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Power law model – Goodness-of-fit tests and estimation methods

Modèle de loi en puissance – Essais d'adéquation et méthodes d'estimation des paramètres



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Power law model – Goodness-of-fit tests and estimation methods

Modèle de loi en puissance – Essais d'adéquation et méthodes d'estimation des paramètres

INTERNATIONAL
ELECTROTECHNICAL
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COMMISSION
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INTERNATIONALE

PRICE CODE **XA**
CODE PRIX

ICS 03.120.01; 03.120.30

ISBN 978-2-83220-797-0

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INTERNATIONAL ELECTROTECHNICAL COMMISSION

**POWER LAW MODEL –
GOODNESS-OF-FIT TESTS
AND ESTIMATION METHODS**

FOREWORD

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International Standard IEC 61710 has been prepared by IEC technical committee 56: Dependability.

This second edition cancels and replaces the first edition, published in 2000, and constitutes a technical revision.

The main changes with respect to the previous edition are listed below:

- the inclusion of an additional Annex C on Bayesian estimation for the power law model.

The text of this standard is based on the following documents:

FDIS	Report on voting
56/1500/FDIS	56/1508/RVD

Full information on the voting for the approval of this standard can be found in the report on voting indicated in the above table.

This publication has been drafted in accordance with the ISO/IEC Directives, Part 2.

The committee has decided that the contents of this publication will remain unchanged until the stability date indicated on the IEC web site under "<http://webstore.iec.ch>" in the data related to the specific publication. At this date, the publication will be

- reconfirmed,
- withdrawn,
- replaced by a revised edition, or
- amended.

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INTRODUCTION

This International Standard describes the power law model and gives step-by-step directions for its use. There are various models for describing the reliability of repairable items, the power law model being one of the most widely used. This standard provides procedures to estimate the parameters of the power law model and to test the goodness-of-fit of the power law model to data, to provide confidence intervals for the failure intensity and prediction intervals for the length of time to future failures. An input is required consisting of a data set of times at which relevant failures occurred, or were observed, for a repairable item or a set of copies of the same item, and the time at which observation of the item was terminated, if different from the time of final failure. All output results correspond to the item type under consideration.

Some of the procedures can require computer programs, but these are not unduly complex. This standard presents algorithms from which computer programs should be easy to construct.

POWER LAW MODEL – GOODNESS-OF-FIT TESTS AND ESTIMATION METHODS

1 Scope

This International Standard specifies procedures to estimate the parameters of the power law model, to provide confidence intervals for the failure intensity, to provide prediction intervals for the times to future failures, and to test the goodness-of-fit of the power law model to data from repairable items. It is assumed that the time to failure data have been collected from an item, or some identical items operating under the same conditions (e.g. environment and load).

2 Normative references

The following documents, in whole or in part, are normatively referenced in this document and are indispensable for its application. For dated references, only the edition cited applies. For undated references, the latest edition of the referenced document (including any amendments) applies.

IEC 60050-191:1990, *International Electrotechnical Vocabulary (IEV) – Chapter 191: Dependability and quality of service*

3 Terms and definitions

For the purposes of this document, the terms and definitions of IEC 60050-191 apply.

4 Symbols and abbreviations

The following symbols and abbreviations apply:

β	shape parameter of the power law model
$\hat{\beta}$	estimated shape parameter of the power law model
β_{LB}, β_{UB}	lower, upper confidence limits for β
C^2	Cramer-von-Mises goodness-of-fit test statistic
$C_{1-\gamma}^2(M)$	critical value for the Cramer-von-Mises goodness-of-fit test statistic at γ level of significance
χ^2	Chi-square goodness-of-fit test statistic
$\chi_\gamma^2(\nu)$	γ th fractile of the χ^2 distribution with ν degrees of freedom
d	number of intervals for groups of failures
$E[N(t)]$	expected accumulated number of failures up to time t
$E[t_j]$	expected accumulated time to j th failure

$\hat{E}[N[t(i)]]$	estimated expected accumulated number of failures up to $t(i)$
$\hat{E}[t_j]$	estimated expected accumulated time to j th failure
$F_\gamma(v_1, v_2)$	γ th fractile for the F distribution with (v_1, v_2) degrees of freedom
i	general purpose indicator
j	general purpose indicator
k	number of items
L, U	multipliers used in calculation of confidence intervals for failure intensity
λ	scale parameter of the power law model
$\hat{\lambda}$	estimated scale parameter of the power law model
M	parameter for Cramer-von-Mises statistical test
N	number of relevant failures
N_j	number of failures for j th item
$N(t)$	accumulated number of failures up to time t
$N[t(i)]$	accumulated number of failures up to time $t(i)$
R	difference between the order number of future (predicted) failure and order number of last (observed) failure
T	accumulated relevant time
T^*	total accumulated relevant time for time terminated test
T_j	total accumulated relevant time for j th item
T_{RL}, T_{RU}	lower, upper prediction limits for the length of time to the R th future failure
$\hat{T}_{N+1}$	estimated median time to $(N+1)$ th failure
t_i	accumulated relevant time to the i th failure
t_{ij}	i th failure time for j th item
t_N	total accumulated relevant time for failure terminated test
t_{Nj}	total accumulated relevant time to N th failure of j th item
$t(i-1), t(i)$	endpoints of i th interval of time for grouped failures
$z(t)$	failure intensity at time t
$\hat{z}(t)$	estimated failure intensity at time t
z_{LB}, z_{UB}	lower, upper confidence limits for failure intensity

5 Power law model

The statistical procedures for the power law model use the relevant failure and time data from the test or field studies. The basic equations for the power law model are given in this clause. Background information on the model is given in Annex A and examples of its application are given in Annex B.

The expected accumulated number of failures up to test time t is given by:

$$E[N(t)] = \lambda t^\beta \text{ with } \lambda > 0, \beta > 0, t > 0$$

where

λ is the scale parameter;

β is the shape parameter ($0 < \beta < 1$ corresponds to a decreasing failure intensity; $\beta = 1$ corresponds to a constant failure intensity; $\beta > 1$ corresponds to an increasing failure intensity).

The failure intensity at time t is given by:

$$z(t) = \frac{d}{dt} E[N(t)] = \lambda \beta t^{\beta-1} \text{ with } t > 0$$

Thus the parameters λ and β both affect the failure intensity in a given time.

Methods are given in 7.2 for maximum likelihood estimation of the parameters of λ and β . Subclause 7.3 gives goodness-of-fit tests for the model and 7.4 and 7.5 give confidence interval procedures. Subclause 7.6 gives prediction interval procedures and 7.7 gives tests for the equality of the shape parameters. The model is simple to evaluate. However when $\beta < 1$, theoretically $z(0) = \infty$ (i.e. $z(t)$ tends to infinity as t tends to zero) and $z(\infty) = 0$ (i.e. $z(t)$ tends to zero as t tends to infinity); but this theoretical limitation does not generally affect its practical use.

