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Analysis and Synthesis of Reset Control Systems

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Contents

I	Background	3
1	Introduction	4
1.1	Historical overview	4
1.2	The Clegg integrator circuit	7
1.3	Modeling issues with reset control	11
1.4	Using thin jump sets: Existence of solutions and robustness	13
1.5	An overview of recent reset systems results	16
2	A First Planar Closed Loop	20
2.1	Hybrid representation for solutions	20
2.2	A motivating example	23
2.3	An overview of this survey	27
II	Non-planar Reset Systems	30
3	Nonlinear Extensions of Clegg's Integrator	31
3.1	A generalization of the FORE reset rule	31
3.2	Hybrid stabilization with reset loops: Nonlinear case	34
3.3	A nonlinear control system stabilized by adding a hybrid loop	38

3.4	Hybrid stabilization with reset loops: Linear case	41
3.5	Examples of linear systems involving hybrid loops	47
3.6	Chapter conclusion	55
4	Dwell-Time Logic and Observer-Based Hybrid Loops	57
4.1	Overcoming two drawbacks of the hybrid loops of Chapter 3	57
4.2	Controller architecture and output feedback stabilization	62
4.3	Simulations	73
4.4	Fundamental properties of temporally regularized systems	80
4.5	Chapter conclusion	86
5	LMI-Based Stability and Performance Analysis	88
5.1	Introducing continuous-time $\mathcal{L}_2$ gains	88
5.2	Lyapunov-based t - $\mathcal{L}_2$ stability conditions	91
5.3	LMI-based t - $\mathcal{L}_2$ stability conditions	98
5.4	Application to SISO control loops with FORE	100
5.5	Numerical examples	112
5.6	Chapter conclusion	118
6	Towards Reset $\mathcal{H}_\infty$ Control Design	120
6.1	Overview	120
6.2	Full state availability	121
6.3	Partial state availability	128
6.4	Illustrative examples	129
6.5	Chapter conclusion	138
III	Planar Reset Systems	139
7	Planar SISO Systems with First Order Reset Elements	140
7.1	A modified model of FORE and its Lyapunov implications	140
7.2	Necessary and sufficient conditions for exponential stability	144
7.3	Extension to minimum phase relative degree one linear SISO plants	150
7.4	Set-point regulation	154
7.5	Reference tracking	163
7.6	Chapter conclusion	166

8	FORE Control of an EGR Valve	167
8.1	Overview	167
8.2	Model parameters identification	168
8.3	Proposed regulation scheme	171
8.4	Results with adaptive feedforward	172
8.5	Results with parameterized feedforward	175
8.6	Tracking a time-varying position reference	178
8.7	Comparison to time-optimal and PI solutions	179
8.8	Implementation on the Diesel engine testbench	182
8.9	Experimental results	185
8.10	Chapter conclusion	189
9	FORE Control of an Electromechanical Valve	191
9.1	Overview	191
9.2	Application to the valve model and simulation results	194
9.3	Experimental tests	199
9.4	Chapter conclusion	203
10	Conclusions and Perspectives	204
10.1	Conclusions	204
10.2	Perspectives	205
	Acknowledgements	208
	References	210

Analysis and Synthesis of Reset Control Systems

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ABSTRACT

This survey monograph overviews a large core of research results produced by the authors in the past decade about reset controllers for linear and nonlinear plants. The corresponding feedback laws generalize classical dynamic controllers because of the interplay of mixed continuous/discrete dynamics. The obtained closed-loop system falls then within the category of hybrid dynamical systems, with the specific feature that the hybrid nature arises from the nature of the controller, rather than the nature of the plant, which is purely continuous-time. Due to this fact, the presented results focus on performance and stability notions that prioritize continuous-time evolution as compared to the discrete-time one. Dwell-time logics (namely, conditions preventing consecutive jumps that are too close to each other) are indeed enforced on solutions, to ensure that the continuous evolution of solutions is complete (no Zeno solutions occur).

After presenting a historical motivation and an overview of the results on this topic in Part I, several results on stability

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and performance analysis and on control design for general linear continuous-time plants are developed in Part II. These results are developed by exploiting the well-established formalism for nonlinear hybrid dynamical systems introduced by Andy Teel and co-authors around 2004. With this formalism, by ensuring sufficient regularity of the reset controller dynamics, we ensure robustness of stability with respect to small disturbances and uncertainties together with suitable continuity of solutions, generally regarded as well-posedness of the hybrid closed loop. Throughout Part II, we provide several simulation studies showing that reset control strategies may allow to attain better performance with respect to the optimal ones obtained by classical continuous-time controllers.

