



ESDU TECHNICAL MEMORANDUM

Wing lift-curve slope in inviscid subsonic flow: Improvements to the Helmbold-Diederich equation and comparison with data from ESDU 70011

Mr P.D. Chappell

ESDU TECHNICAL MEMORANDA

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* Corresponding Member

The development and preparation of this Technical Memorandum were undertaken by

Mr P.D. Chappell — Independent.

The person with overall responsibility for the work in this subject area is Mr A.J. Clarke, Head of Aerodynamics Group.

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1. NOTATION AND UNITS

		<i>SI</i>	<i>British</i>
A	wing aspect ratio		
a, b	$a = \beta A$, $b = A \tan \Lambda_{1/2}$ in Sketches 4.2 and 4.4		
a_1	wing lift-curve slope	rad^{-1}	rad^{-1}
$(a_1/A)_{\text{Eqn (3.17)}}$	value of a_1/A from Equation (3.17)	rad^{-1}	rad^{-1}
$(a_1/A)_{\text{Eqn (3.18)}}$	value of a_1/A from Equation (3.18)	rad^{-1}	rad^{-1}
$(a_1/A)_{\text{Eqn (4.3)}}$	value of a_1/A from Equation (4.3)	rad^{-1}	rad^{-1}
$(a_1/A)_{\text{Eqn (4.11)}}$	value of a_1/A from Equation (4.11)	rad^{-1}	rad^{-1}
$(a_1/A)_{\text{Eqn (4.20)}}$	value of a_1/A from Equation (4.20)	rad^{-1}	rad^{-1}
$(a_1/A)_{70011}$	value of a_1/A from ESDU 70011	rad^{-1}	rad^{-1}
$(a_1)_{M=0}$	value of a_1 for incompressible flow	rad^{-1}	rad^{-1}
$(a_1)_0$	section lift-curve slope, 2π for thin section in incompressible flow	rad^{-1}	rad^{-1}
$(a_1)_{0\Lambda}$	value of $(a_1)_0$ for infinite swept wing, Equation (3.3)	rad^{-1}	rad^{-1}
$F_{\Lambda 1/2}$	wing sweep factor, Equation (4.7)		
$F_{\Lambda 1/2\beta}$	value of $F_{\Lambda 1/2}$ for compressible flow, Equation (4.12)		
f_λ	taper ratio correction, Equation (4.19)		
k_1, k_2	correction factors in Equation (4.13) to give exact agreement with $(a_1/A)_{70011}$, Equations (4.17), (4.14)		
M	Mach number		
P	coefficient in Equation (4.17) for k_1 , Equation (4.18)		
β	Prandtl-Glauert compressibility factor, Equation (3.10)		

β_n	value of β based on Mach number normal to wing sweep, Equation (3.7)		
Λ	wing sweep angle	deg	deg
$\Lambda_{1/4}$	value of Λ for wing quarter-chord line	deg	deg
$\Lambda_{1/2}$	value of Λ for wing half-chord line	deg or rad	deg rad
λ	wing taper ratio, tip chord/root chord		

2. INTRODUCTION

Appendix B of ESDU 70011 (Reference 13, as at Amendment I) introduced the semi-empirical so-called Helmbold-Diederich equation for the subsonic lift-curve slope of a straight-tapered wing, in which the wing sweep is taken as the half-chord value, $\Lambda_{1/2}$. Similarly, ESDU 87031 (Reference 17, as at Amendment A) used a form of the equation applicable to incompressible flows. In that application the sweep was taken as the quarter-chord value, $\Lambda_{1/4}$.

The Helmbold-Diederich equation was given in ESDU 70011 purely for information, as a record of a simple analytical expression which was widely used prior to the advent of more reliable, but complex, methods necessitating the use of a computer for solution in practice. Such methods include the lifting-surface theory which was used to derive the data given in ESDU 70011. However, no comparisons between predictions from the Helmbold-Diederich equation and that of ESDU 70011 were given in Appendix B of the Item. With regard to ESDU 87031, the Helmbold-Diederich equation was used in preference to ESDU 70011 mainly because of its simplicity. A brief reference to the accuracy of the simple method compared to that of ESDU 70011 was given.