6 Data requirements

6.1 General

6.1.1 Case 1 – Time data for every relevant failure for one or more copies from the same population

The normal evaluation methods assume the observed times to be exact times of failure of a single repairable item or a set of copies of the same repairable item. The figures below illustrate how the failure times are calculated for three general cases.

6.1.2 Case 1a) – One repairable item

For one repairable item observed from time 0 to time T , the relevant failure time, t_i , is the elapsed operating time (that is, excluding repair and other down times) until the occurrence of the i -th failure as shown in Figure 1.

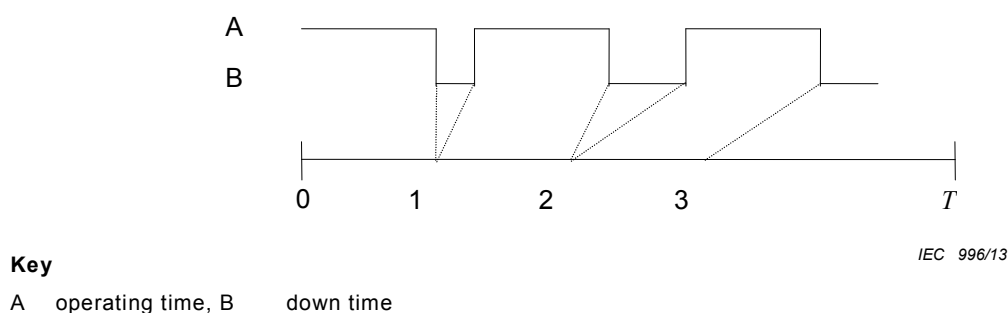
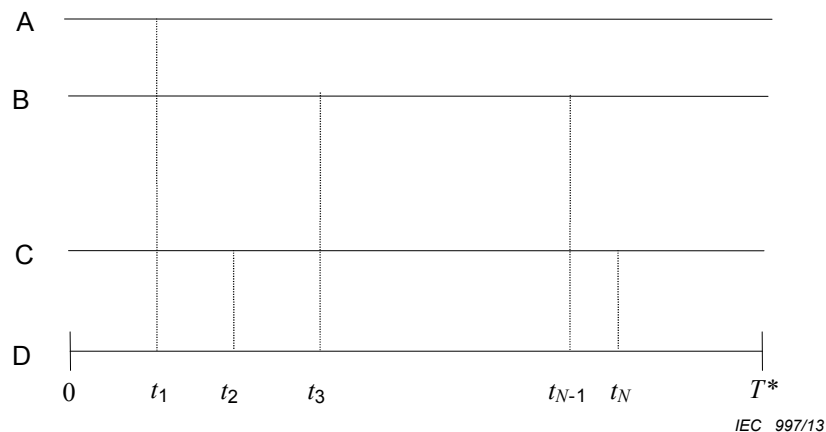


Figure 1 – One repairable item

Time terminated data are observed to T^* , which is not a failure time, and failure terminated data are observed to t_N , which is the time of the N th failure. Time terminated and failure terminated data use slightly different formulae.

6.1.3 Case 1b) – Multiple items of the same kind of repairable item observed for the same length of time

It is assumed there are k items, which all represent the same population. That is, they are nominally identical items operating under the same conditions (e.g. environment and load). When all items are observed to time T^* , which is not a failure time (i.e. time terminated data), then the failure time data are combined by superimposing failure times $(t_i, i = 1, 2, \dots, N)$ for all k items on the same time line as shown in Figure 2.



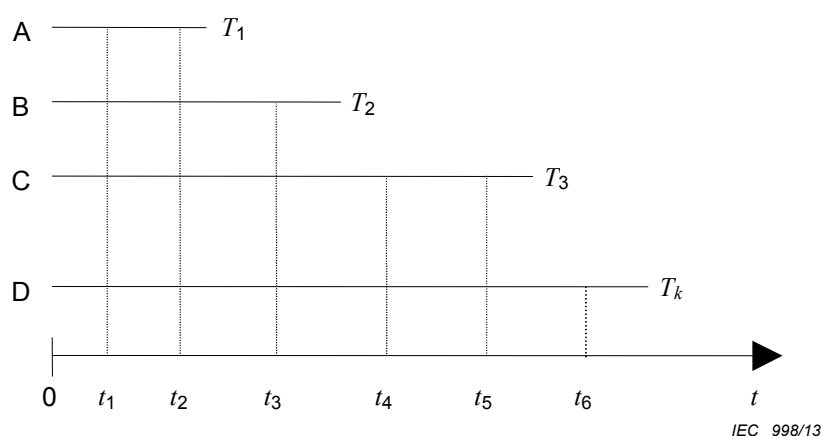
Key

- A item 1
- B item 2
- C item k
- D superimposed process

Figure 2 – Multiple items of the same kind of repairable item observed for same length of time

6.1.4 Case 1c) – Multiple repairable items of the same kind observed for different lengths of time

When all items do not operate for the same period of time, then the time at which observation of the j th item is terminated $T_j (j = 1, 2, \dots, k)$, where $T_1 < T_2 < \dots < T_k$, is noted. The failure data are combined by superimposing all the failure times for all k items on the same time line as shown in Figure 3. The times to failure are $t_i, i = 1, 2, \dots, N$, where N = the total number of failures observed accumulated over the k items.



Key

- A item 1
- B item 2
- C item 3
- D item k
- t time

Figure 3 – Multiple repairable items of the same kind observed for different lengths of time

If each item is a software system then the repair action should be done to the other systems which did not fail at that time.

6.2 Case 2 – Time data for groups of relevant failures for one or more repairable items from the same population

This alternative method is used when there is at least one copy of an item and the data consist of known time intervals, each containing a known number of failures.

The observation period is over the interval $(0, T)$ and is partitioned into d intervals at times $0 < t(1) < t(2) < \dots < t(d)$. The i th interval is the time period between $t(i-1)$ and $t(i)$, where $i = 1, 2, \dots, d$, $t(0) = 0$ and $t(d) = T$. It is important to note that the interval lengths and the number of failures per interval need not be the same.

6.3 Case 3 – Time data for every relevant failure for more than one repairable item from different populations

It is assumed there are k items which do not represent the same population and are to be compared. It should be noted that if each item is to be considered individually then it is appropriate to use case 1a) in 6.1.2.

If direct comparisons of the items are to be made then as an extension of 6.1 the following notation is used:

- t_{ij} denotes the i th failure time for the process corresponding to the j th item;
- N_j denotes the number of failures observed for the j th item;
- t_{N_j} is the time of the N th failure for the j th item;

where $i = 0, 1, 2, \dots, N_j$ and $j = 1, 2, \dots, k$.

7 Statistical estimation and test procedures

7.1 Overview

In case 1 – time data for every relevant failure – the formulae given for failure terminated data assume one repairable item, that is $k=1$. All output results correspond to that item. The formulae given for time terminated data assume k copies of the item observed for the same length of time. If there is only one repairable item then $k=1$. The point estimation procedures for all the aforementioned cases are given in 7.2.1. The appropriate procedures for the case when all copies are observed for different lengths of time are given in 7.2.2. Procedures for the case of time data for groups of relevant failures are given in 7.2.3.

