



ESDU TECHNICAL MEMORANDUM

A Rayleigh-Ritz method of analysis for vibration of orthotropic plates under static in-plane loading (including shear)

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THE PREPARATION OF THIS TECHNICAL MEMORANDUM

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SUMMARY

This Memorandum describes a computation procedure for the estimation of natural frequencies and mode shapes of isotropic and orthotropic rectangular plates under static in-plane loading. The static in-plane loading, uniformly distributed along the plate edges, may be any combination of direct and shear loading that does not cause the plate to buckle. The document provides details of the theory and method of computation used in the program listed in ESDU Data Item No. 90016. The development of the frequency equation is outlined, the orthogonal polynomial functions used in the displacement equation are defined, and the computations involving the orthogonal polynomials are considered. Notes are also provided on the computation procedures used in Item No. 90016 for the evaluation of the eigenvalues and eigenvectors.

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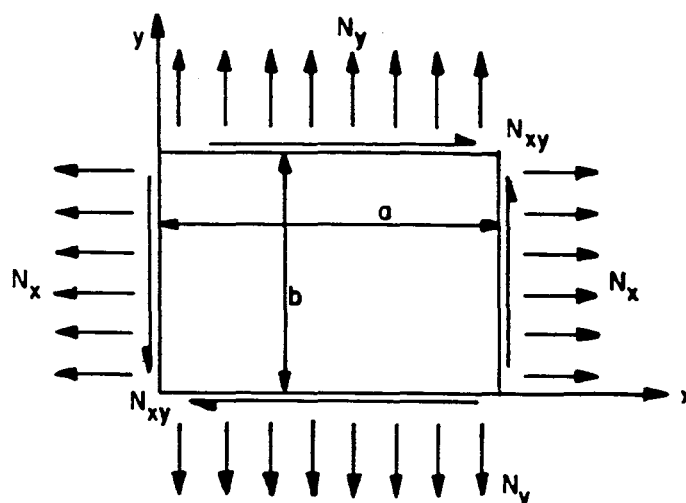
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1. NOTATION

In this notation the major parameters only are defined.

A_{mn}	displacement parameter [see Equation (3.2)]	m
a	length of plate in x -direction	m
a_{ij}	coefficients in displacements functions (see Section 5)	
B_k	coefficient in orthogonal polynomial function [see Equation (4.13)]	
b	length of plate in y -direction	m
C_k	coefficient in orthogonal polynomial function [see Equation (4.14)]	
D_x, D_y, D_{xy}	plate stiffness parameters [see Equation (3.3)]	Nm
E_x, E_y	orthotropic plate moduli of elasticity in x - and y - directions respectively	N/m ²
$E_{mi}^{(r,s)}, F_{nj}^{(r,s)}$	integrals defined by Equation (3.7)	
G_{xy}	shear stiffness of orthotropic plate in (x,y) plane	N/m ²
H	stiffness parameter defined by Equation (3.6)	Nm
m, n	number of terms used in displacement expression in x - and y - direction respectively	
N_x, N_y	static force per unit width in x - and y -directions respectively (tensile forces positive)	N/m
N_{xy}	static shear force per unit width (see Sketch 1.1)	N/m
T_{max}	maximum kinetic energy in plate	kg m ² /s ²
t	plate thickness	m
V_{max}	maximum energy in plate	kg m ² /s ²
W	displacement parameter [see Equation (3.2)]	m
w	out-of-plane deflection	m
x, y	plate coordinate system	
α, β	non-dimensionalised plate dimensions x/a and y/b respectively	
λ	non-dimensional frequency parameter defined by Equation (3.6)	
ν_{xy}	Poisson's ratio of orthotropic plate associated with load applied in x - direction	

ν_{yx}	Poisson's ratio of orthotropic plate associated with load applied in y- direction	
ρ	density of plate material	kg/m ³
τ	time	s
Φ	energy term [see Equation (3.4)]	kg m ² /s ²
$\phi_m(\alpha), \psi_n(\beta)$	orthogonal polynomials representing out-of-plane displacements in x- and y- directions respectively	
ω	circular natural frequency	rad/s



