$

for a perfect gas, where c_v is the specific heat capacity at constant volume and is related to c_p , the specific heat capacity at constant pressure, by the equation

$$c_p - c_v = R. \quad (2.4)$$

For an isothermal process dT is zero so that the internal energy is constant.

For unit mass, entropy s is defined by the equation

$$ds = \frac{dQ}{T}. \quad (2.5)$$

If the change of entropy is zero the process is called *isentropic*. Thus, by Equation (2.2) an *adiabatic reversible* process is also isentropic. This term is sometimes restricted to a steady flow in which the entropy is constant along streamlines; the term *homentropic* may then be used to describe a flow in which the entropy is constant everywhere. Such a flow is irrotational if the total energy (see below) is also constant everywhere. The requirement of adiabatic flow means that there is zero conduction between the fluid and surrounding fluid elements and the requirement of reversible processes means that the dissipative effects of viscosity must be negligible.

Thus isentropic flow is strictly realised only in the absence of conduction and viscosity. In real flows at the Reynolds numbers commonly encountered in aerodynamics, the effects of conduction and viscosity are largely confined to boundary layers, wakes and shock waves. Thus there remain in practice large regions of the flow in which the motion is sensibly isentropic.

Using Equations (2.1), (2.3) and (2.4), Equation (2.2) may be written

$$\frac{dQ}{T} = c_p \frac{dT}{T} - R \frac{dp}{p}. \quad (2.6)$$

This (by definition) is equal to the change of entropy, and may be integrated to give the change of entropy between conditions denoted by subscript 0 and without subscript, in the form

$$s - s_0 = c_p \log \frac{T}{T_0} - R \log \frac{p}{p_0}. \quad (2.7)$$

In an isentropic process this is equal to zero and hence, using Equations (2.1) and (2.4), the following relation between the pressure and density is obtained for isentropic flow

$$\frac{p_0}{\rho_0^\gamma} = \frac{p}{\rho^\gamma} = \text{constant}, \quad (2.8)$$

where $\gamma = \frac{c_p}{c_v}$.

The *second law of thermodynamics* states that the entropy of a closed system cannot decrease and this criterion is often used if there is any doubt as to the direction in which a process can proceed. The *enthalpy* or *total heat i* per unit mass is defined by

$$i = E + \frac{p}{\rho} = c_p T. \quad (2.9)$$

Thus using Equation (2.2)

$$di = dQ + \frac{dp}{\rho}, \quad (2.10)$$

which for an isentropic process becomes

$$di = \frac{dp}{\rho}. \quad (2.11)$$

The *total energy* of a fluid is the sum of the internal, kinetic and potential energies. If the total energy is constant the process is referred to as *isenergetic*. This term is sometimes restricted to a steady flow in which the total energy is constant along stream lines; the term *homenergetic* may then be used to describe flows in which the total energy is constant throughout the field. If the effects of gravity are neglected, and the process is adiabatic and steady, the energy equation, for unit mass, may be written

$$\frac{V^2}{2} + E \frac{p}{\rho} = i_H = \text{constant}, \quad (2.12)$$

or for a perfect gas

$$\frac{V^2}{2} + c_p T = c_p T_H = \text{constant}, \quad (2.12a)$$

where V is the (resultant) velocity of the flow, and subscript H is used to denote the condition of zero velocity. Thus the *stagnation* or *total temperature*, T_H , is a measure of the total energy of the fluid and remains constant in an homenergetic flow.

For isentropic flow Equations (2.11) and (2.12a) give

$$VdV + \frac{dp}{\rho} = 0, \quad (2.13)$$

which is a form of Bernoulli's equation for steady motion, and could have been derived alternatively from kinetic considerations on the assumption that viscosity is absent.

Equation (2.12) shows that the kinetic energy is obtained at the expense of the enthalpy, and that the velocity V cannot exceed the value obtained when the (absolute) temperature T becomes zero. This maximum velocity is given by

$$V_{max} = \sqrt{2c_p T_H} \quad (2.14)$$

or roughly 760 m/s (2 500 ft/s) for air with a stagnation temperature of 288.15 K (15°C).

The speed of sound, a , is the speed of propagation of pressure disturbances of infinitesimal amplitude. It is given by

$$a = \sqrt{\frac{dp}{d\rho}} \text{ and for a perfect gas } a = \sqrt{\frac{\gamma p}{\rho}} = \sqrt{\gamma R T}. \quad (2.15)$$

For a given gas, therefore, the speed of sound depends only on the temperature. Its value at a particular point in the flow is termed the *local speed of sound*.

Equation (2.12) may be written

$$\frac{V^2}{2} + \frac{1}{\gamma - 1} a^2 = \frac{V_{max}^2}{2} = \frac{1}{\gamma - 1} a_H^2, \quad (2.16)$$

where a_H is the speed of sound at the stagnation temperature. The *critical speed of sound*, a^* , is defined as the local speed of sound in the particular conditions in which it is equal to the fluid velocity (i.e., $a^* = a$ when $a = V$). Hence from Equation (2.16)

$$\frac{1}{\gamma - 1} a_H^2 = \frac{V_{max}^2}{2} = \frac{\gamma + 1}{2(\gamma - 1)} a^{*2} = c_p T_H. \quad (2.17)$$

The Mach number, M , is defined as the ratio of the speed of the fluid to the local speed of sound,

$$M = \frac{V}{a}. \quad (2.18)$$

It should be noted that, since the local speed of sound is zero when the velocity reaches its maximum value, V_{max} (i.e. $T = 0$), the corresponding Mach number is infinite.

2.2 Expressions for Steady Isentropic Flow in Terms of the Total or Stagnation Conditions

Equations (2.15) and (2.16) give

$$\left(\frac{a}{a_H}\right)^2 = \frac{T}{T_H} = \frac{2}{2 + (\gamma - 1)M^2}, \quad (2.19)$$

and for isentropic process, Equations (2.1) and (2.8) give

$$\frac{p}{p_H} = \left(\frac{T}{T_H}\right)^{\gamma/(\gamma - 1)} \quad (2.20)$$

$$\text{and} \quad \frac{\rho}{\rho_H} = \left(\frac{T}{T_H}\right)^{1/(\gamma - 1)}, \quad (2.21)$$

where p_H is used to denote the pressure after the flow has been brought to rest isentropically (i.e. the *total head*). In isentropic flow p_H is constant. Thus, the following relations are obtained for steady isentropic flow; the expressions at the right hand side are for $\gamma = 1.40$, the value customarily adopted for air at temperatures between 200 K and 400 K and pressures between 2 and 0.1 atmospheres, (see Item No. Aero 00.01.08).

$$\frac{T}{T_H} = \frac{2}{2 + (\gamma - 1)M^2} = \frac{5}{5 + M^2}, \quad (2.22)$$

$$\frac{p}{p_H} = \left(\frac{2}{2 + (\gamma - 1)M^2}\right)^{\gamma/(\gamma - 1)} = \left(\frac{5}{5 + M^2}\right)^{7/2}, \quad (2.23)$$

$$\frac{\rho}{\rho_H} = \left(\frac{2}{2 + (\gamma - 1)M^2}\right)^{1/(\gamma - 1)} = \left(\frac{5}{5 + M^2}\right)^{5/2}, \quad (2.24)$$

$$\text{and} \quad \frac{a}{a_H} = \left(\frac{2}{2 + (\gamma - 1)M^2}\right)^{1/2} = \left(\frac{5}{5 + M^2}\right)^{1/2}. \quad (2.25)$$