An appropriate goodness-of-fit test, as described in 7.3 shall be performed after the parameter estimation procedures of 7.2. Note that these tests, and the procedures given in 7.4 to 7.7 for constructing interval estimates and carrying out statistical tests, distinguish only between the cases of time data for every relevant failure (i.e. all instances of case 1 data – 1a), 1b) and 1c)) and time data for groups of relevant failures (i.e. case 2)).

The inference procedures that follow provide approximate estimates in some circumstances and so caution is required if they are to be applied if the number of observed failures is less than 10.

7.2 Point estimation

7.2.1 Case 1a) and 1b) – Time data for every relevant failure

This method applies only when the time of failure has been logged for every failure as described in 6.1.2 and 6.1.3.

Step 1: Calculate the summation:

$$S_1 = \sum_{j=1}^N \ln \left(\frac{T^*}{t_j} \right) \quad (\text{time terminated})$$

$$S_2 = \sum_{j=1}^N \ln \left(\frac{t_N}{t_j} \right) \quad (\text{failure terminated})$$

Step 2: Calculate the (unbiased) estimate of the shape parameter β from the formula:

$$\hat{\beta} = \frac{N-1}{S_1} \quad (\text{time terminated})$$

$$\hat{\beta} = \frac{N-2}{S_2} \quad (\text{failure terminated})$$

Step 3: Calculate the estimate of the scale parameter λ from the formula:

$$\hat{\lambda} = \frac{N}{k(T^*)^{\hat{\beta}}} \quad (\text{time terminated})$$

$$\hat{\lambda} = \frac{N}{k(t_N)^{\hat{\beta}}} \quad (\text{failure terminated})$$

Step 4: Calculate the estimate of the failure intensity $z(t)$, for any time $t > 0$, from the formula:

$$\hat{z}(t) = \hat{\lambda} \hat{\beta} t^{\hat{\beta}-1}$$

$\hat{z}(t)$ estimates the current failure intensity for t over the range represented by the data. "Extrapolated" estimates for a future t may be obtained similarly, but should be used with the usual caution associated with extrapolation.

Step 5: Given N observed failures the last of which occurred at t_N , the median time to the $(N+1)$ th failure can be estimated from the formula:

$$\hat{T}_{N+1} = t_N \exp \left[\frac{0,5^{\frac{-1}{N+1}} - 1}{\hat{N}\hat{\beta} / (N-1)} \right] \quad (\text{time terminated})$$

$$\hat{T}_{N+1} = t_N \exp \left[\frac{0,5^{\frac{-1}{N+1}} - 1}{\hat{N}\hat{\beta} / (N-2)} \right] \quad (\text{failure terminated})$$

7.2.2 Case 1c) – Time data for every relevant failure

This method applies only when the time of failure has been logged for every failure as described in 6.1.4.

Step 1: Assemble the data into the times to failure, t_i , $i=1,2,...,N$, where N is the total number of failures over the k copies and T_j , $j=1,2,...,k$, is the end of the observation period for the j th copy.

Step 2: The maximum likelihood estimate of the shape parameter β is the value of $\hat{\beta}$ which satisfies the formula:

$$\frac{N}{\hat{\beta}} + \sum_{i=1}^N \ln t_i - \frac{N \sum_{j=1}^k T_j^{\hat{\beta}} \ln T_j}{\sum_{j=1}^k T_j^{\hat{\beta}}} = 0$$

An iterative method shall be used to solve the formula for $\hat{\beta}$.

Step 3: Calculate the estimate of the scale parameter λ from the formula:

$$\hat{\lambda} = \frac{N}{\sum_{j=1}^k T_j^{\hat{\beta}}}$$

Step 4: Calculate the estimate of the failure intensity $z(t)$, for any time $t > 0$, from the formula:

$$\hat{z}(t) = \hat{\lambda} \hat{\beta} t^{\hat{\beta}-1}$$

$\hat{z}(t)$ estimates the current failure intensity for t over the range represented by the data. "Extrapolated" estimates for a future t may be obtained similarly, but should be used with the usual caution associated with extrapolation.

7.2.3 Case 2 – Time data for groups of relevant failures

This method applies when the data set consists of known time intervals, each containing a known number of failures as described in 6.2.

Step 1: Assemble into a data set the number of relevant failures N_i recorded in the i th interval $[t(i-1), t(i)]$, $i = 1, 2, \dots, d$. The total number of relevant failures is

$$N = \sum_{i=1}^d N_i$$

Step 2: The maximum likelihood estimate of the shape parameter β is the value of $\hat{\beta}$ which satisfies the formula:

$$\sum_{i=1}^d N_i \left[\frac{[t(i)]^{\hat{\beta}} \ln t(i) - [t(i-1)]^{\hat{\beta}} \ln t(i-1)}{[t(i)]^{\hat{\beta}} - [t(i-1)]^{\hat{\beta}}} - \ln t(d) \right] = 0$$

Note that $[t(0)]^{\hat{\beta}} = 0$ and $[t(0)]^{\hat{\beta}} \ln t(0) = 0$. All $t(\cdot)$ terms may be normalized with respect to $t(d)$ and then the final term $\ln[t(d)]$ disappears. An iterative method shall be used to solve the formula for $\hat{\beta}$.

Step 3: Calculate the estimate of the scale parameter λ from the formula:

$$\hat{\lambda} = \frac{N}{t(d)^{\hat{\beta}}}$$

Step 4: Calculate the estimate of the failure intensity $z(t)$, for any test time $t > 0$, from the formula:

$$\hat{z}(t) = \hat{\lambda} \hat{\beta} t^{\hat{\beta}-1}$$

$\hat{z}(t)$ estimates the current failure intensity for t over the range represented by the data. "Extrapolated" estimates for a future t may be obtained similarly, but should be used with the usual caution associated with extrapolation.

7.3 Goodness-of-fit tests

7.3.1 Case 1 – Time data for every relevant failure

7.3.1.1 Cramer-von-Mises test

Step 1: Calculate $\hat{\beta}$ from step 2 in 7.2.1 or step 2 in 7.2.2.

Step 2: Calculate the Cramer-von-Mises goodness-of-fit test statistic given by the formula:

$$C^2 = \frac{1}{12M} + \sum_{j=1}^M \left[\left(\frac{t_j}{T} \right)^{\hat{\beta}} - \left(\frac{2j-1}{2M} \right) \right]^2$$

where

$M = N$ and $T = T^*$ (time terminated)

$M = N - 1$ and $T = t_N$ (failure terminated)

Step 3: Select the critical value $C_{0,90}^2(M)$ for the Cramer-von-Mises test corresponding to M from Table 1, which gives critical values at a 10 % significance level.

Step 4: If:

$$C^2 > C_{0,90}^2(M)$$

then the hypothesis that the power law model fits the data cannot be accepted. Otherwise, on the basis of the data analysed, the power law model can be used as a working hypothesis.