Sketch 1.1 Plate geometry and positive loading system

2. INTRODUCTION

This note describes the method of analysis and computation procedure used in Item No. 90016 (Reference 1) for the estimation of natural frequencies of flat rectangular plates. The method of analysis is applicable to the computation of the lower natural frequencies, and mode shapes, of isotropic or specially orthotropic plates. Plates are specially orthotropic when their principal axes of orthotropy are parallel to the plate edges. The method of analysis is based on that described in Reference 2 and uses the Rayleigh-Ritz method in which the displacements are assumed to be represented by a series of orthogonal polynomial functions. The polynomial functions are chosen to satisfy the plate boundary conditions. In the analysis, provision is made for constant in-plane forces, acting on the middle surface of the plate, in both the x- and y- directions together with a constant in-plane shear force.

3. DEVELOPMENT OF FREQUENCY EQUATION

For the free vibration of a plate, the out-of-plane deflection w may be represented by the expression

$$w = W \sin \omega t \quad (3.1)$$

where

$$W = \sum_m \sum_n A_{mn} \phi_m \left(\frac{x}{a} \right) \psi_n \left(\frac{y}{b} \right) \quad (3.2)$$

and $\phi_m(x/a)$ and $\psi_n(y/b)$ are polynomial functions that satisfy the geometric boundary conditions of the plate.

The maximum strain energy of an orthotropic rectangular plate, V_{max} , is given by

$$V_{max} = \frac{1}{2} \int_0^a \int_0^b \left[D_x \left(\frac{\partial^2 W}{\partial x^2} \right)^2 + 2 \nu_{xy} D_y \left(\frac{\partial^2 W}{\partial x^2} \right) \left(\frac{\partial^2 W}{\partial y^2} \right) \right. \\ \left. + D_y \left(\frac{\partial^2 W}{\partial y^2} \right)^2 + 4 D_{xy} \left(\frac{\partial^2 W}{\partial x \partial y} \right)^2 + N_x \left(\frac{\partial W}{\partial x} \right)^2 \right. \\ \left. + N_y \left(\frac{\partial W}{\partial y} \right)^2 + 2 N_{xy} \frac{\partial W}{\partial x} \frac{\partial W}{\partial y} \right] dy dx ,$$

where

$$D_x = \frac{E_x t^3}{12(1 - \nu_{xy} \nu_{yx})} , \quad D_y = \frac{D_x E_y}{E_x} \quad (3.3)$$

and

$$D_{xy} = \frac{G_{xy} t^3}{12} .$$

Also,

$$\nu_{yx} = \nu_{xy} \frac{E_y}{E_x} .$$

The maximum kinetic energy of the plate, T_{max} , is given by

$$T_{max} = \frac{\rho t \omega^2}{2} \int_0^a \int_0^b W^2 dy dx .$$

In order to solve for the coefficients, A_{mn} , in the displacement expression it is necessary to consider the variation of the difference $V_{max} - T_{max}$ (denoted by Φ) and apply the condition $\partial \Phi / \partial A_{mn} = 0$.

$$\Phi = \frac{1}{2} \int_0^a \int_0^b \left[D_x \left(\frac{\partial^2 W}{\partial x^2} \right)^2 + 2 \nu_{xy} D_y \left(\frac{\partial^2 W}{\partial x^2} \right) \left(\frac{\partial^2 W}{\partial y^2} \right) \right. \\ \left. + D_y \left(\frac{\partial^2 W}{\partial y^2} \right)^2 + 4 D_{xy} \left(\frac{\partial^2 W}{\partial x \partial y} \right)^2 + N_x \left(\frac{\partial W}{\partial x} \right)^2 + N_y \left(\frac{\partial W}{\partial y} \right)^2 \right. \\ \left. + 2 N_{xy} \frac{\partial W}{\partial x} \frac{\partial W}{\partial y} - \rho t \omega^2 W^2 \right] dy dx . \quad (3.4)$$

Making the substitution

$$\alpha = \frac{x}{a} \text{ and } \beta = \frac{y}{b} , \\ dx = a d\alpha \text{ and } dy = b d\beta ,$$