Two questions remain from the foregoing:

- (i) How accurate is the compressible form of the Helmbold-Diederich equation when compared with the data from ESDU 70011?
- (ii) How do the values of lift-curve slope using $\Lambda_{1/4}$ or $\Lambda_{1/2}$ in that equation compare with those from ESDU 70011?

The objects of this Memorandum are twofold. Firstly, to provide a brief background to the development of the Helmbold-Diederich equation, in Section 3, and secondly, to answer the questions (i) and (ii) above, in Section 4. In particular, Section 4.1 provides a simple means of improving the accuracy of the Helmbold-Diederich equation. The conclusions of the study are given in Section 5 and the References used are listed in Section 6. Appendix A provides tabulated data used in the comparisons and Appendix B gives a brief background to the developments of lifting-line and lifting-surface theories.

3. BACKGROUND TO THE HELMBOLD-DIEDERICH EQUATION

In Reference 8 Diederich initially presents the lift-curve slope of an unswept elliptic wing in incompressible flow, $(a_1)_{M=0}$, which from lifting-line theory can be given as

$$\frac{(a_1)_{M=0}}{A} = \frac{(a_1)_0}{\frac{(a_1)_0}{\pi} + A}, \quad (3.1)$$

in which $(a_1)_0$ is the section lift-curve slope and A is the wing aspect ratio.

According to Diederich, in order to correct the over-prediction of Equation (3.1) at low values of A , Helmbold (Reference 2) modified it using lifting-surface theory for elliptic wings. In Reference 8 Diederich interpreted Helmbold's results in general terms as

$$\frac{(a_1)_{M=0}}{A} = \frac{(a_1)_0}{\frac{(a_1)_0}{\pi} + A \sqrt{1 + \frac{1}{A^2} \left(\frac{(a_1)_0}{\pi} \right)^2}}. \quad (3.2)$$

Comparison of Equation (3.2) with (3.1) shows that the correction for low aspect ratio amounts to the addition of the square root term modifying A in the denominator of Equation (3.2).

For an infinite swept wing, with a section lift-curve slope given by

$$(a_1)_{0\Lambda} = (a_1)_0 \cos \Lambda, \quad (3.3)$$

Equation (3.2) now becomes

$$\frac{(a_1)_{M=0}}{A} = \frac{(a_1)_0 \cos \Lambda}{\frac{(a_1)_0}{\pi} \cos \Lambda + A \sqrt{1 + \left(\frac{(a_1)_0}{\pi} \right)^2 \left(\frac{\cos \Lambda}{A} \right)^2}}. \quad (3.4)$$

Following the division of the numerator and denominator of the right-hand-side by $\cos \Lambda$, Equation (3.4) becomes, with some rearrangement,

$$\frac{(a_1)_{M=0}}{A} = \frac{(a_1)_0}{\frac{(a_1)_0}{\pi} + \sqrt{\left(\frac{A}{\cos \Lambda} \right)^2 + \left(\frac{(a_1)_0}{\pi} \right)^2}}. \quad (3.5)$$

Now, multiplying the numerator and denominator by $\pi/(a_1)_0$, gives

$$\frac{(a_1)_{M=0}}{A} = \frac{\pi}{1 + \sqrt{\left(\frac{\pi}{(a_1)_0}\right)^2 \left(\frac{A}{\cos \Lambda}\right)^2} + 1}. \quad (3.6)$$

To account for compressibility, Diederich suggests that $(a_1)_0$ can be divided by a Prandtl-Glauert factor, β_n , based on Mach number normal to the wing sweep, *i.e.*

$$\beta_n = \left[1 - (M \cos \Lambda)^2\right]^{1/2}, \quad (3.7)$$

and Equation (3.6) becomes

$$\frac{a_1}{A} = \frac{\pi}{1 + \sqrt{\left(\frac{\pi}{(a_1)_0}\right)^2 \left(\frac{A}{\cos \Lambda}\right)^2 \left[1 - (M \cos \Lambda)^2\right]} + 1}, \quad (3.8)$$

which can be simplified to

$$\frac{a_1}{A} = \frac{\pi}{1 + \sqrt{\left(\frac{\pi}{(a_1)_0}\right)^2 \left(\frac{A}{\cos \Lambda}\right)^2 - \left(\frac{\pi}{(a_1)_0}\right)^2 A^2 M^2} + 1}. \quad (3.9)$$