7.3.1.2 Graphical procedure

When the failure times are known, the graphical procedure described below may be used to obtain additional information about the correspondence between the model and the data. This involves plotting the expected time to the j th failure, $E(t_j)$, against the observed time to the j th failure. Further details about the approach are given in Annexes A and B.

Step 1: Calculate $\hat{\beta}$ from step 2 in 7.2.1 and $\hat{\lambda}$ from step 3 in 7.2.1.

Step 2: Calculate the estimate of the expected time to the j th failure, $j=1,2,...,N$, from the formula:

$$\hat{E}(t_j) = \left(\frac{j}{\hat{\lambda}} \right)^{\frac{1}{\hat{\beta}}}$$

Step 3: Plot $\hat{E}(t_j)$ against t_j on identical linear scales. The visual agreement of these points with a line of 45 ° through the origin is a subjective measure of the applicability of the model.

7.3.2 Case 2 – Time data for groups of relevant failures

7.3.2.1 Chi-square test

Step 1: Calculate $\hat{\beta}$ from step 2 in 7.2.3 and $\hat{\lambda}$ from step 3 in 7.2.3.

Step 2: Calculate the expected number of failures in the time interval $[t(i-1), t(i)]$ which is approximated by:

$$e_i = \hat{\lambda} \left\{ [t(i)]^{\hat{\beta}} - [t(i-1)]^{\hat{\beta}} \right\}$$

Step 3: For each interval, e_i shall not be less than 5, and if necessary, adjacent intervals should be combined before the test. For d intervals (after combination if necessary) and with N_i the same as in 7.2.3, calculate the statistic:

$$\chi^2 = \sum_{i=1}^d \frac{(N_i - e_i)^2}{e_i}$$

Step 4: Select the critical value from a χ^2 distribution with $(d-2)$ degrees of freedom and a 10 % significance level from Table 2, i.e. $\chi_{0,90}^2(d-2)$.

Step 5: If the test statistic χ^2 exceeds the critical value $\chi_{0,90}^2(d-2)$ then the hypothesis that the power law model fits the data cannot be accepted. Otherwise, on the basis of the data analysed, the power law model can be used as a working hypothesis.

The Chi-square test is a large sample test and so will need large data sets to detect deviations from the power law model that are practically important.

7.3.2.2 Graphical procedure

When the data set consists of known time intervals, each containing a known number of failures, the graphical procedure described below may be used to obtain additional information about the correspondence between the model and the data. This involves plotting the expected number of failures against those observed at each endpoint. Further details of the approach are given in B.5.

Step 1: For each endpoint $t(i)$, calculate the observed number of failures from 0 to $t(i)$ from the formula:

$$N[t(i)] = \sum_{j=1}^i N_j$$

Step 2: Calculate the estimate of the corresponding expected number of failures $E[N[t(i)]]$ from the formula:

$$\hat{E}[N[t(i)]] = \hat{\lambda}_t(i)^{\hat{\beta}}$$

Step 3: Plot $\hat{E}[N[t(i)]]$ against $N[t(i)]$ on identical linear scales. The visual agreement of these points with a line of 45 ° through the origin is a subjective measure of the applicability of the model.

7.4 Confidence intervals for the shape parameter

7.4.1 Case 1 – Time data for every relevant failure

The shape parameter β in the power law model determines if the failure intensity changes with time. If $0 < \beta < 1$, there is decreasing failure intensity; if $\beta = 1$, there is a constant failure intensity; if $\beta > 1$, there is an increasing failure intensity.

For a two-sided confidence interval for β when individual failure times are available, follow the steps below as appropriate for time and failure terminated data.

Two-sided 90 % confidence interval for β – Time terminated data

Step 1: Calculate $\hat{\beta}$ from step 2 in 7.2.1 or from step 2 in 7.2.2.

Step 2: Calculate:

$$D_L = \frac{\chi_{0,05}^2(2N)}{2(N-1)}$$

$$D_U = \frac{\chi_{0,95}^2(2N)}{2(N-1)}$$

where the fractiles of the χ^2 distribution are given in Table 2.

Step 3: Calculate the lower confidence limit for β from the formula:

$$\beta_{LB} = D_L \hat{\beta}$$

and the upper confidence limit for β from the formula:

$$\beta_{UB} = D_U \hat{\beta}$$

Step 4: The two-sided 90 % confidence interval for β is given by (β_{LB}, β_{UB}) .

NOTE One-sided 95 % lower and upper limits for β are β_{LB} and β_{UB} , respectively.

Two-sided 90 % confidence interval for β – Failure terminated data

Step 1: Calculate $\hat{\beta}$ from step 2 in 7.2.1.

Step 2: Calculate:

$$D_L = \frac{\chi_{0,05}^2(2(N-1))}{2(N-2)}$$

$$D_U = \frac{\chi_{0,95}^2(2(N-1))}{2(N-2)}$$

where the fractiles of the χ^2 distribution are given in Table 2.

Step 3: Calculate the lower confidence limit for β from the formula:

$$\beta_{LB} = D_L \hat{\beta}$$

and the upper confidence limit for β from the formula:

$$\beta_{UB} = D_U \hat{\beta}$$

Step 4: The two-sided 90 % confidence interval for β is given by (β_{LB}, β_{UB}) .

NOTE One-sided 95 % lower and upper limits for β are β_{LB} and β_{UB} , respectively.

7.4.2 Case 2 – Time data for groups of relevant failures

Step 1: Calculate $\hat{\beta}$ from step 2 in 7.2.3.

Step 2: Calculate:

$$P(i) = \frac{t(i)}{t(d)} \quad \text{with} \quad i = 1, 2, \dots, d$$

Step 3: Calculate the expression:

$$A = \sum_{i=1}^d \frac{\left[[P(i)]^{\hat{\beta}} \ln[P(i)]^{\hat{\beta}} - [P(i-1)]^{\hat{\beta}} \ln[P(i-1)]^{\hat{\beta}} \right]^2}{\left[[P(i)]^{\hat{\beta}} - [P(i-1)]^{\hat{\beta}} \right]}$$

Step 4: Calculate:

$$C = \frac{1}{\sqrt{A}}$$

Step 5: For an approximate two-sided 90 % confidence interval for β , calculate:

$$S = \frac{1,64C}{\sqrt{N}}$$

where N is the total number of failures.

Step 6: Calculate the lower confidence limit for β from the formula:

$$\beta_{LB} = \hat{\beta}(1 - S)$$

and the upper confidence limit for β from the formula:

$$\beta_{UB} = \hat{\beta}(1 + S)$$

Step 7: The two-sided 90 % confidence interval for β is given by (β_{LB}, β_{UB}) .

NOTE One-sided 95 % lower and upper limits for β are β_{LB} and β_{UB} , respectively.

7.5 Confidence intervals for the failure intensity

7.5.1 Case 1 – Time data for every relevant failure

Step 1: Calculate $\hat{z}(t)$ from step 4 in 7.2.1 or step 4 in 7.2.2.

Step 2: For a two-sided 90 % confidence interval refer to Table 3 (*time terminated*) and Table 4 (*failure terminated*) and locate values of L and U for the appropriate sample size N .

Step 3: Calculate the lower confidence limit for $z(t)$ from the formula:

$$z_{LB} = \frac{\hat{z}(t)}{U}$$

and the upper confidence limit for $z(t)$ from the formula:

$$z_{UB} = \frac{\hat{z}(t)}{L}$$

Step 4: The two-sided 90 % confidence interval for $z(t)$ is given by (z_{LB}, z_{UB}) .

NOTE One-sided 95 % lower and upper limits for $z(t)$ are z_{LB} and z_{UB} , respectively.

7.5.2 Case 2 – Time data for groups of relevant failures

Step 1: Calculate $\hat{z}(t)$ from step 4 in 7.2.3.

Step 2: Calculate:

$$P(i) = \frac{t(i)}{t(d)} \quad \text{with} \quad i = 1, 2, \dots, d$$

Step 3: Calculate:

$$A = \sum_{i=1}^d \frac{\left[[P(i)]^{\hat{\beta}} \ln[P(i)]^{\hat{\beta}} - [P(i-1)]^{\hat{\beta}} \ln[P(i-1)]^{\hat{\beta}} \right]^2}{\left[[P(i)]^{\hat{\beta}} - [P(i-1)]^{\hat{\beta}} \right]}$$

Step 4: Calculate:

$$D = \sqrt{\frac{1}{A} + 1}$$

Step 5: For an approximate two-sided 90 % confidence interval for $z(t)$ calculate:

$$S = \frac{1,64D}{\sqrt{N}}$$

where N is the cumulative number of relevant failures.

Step 6: The lower confidence limit on $z(t)$ is given by:

$$z_{LB} = \frac{\hat{z}(t)}{1+S}$$

and the upper confidence limit on $z(t)$ is given by:

$$z_{UB} = \frac{\hat{z}(t)}{1-S}$$

Step 7: The two-sided 90 % confidence interval for $z(t)$ is given by (z_{LB}, z_{UB}) .

NOTE One-sided 95 % lower and upper limits for $z(t)$ are z_{LB} and z_{UB} , respectively.

7.6 Prediction intervals for the length of time to future failures of a single item

7.6.1 Prediction interval for length of time to next failure for case 1 – Time data for every relevant failure

For a two-sided 90 % prediction interval for the time to the $(N+1)$ th failure T_{N+1} , that is, the next future failure given that N failures have occurred at times $t_1, t_2, \dots, t_N$, follow the steps below as appropriate for time and failure terminated data.

Step 1: Calculate $\hat{\beta}$ from step 2 in 7.2.1 or from step 2 in 7.2.2.

Step 2: Calculate the lower prediction limit for T_{N+1} from the formula:

$$T_{1L} = t_N \exp \left[\frac{0,95^{\frac{-1}{N-1}} - 1}{N\hat{\beta} / (N-1)} \right] \quad (\text{time terminated})$$

$$T_{1L} = t_N \exp \left[\frac{0,95^{\frac{-1}{N-1}} - 1}{N\hat{\beta} / (N-2)} \right] \quad (\text{failure terminated})$$

and the upper prediction limit for T_{N+1} from the formula:

$$T_{1U} = t_N \exp \left[\frac{0,05^{\frac{-1}{N-1}} - 1}{N\hat{\beta} / (N-1)} \right] \quad (\text{time terminated})$$

$$T_{1U} = t_N \exp \left[\frac{0,05^{\frac{-1}{N-1}} - 1}{N\hat{\beta} / (N-2)} \right] \quad (\text{failure terminated})$$

Step 3: The two-sided 90 % prediction interval for T_{N+1} is given by (T_{1L}, T_{1U}) .

NOTE One-sided 95 % lower and upper limits for T_{N+1} are T_{1L} and T_{1U} , respectively.

7.6.2 Prediction interval for length of time to R th future failure for case 1 – Time data for every relevant failure

For an approximate two-sided 90 % prediction interval for the time to the $(N+R)$ th failure T_{N+R} , that is, the R th future failure given that N failures have occurred at times $t_1, t_2, \dots, t_N$, follow the steps below as appropriate for failure and time terminated data.

Step 1: Calculate $\hat{\beta}$ from step 2 in 7.2.1 or from step 2 in 7.2.2.

Step 2: Calculate:

$$G = \left[\frac{(N-0,5)(N+R-0,5)}{NR} \right] \ln \left[\frac{N+R-0,5}{N-0,5} \right]$$

Step 3: Calculate:

$$V = 2NG \ln \left[\frac{N+R-0,5}{N-0,5} \right]$$

Step 4: Calculate the lower prediction limit for T_{N+R} from the formula:

$$T_{RL} = t_N \exp \left[\frac{V}{2NG \hat{\beta} F_{0,95}(2(N-1), V')} \right] \quad (\text{time terminated})$$

$$T_{RL} = t_N \exp \left[\frac{V(N-2)}{2N(N-1)G \hat{\beta} F_{0,95}(2(N-1), V')} \right] \quad (\text{failure terminated})$$

and the upper prediction limit for T_{N+R} from the formula:

$$T_{RU} = t_N \exp \left[\frac{VF_{0,95}(V', 2(N-1))}{2NG \hat{\beta}} \right] \quad (\text{time terminated})$$

$$T_{RU} = t_N \exp \left[\frac{V(N-2)F_{0,95}(V', 2(N-1))}{2N(N-1)G \hat{\beta}} \right] \quad (\text{failure terminated})$$

where the fractiles of the F distribution are given in Table 5 and V' is the rounded integer value of V .

Step 5: The two-sided 90 % prediction interval for T_{N+R} is given by (T_{RL}, T_{RU}) .

NOTE One-sided 95 % lower and upper limits for T_{N+R} are T_{RL} and T_{RU} , respectively.

7.7 Test for the equality of the shape parameters $\beta_1, \beta_2, \dots, \beta_k$

7.7.1 Case 3 – Time data for every relevant failure for two items from different populations

Step 1: Calculate $\hat{\beta}_1$ for item 1 and $\hat{\beta}_2$ for item 2 from step 2 in 7.2.1.

Step 2: Calculate:

$$S_1 = \sum_{i=1}^{N_1-1} \ln \left[\frac{t_{N_1}}{t_{i1}} \right]$$

and

$$S_2 = \sum_{i=1}^{N_2-1} \ln \left[\frac{t_{N_2}}{t_{i2}} \right]$$

Step 3: Calculate:

$$F = \frac{S_1(N_2-1)}{S_2(N_1-1)}$$

Step 4: If

$$\frac{1}{F_{0,95}(2(N_2 - 1), 2(N_1 - 1))} < F < F_{0,95}(2(N_1 - 1), 2(N_2 - 1))$$

where the fractiles of the F distribution are given in Table 5, then the null hypothesis that the β values are the same cannot be rejected at the 10 % significance level. Otherwise, conclude that the shape parameters of the models fitted to the data for the two items are statistically different.