The conventional Prandtl-Glauert compressibility factor

$$\beta = \left(1 - M^2\right)^{1/2} \quad (3.10)$$

can be arranged to give

$$-M^2 = \beta^2 - 1. \quad (3.11)$$

Then, with the substitution of $-M^2$ given by Equation (3.11), Equation (3.9) becomes

$$\frac{a_1}{A} = \frac{\pi}{1 + \sqrt{\left(\frac{\pi}{(a_1)_0}\right)^2 \left[(\beta A)^2 + \left(\frac{A}{\cos \Lambda}\right)^2 - A^2 \right]} + 1}, \quad (3.12)$$

which, making use of the relationship

$$\tan^2 \Lambda = \sec^2 \Lambda - 1 = \frac{1}{\cos^2 \Lambda} - 1,$$

becomes

$$\frac{a_1}{A} = \frac{\pi}{1 + \sqrt{\left(\frac{\pi}{(a_1)_0}\right)^2 [(\beta A)^2 + (A \tan \Lambda)^2] + 1}}. \quad (3.13)$$

If it is assumed that, for a thin section in inviscid incompressible flow,

$$(a_1)_0 = 2\pi, \quad (3.14)$$

then Equation (3.13) reduces to

$$\boxed{\frac{a_1}{A} = \frac{2\pi}{2 + \sqrt{(\beta A)^2 + (A \tan \Lambda)^2 + 4}}}, \quad (3.15)$$

which is the general form of the Helmbold-Diederich equation.

For incompressible flow with $\beta = 1 (M = 0)$, Equation (3.15) further reduces to

$$\boxed{\frac{(a_1)_{M=0}}{A} = \frac{2\pi}{2 + \sqrt{\left(\frac{A}{\cos \Lambda}\right)^2 + 4}}}, \quad (3.16)$$

which, after some rearrangement, is identical to Equation (3.5) with $(a_1)_0 = 2\pi$.

Equations (3.15) and (3.16) are compressible and incompressible forms, respectively, of the Helmbold-Diederich equation.

The foregoing development, based on that due to Diederich (Reference 8) in 1951, is consistent with the later work (in 1957) by Lowry and Polhamus (Reference 11), who developed an equation* which can be shown to be identical to Equation (3.9)

3.1 The Effect of Wing Taper

As noted in an appendix to Reference 11, no term relating to wing taper appears in the development so far, it being based heavily on lifting-line theory for elliptic wings of moderate to high aspect ratio, with simple corrections for low aspect ratio, sweep and compressibility. However, Lowry and Polhamus point out that the reverse flow theorem for thin wings (Reference 7) suggests that the use of the half-chord sweep, $\Lambda_{1/2}$,

* In Reference 11 it is claimed that Polhamus modified Helmbold's equation, Equation (3.2), for sweep and compressibility in 1950, the year prior to the publication of Reference 8. So, perhaps Equation (3.15) could justifiably be referred to as the Helmbold-Polhamus equation?

rather than any other – the commonly used quarter-chord sweep, $\Lambda_{1/4}$, for example, should eliminate the main effects of wing taper in the determination of lift-curve slope. Consequently, the use of Equation (3.15) with $\Lambda = \Lambda_{1/2}$, *i.e.*

$$\frac{a_1}{A} = \frac{2\pi}{2 + \sqrt{(\beta A)^2 + (A \tan \Lambda_{1/2})^2} + 4}, \quad (3.17)$$

would be expected to correlate the effects of wing taper better than that with $\Lambda = \Lambda_{1/4}$, or

$$\frac{a_1}{A} = \frac{2\pi}{2 + \sqrt{(\beta A)^2 + (A \tan \Lambda_{1/4})^2} + 4}. \quad (3.18)$$

This expectation is investigated in Section 4.