7.7.2 Case 3 – Time data for every relevant failure for three or more items from different populations

Step 1: Calculate $\hat{\beta}_j$ for item $j, j = 1, 2, \dots, k$ from step 2 in 7.2.1.

Step 2: Calculate:

$$S_j = \sum_{i=1}^{N_j-1} \ln \left[\frac{t_{N_j}}{t_{ij}} \right]$$

Step 3: Calculate:

$$N = \sum_{j=1}^k N_j \text{ where } k \text{ denotes the number of items of the same type}$$

Step 4: Calculate:

$$W = 1 + \frac{1}{3(k-1)} \left[\sum_{j=1}^k \frac{1}{2(N_j - 1)} - \frac{1}{2(N - k)} \right]$$

Step 5: Calculate:

$$Y = 2(N - k) \ln \left[\left(\frac{1}{N - k} \right) \sum_{j=1}^k S_j \right] - \sum_{j=1}^k 2(N_j - 1) \ln \left[\frac{S_j}{N_j - 1} \right]$$

Step 6: If

$$\frac{Y}{W} < \chi_{0,90}^2(k - 1)$$

where the fractiles of the χ^2 distribution are given in Table 2, then the null hypothesis that the β values are the same cannot be rejected at a 10 % significance level. Otherwise, conclude that the shape parameter of the models fitted to the different items is statistically different.

**Table 1 – Critical values for Cramer-von-Mises goodness-of-fit test
at 10 % level of significance**

M	Critical value of statistic $C_{0,90}^2(M)$
3	0,154
4	0,155
5	0,160
6	0,162
7	0,165
8	0,165
9	0,167
10	0,167
11	0,169
12	0,169
13	0,169
14	0,169
15	0,169
16	0,171
17	0,171
18	0,171
19	0,171
20	0,172
30	0,172
≥ 60	0,173
NOTE 1 For time terminated tests, $M = N$.	
NOTE 2 For failure terminated tests, $M = N-1$.	

Table 2 – Fractiles of the Chi-square distribution

Degrees of freedom ν	$\chi^2_{0,05}(\nu)$	$\chi^2_{0,90}(\nu)$	$\chi^2_{0,95}(\nu)$
2	0,10	4,61	5,99
4	0,71	7,78	9,49
6	1,64	10,65	12,59
8	2,73	13,36	15,51
10	3,94	15,98	18,31
12	5,23	18,55	21,03
14	6,57	21,06	23,69
16	7,96	23,54	26,30
18	9,39	25,99	28,87
20	10,85	28,41	31,41
22	12,34	30,81	33,92
24	13,85	33,20	36,42
26	15,38	35,56	38,89
28	16,92	37,92	41,34
30	18,49	40,26	43,77
32	20,09	42,57	46,17
34	21,70	44,88	48,57
36	23,30	47,19	50,96
38	24,91	49,50	53,36
40	26,51	51,81	55,76
42	28,16	54,08	58,11
50	34,76	63,17	67,51
52	36,45	65,42	69,82
60	43,19	74,40	79,08
62	44,90	76,63	81,37
70	51,74	85,53	90,53
72	53,47	87,74	92,80
80	60,39	96,58	101,88
82	62,14	98,78	104,13
90	69,13	107,57	113,15
92	70,89	109,76	115,39
100	77,93	118,50	124,34
102	79,70	120,68	126,57
110	86,79	129,38	135,48
112	88,57	131,56	137,70
120	95,71	140,23	146,57
122	97,49	142,40	148,78
200	168,28	226,02	233,99
z_p	-1,64	+1,28	+1,64

NOTE 1 Linear interpolation of intermediate values is sufficiently accurate.

NOTE 2 For higher values of ν use $\chi^2_p = \left[\left(z_p + \sqrt{2\nu - 1} \right)^2 \right] / 2$ where z_p is the corresponding fractile of the standard normal distribution.

Table 3 – Multipliers for two-sided 90 % confidence intervals for intensity function for time terminated data

<i>N</i>	<i>L</i>	<i>U</i>		<i>N</i>	<i>L</i>	<i>U</i>
3	0,175	6,490		21	0,570	1,738
4	0,234	4,460		22	0,578	1,714
5	0,281	3,613		23	0,586	1,692
6	0,320	3,136		24	0,593	1,672
7	0,353	2,826		25	0,600	1,653
8	0,381	2,608		26	0,606	1,635
9	0,406	2,444		27	0,612	1,619
10	0,428	2,317		28	0,618	1,604
11	0,447	2,214		29	0,623	1,590
12	0,464	2,130		30	0,629	1,576
13	0,480	2,060		35	0,652	1,520
14	0,494	1,999		40	0,672	1,477
15	0,508	1,947		45	0,689	1,443
16	0,521	1,902		50	0,703	1,414
17	0,531	1,861		60	0,726	1,369
18	0,543	1,825		70	0,745	1,336
19	0,552	1,793		80	0,759	1,311
20	0,561	1,765		100	0,783	1,273

NOTE 1 For $N > 100$

$$L \cong \frac{N-1}{N} \left(1 + 1,64 \sqrt{\frac{1}{2N}} \right)^{-2}$$

$$U \cong \frac{N-1}{N} \left(1 - 1,64 \sqrt{\frac{1}{2N}} \right)^{-2}$$

NOTE 2 Linear interpolation of intermediate values is sufficiently accurate.

Table 4 – Multipliers for two-sided 90 % confidence intervals for intensity function for failure terminated data

<i>N</i>	<i>L</i>	<i>U</i>		<i>N</i>	<i>L</i>	<i>U</i>
3	0,1712	4,746		21	0,6018	1,701
4	0,2587	3,825		22	0,6091	1,680
5	0,3174	3,254		23	0,6160	1,659
6	0,3614	2,892		24	0,6225	1,641
7	0,3962	2,644		25	0,6286	1,623
8	0,4251	2,463		26	0,6344	1,608
9	0,4495	2,324		27	0,6400	1,592
10	0,4706	2,216		28	0,6452	1,578
11	0,4891	2,127		29	0,6503	1,566
12	0,5055	2,053		30	0,6551	1,553
13	0,5203	1,991		35	0,6763	1,501
14	0,5337	1,937		40	0,6937	1,461
15	0,5459	1,891		45	0,7085	1,428
16	0,5571	1,876		50	0,7212	1,401
17	0,5674	1,814		60	0,7422	1,360
18	0,5769	1,781		70	0,7587	1,327
19	0,5857	1,752		80	0,7723	1,303
20	0,5940	1,726		100	0,7938	1,267
NOTE 1 For $N > 100$ $L \cong \frac{N-2}{N} \left(1 + 1,64 \sqrt{\frac{2}{N}} \right)^{-1}$ $U \cong \frac{N-2}{N} \left(1 - 1,64 \sqrt{\frac{2}{N}} \right)^{-1}$ NOTE 2 Linear interpolation of intermediate values is sufficiently accurate.						

Table 5 – 0,95 fractiles of the F distribution

$F_{0,95}(v_1, v_2)$	v_1										
v_2	2	4	6	8	10	20	30	40	60	120	∞
2	19,00	19,20	19,30	19,40	19,40	19,40	19,50	19,50	19,50	19,50	19,50
4	6,94	6,39	6,16	6,04	5,96	5,80	5,75	5,72	5,69	5,66	5,63
6	5,14	4,53	4,28	4,15	4,06	3,87	3,81	3,77	3,74	3,70	3,67
8	4,46	3,84	3,58	3,44	3,35	3,15	3,08	3,04	3,01	2,97	2,93
10	4,10	3,48	3,22	3,07	2,98	2,77	2,70	2,66	2,62	2,58	2,54
12	3,89	3,26	3,00	2,85	2,75	2,54	2,47	2,43	2,38	2,34	2,30
14	3,74	3,11	2,85	2,70	2,60	2,39	2,31	2,27	2,22	2,18	2,13
16	3,63	3,01	2,74	2,59	2,49	2,28	2,19	2,15	2,11	2,06	2,01
18	3,55	2,93	2,66	2,51	2,41	2,19	2,11	2,06	2,02	1,97	1,92
20	3,49	2,87	2,60	2,45	2,35	2,12	2,04	1,99	1,95	1,90	1,84
30	3,32	2,69	2,42	2,27	2,16	1,93	1,84	1,79	1,74	1,68	1,62
40	3,23	2,61	2,34	2,18	2,08	1,84	1,74	1,69	1,64	1,58	1,51
60	3,15	2,53	2,25	2,10	1,99	1,75	1,65	1,59	1,53	1,47	1,39
120	3,07	2,45	2,18	2,02	1,91	1,66	1,55	1,49	1,43	1,35	1,25
∞	3,00	2,37	2,10	1,94	1,83	1,57	1,46	1,39	1,32	1,22	1,00

NOTE Linear interpolation for intermediate values is sufficiently accurate.

Annex A (informative)

The power law model – Background information

The power law model is widely used to analyse the reliability of repairable items. It is particularly useful for those items classified as 'bad-as-old' when repair is minimal and so reliability of the item remains essentially unchanged after failure and repair. It is also appropriate for those items whose reliability is likely to improve. Indeed the power law model was first considered by L.H. Crow [2]¹ in 1974 to describe the power law growth pattern first reported by J.T. Duane in 1964 [5]. Methods for reliability growth analysis based on the power law model are given in IEC 61164 [6].

Crow [2] formulated the underlying probabilistic model for failures as a non-homogeneous Poisson process (NHPP), $\{N(t), t > 0\}$, with an expected value of:

$$E[N(t)] = \lambda t^\beta$$

and the failure intensity is given by

$$z(t) = \lambda \beta t^{\beta-1}$$

The NHPP model gives the Poisson probability that $N(t)$ will assume a particular value, that is:

$$\Pr[N(t) = n] = \frac{(\lambda t^\beta)^n e^{-\lambda t^\beta}}{n!} \quad \text{with } n = 0, 1, 2, \dots$$

Also, under this model

$$E[t_j^\beta] = j \quad \text{with } j = 1, 2, \dots$$

where t_j is the accumulated time to the j th failure. This gives the useful first-order approximation

$$E[t_j] = \left(\frac{j}{\lambda}\right)^{1/\beta} \quad \text{with } j = 1, 2, \dots$$

for the expected time to the j th failure.

When $\beta = 1$, then $z(t) = \lambda$ and the times between successive failures follow an exponential distribution with mean $1/\lambda$ (homogeneous Poisson process), indicating a constant failure intensity. The intensity function $z(t)$ is decreasing for $\beta < 1$ (reliability growth), and increasing for $\beta > 1$ (reliability deterioration).

¹ Figures in square brackets refer to the bibliography.

Annex B (informative)

Numerical examples

B.1 Background information

The following numerical examples show the use of the procedures discussed in Clause 7. Example 1 considers time data for every relevant failure for a single repairable item when observation is failure terminated. Example 2 considers time data for every relevant failure for multiple repairable items of the same kind when observation is time terminated. Example 3 considers time data for every relevant failure of two repairable items from different populations. Example 4 considers groups of relevant failures for a single repairable item. All examples illustrate the use of appropriate estimation methods. Goodness-of-fit tests are applied when appropriate. These examples may be used to validate computer programs designed to implement the methods given in Clause 7.

Note that all the calculations in the examples were carried out using a spreadsheet software package. Although the final figures are presented to two or three decimal places, the intermediate calculations were carried out in double precision. If intermediate calculations are computed with less precision, then the final figures might differ slightly from those presented due to rounding errors.

It should also be pointed out that all the confidence intervals presented are at a 90 % confidence level and similarly all the statistical tests are conducted at a 10 % significance level. These correspond to the values given in Tables 1 to 5. However, if the appropriate values are taken from corresponding tables reported elsewhere or are generated from software, then alternative values for the confidence and significance levels can be chosen according to users' requirements.

B.2 Example 1

The successive failure times (in hours) of a piece of software developed as part of a large system are given in Table B.1.

Table B.1 – All relevant failures and accumulated times for software system

0,2	4,2	4,5	5	5,4	6,1	7,9	14,8	19,2	48,6	85,8	108,9	127,2
129,8	150,1	159,7	227,4	244,7	262,7	315,3	329,6	404,3	486,2			
NOTE $t_N = 486,2\text{h}$ $N = 23$.												

Plot of accumulated failures against time

The concave down pattern in Figure B.1 indicates a decreasing failure intensity.