4. COMPARISONS BETWEEN THE USE OF EQUATION (3.17) OR (3.18) AND THE DATA OF ESDU 70011

In their appendix to Reference 11, Lowry and Polhamus carried out a limited study of the effect of the use of either $\Lambda_{1/4}$ or $\Lambda_{1/2}$ for incompressible flow using Equation (3.16) when compared with lift-curve slopes using the Weissinger 15-point method* (Reference 3). The results, for both swept-back and swept-forward wings with taper ratios, $\lambda = 0, 1$ and 1.5 (*i.e.* inverse taper), clearly indicated the benefit of the use of $\Lambda_{1/2}$ in providing a much better correlation of lift-curve slope.

ESDU 70011 was developed using subsonic lifting-surface theory: for details see Section 3 of that Data Item and Appendix B here. Initially, calculations were performed by Garner and Inch (Reference 12) for 64 wing planforms covering every combination of the parameters:

$$\beta A = 1.5, 3, 5 \text{ and } 8,$$

$$A \tan \Lambda_{1/2} = 0, 2, 4, \text{ and } 6$$

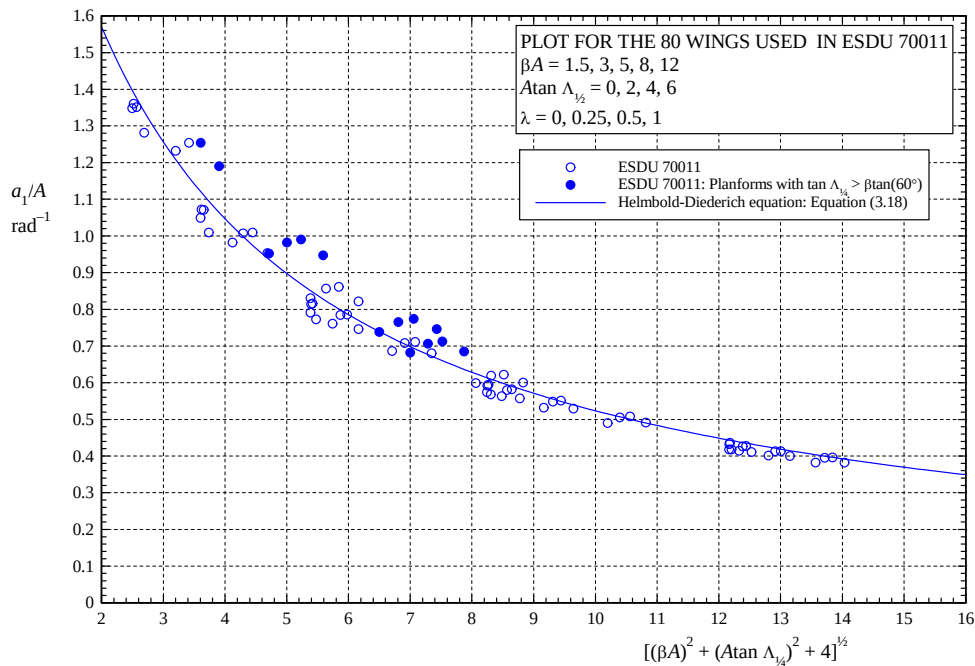
$$\text{and } \lambda = 0, 0.25, 0.5 \text{ and } 1.$$

Subsequently, 16 additional cases were added to extend the range of βA up to $\beta A = 12$. The results, in terms of a_1/A , defined here as $(a_1/A)_{70011}$, are given in Table A1.1 of Appendix A of this Technical Memorandum.

[In order to cover the complete range down to $\beta A = 0$ for the charts in ESDU 70011, sonic slender-wing theory (Reference 9) was used. However, for the present purpose the comparisons will be restricted to $\beta A \geq 1.5$. Although the Helmbold-Diederich equation provides the correct slender-wing value of $\pi/2$ for $A \tan \Lambda_{1/2} = 0$, it would be unreasonable to expect it to give good results for larger values, and comparisons with data from ESDU 70011 confirm that.]

Sketch 4.1 presents a correlation of the data from Table A1.1 using Equation (3.18), *i.e.* the Helmbold-Diederich equation in terms of $\Lambda_{1/4}$. The abscissal parameter, $[(\beta A)^2 + (A \tan \Lambda_{1/4})^2 + 4]^{1/2}$, is that part of Equation (3.18) that contains the two planform parameters used in it.

* The Weissinger L-method, a modified lifting-line theory based on the three-quarter chord theorem, see Appendix B.