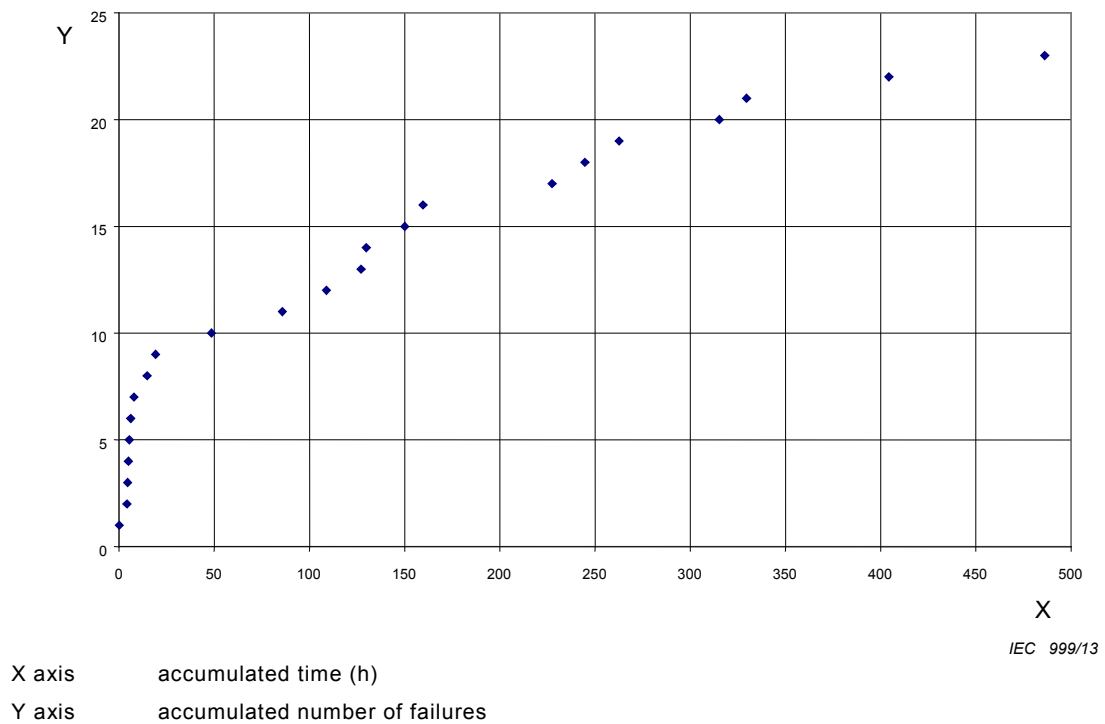


Figure B.1 – Accumulated number of failures against accumulated time for software system

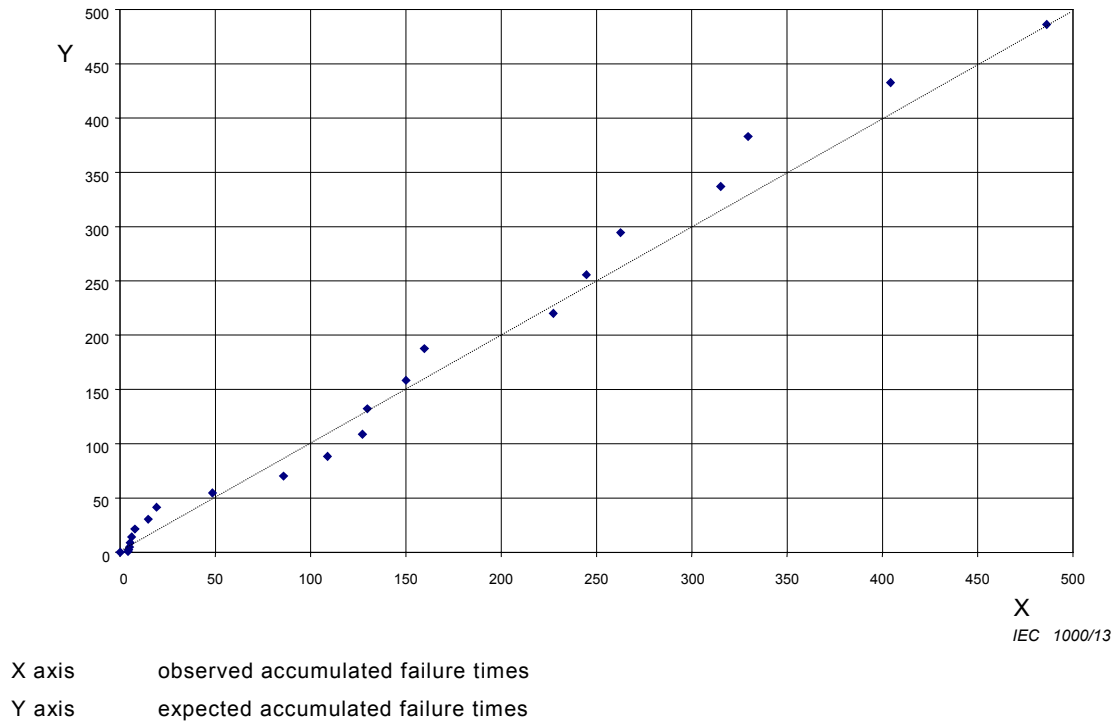


Figure B.2 – Expected against observed accumulated times to failure for software system

Parameter estimation

From 7.2.1, the estimated parameters of the power law model are as follows:

$$\hat{\lambda} = 2,17$$

$$\hat{\beta} = 0,38$$

Goodness-of-fit

From 7.3.1.2, the plot of expected against observed failure times in Figure B.2 displays a random scatter around the 45° line indicating a good fit of the power law model to the data. Table B.2 shows the workings for the expected and observed failure times plotted in Figure B.2 where the expected failure times are computed from Step 2 in 7.3.1.2.

From 7.3.1.1, $C^2 = 0,063$ with $M = 22$. At a 10 % significance level, the critical value from Table 1 is 0,172. Since $0,063 < 0,172$, it can be concluded that the hypothesis that the power law model is a good fit to the data cannot be rejected.

Table B.2 – Calculation of expected accumulated times to failure for Figure B.2

Failure	Observed failure time (h)	Expected failure time (h)
1	0,2	0,130
2	4,2	0,803
3	4,5	2,326
4	5,0	4,946
5	5,4	8,881
6	6,1	14,326
7	7,9	21,465
8	14,8	30,468
9	19,2	41,496
10	48,6	54,705
11	85,8	70,242
12	108,9	88,250
13	127,2	108,866
14	129,8	132,224
15	150,1	158,454
16	159,7	187,681
17	227,4	220,028
18	244,7	255,617
19	262,7	294,564
20	315,3	336,983
21	329,6	382,989
22	404,3	432,692
23	486,2	486,200

Confidence interval for β

From 7.4.1, a two-sided 90 % confidence interval for β is (0,27; 0,55). Since all values in this interval are less than one, it indicates a decreasing failure intensity.

Confidence interval for failure intensity

From 7.5.1, a two-sided 90 % confidence interval for the failure intensity at $t = 450$ h is (0,011; 0,031) failures/h.

Prediction interval for the time to future failures

From 7.6.1, a two-sided 90 % prediction interval for the failure time of the 24th failure is (488,93; 690,30) h. From 7.6.2, a two-sided 90 % prediction interval for the time to the 25th failure is (504,68; 845,30) h.

B.3 Example 2

Five copies of a system were put into operation at the same time under identical conditions. When a system failed it was repaired immediately and returned to operation. The repair time is insignificant compared with the time in operation. Each copy of the system was observed for 1 850 h of operation. The accumulated times to failure are given in Table B.3.

**Table B.3 – Accumulated times for all relevant failures
for five copies of a system (labelled A, B, C, D, E)**

A	B	C	D	E
96	552	1 056	1 560	
1 224	1 225			
1 392	1 570			

The data to be analysed consist of the superimposition of the failure times, which are presented in Table B.4, i.e. the accumulated times for all systems are combined into one data set and presented in their order of occurrence from smallest to largest.

**Table B.4 – Combined accumulated times
for multiple items of the same kind of a system**

Failure	Accumulated time (h)
1	96
2	552
3	1 056
4	1 224
5	1 225
6	1 392
7	1 560
8	1 570

Plot of accumulated failures against time

The concave up pattern in Figure B.3 indicates a possible increasing failure intensity.

Parameter estimation