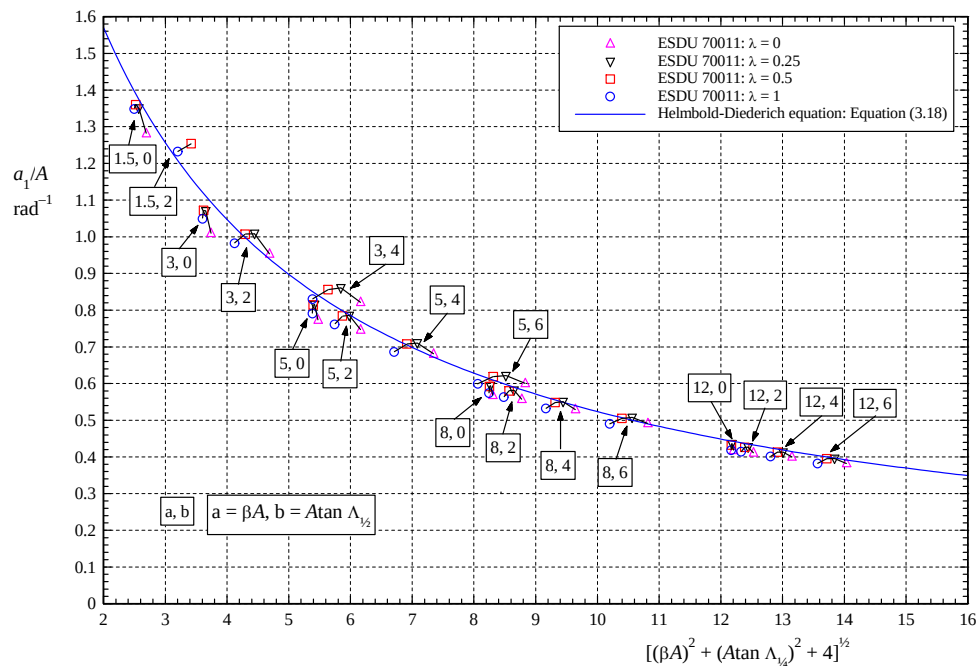
Sketch 4.1 Comparison of Equation (3.18) with the subsonic lifting-surface theory data from Table A1.1

If the data are taken as a whole, it is seen that they are not perfectly correlated, but they do have a fairly even scatter. However, the data at high values of the abscissal parameter, beyond about 11, are all over-predicted by Equation (3.18). The worst scatter is created by planforms with extreme values of $\Lambda_{1/4}$, as shown by the solid symbols in Sketch 4.1. Those points were selected using the somewhat arbitrary relationship

$$\frac{A \tan \Lambda_{1/4}}{\beta A} = \frac{\tan \Lambda_{1/4}}{\beta} > \tan(60^\circ) \quad (4.1)$$

or $\tan \Lambda_{1/4} > \beta \tan(60^\circ) (=1.732\beta), \quad (4.2)$

in which $\Lambda_{1/4} = 60^\circ$ was taken as an upper limit for low speeds. Even that might be considered conservative in terms of most current practical planforms. Removal of the data for those (14) extreme planforms gives the plot shown in Sketch 4.2, in which the taper ratio is now identified.



Sketch 4.2 Sketch 4.1 with the 14 planforms for $\tan \Lambda_{1/4}/\beta > 1.732$ removed

The removal of the extreme planforms provides a reduced and more evenly distributed scatter. In the scatter that remains it is now possible to identify the effect of the planform parameters. For a given value of βA it is seen that Equation (3.18) provides a gradual improvement in agreement with the data from ESDU 70011 as $A \tan \Lambda_{1/2}$ is increased from zero. Contrary to expectations, the effect of λ is not too significant, at least for the larger values of βA . Overall, bearing in mind the simple origins of the Helmbold-Diederich equation, its use with $\Lambda_{1/4}$ provides good agreement with subsonic lifting-surface theory data.

Sketches 4.3 and 4.4 are plots similar to those in Sketches 4.1 and 4.2, respectively, with $\Lambda_{1/4}$ in the abscissal parameter being replaced by $\Lambda_{1/2}$. The corresponding correlating equation is Equation (3.17).