Finally, in Part III we focus on planar systems, that is reset closed loops involving a one-dimensional linear plant and a one-dimensional reset controller. For this simple interconnection interesting stability conditions can be drawn and relevant extensions addressing the reference tracking problem are introduced, illustrating them on a few relevant case studies emerging in the automotive field.

Part I

Background

1

Introduction

1.1 Historical overview

Reset controllers were proposed for the first time more than 50 years ago by Clegg (1958), with the aim at providing more flexibility in linear controller designs and at potentially removing fundamental performance limitations of linear controllers (see, e.g., the motivating example in Section 2.2). The first systematic designs for reset controllers were reported in the 1970s by Krishnan and Horowitz (1974) and Horowitz and Rosenbaum (1975) and there has been a renewed interest in this class of systems in the late 1990s with Beker *et al.* (1999b), Beker *et al.* (1999a), Beker *et al.* (2001b), Beker *et al.* (2004), Chait and Hollot (2002), Chen *et al.* (2000a), Chen *et al.* (2000b), Chen *et al.* (2001), Haddad *et al.* (2000), Hollot *et al.* (1997), Hollot *et al.* (2001), Hu *et al.* (1997), and Zheng *et al.* (2000).

More specifically, a reset controller, according to its historical definition, is a linear controller whose output is reset to zero whenever its input and output satisfy an appropriate algebraic relationship (even though generalizations not necessarily resetting to zero will be also considered in this survey). For instance, in Beker *et al.* (2004) and the references cited therein, a class of reset controllers was considered where

the output of the controller is reset to zero whenever its input is equal to zero. The Clegg integrator introduced in Clegg (1958) acts like a linear integrator whenever its input and output have the same sign and it resets its output to zero otherwise (see Section 1.2). Consequently, its describing function has the same magnitude plot as the linear integrator but it has 51.9° less phase lag. This feature of the Clegg integrator was used for the first time in Krishnan and Horowitz (1974) to provide a systematic procedure for controller design exploiting this device. Subsequently, a new reset device called the *First Order Reset Element (FORE)* was introduced in Horowitz and Rosenbaum (1975), essentially generalizing the Clegg integrator's base linear dynamics by also allowing for a nonzero real pole. Horowitz and Rosenbaum (1975) also proposed a design procedure consisting of two steps. First, a non-reset part of the controller was designed to achieve all design specifications except for the overshoot. Then, in the second step, the pole of the FORE was selected to reduce the overshoot. It was illustrated through examples and simulations that the controller in the first step of the procedure could indeed be designed with lower phase margin, which provided more design flexibility. A nice account of these results and their relation to more recent developments in reset control are given in Chait and Holot (2002).

The first example that clearly illustrated the advantages of reset over linear controllers was presented in Beker *et al.* (2001a) where a reset controller was designed to achieve design specifications that are impossible to achieve by any linear controller (see also Feuer *et al.*, 1997). Indeed, for linear plants including an integral action, if the desired rise time is sufficiently small, then the output must overshoot with any linear controller. However, a reset controller is designed in Beker *et al.* (2001a) that overcomes this fundamental performance limitation of linear controllers. To date, this appears to be the only real situation where reset designs have been shown to outperform the best possible classical design, nevertheless, practical experience reveals that desirable closed-loop behavior is obtained when suitably embedding resets in otherwise continuous-time control devices. Examples of such experiences can be found in the experimental applications reported in Zheng *et al.* (2007), Fernandez *et al.* (2008), Wu *et al.* (2007), Li *et al.*

(2009), Bakkeheim *et al.* (2008), Guo *et al.* (2009), Panni *et al.* (2014), Cordioli *et al.* (2015), Zheng *et al.* (2000), Carrasco and Baños (2012), Li *et al.* (2011), Vidal and Baños (2010), and Villaverde *et al.* (2011). The difficulty in proving rigorous statements with reset systems was due to the lack of suitable stability and performance analysis tools for systems whose solutions may experience instantaneous jumps (this is the case for an integrator reset to zero).

From a theoretical viewpoint, first attempts to rigorously analyze stability of reset systems with Clegg integrators can be found in Hu *et al.* (1997) and Hollot *et al.* (1997). In particular, an integral quadratic constraint was proposed in Hollot *et al.* (1997) to analyze stability of a class of reset systems. However, the proposed condition was conservative as it was independent of reset times. BIBO stability analysis of reset systems consisting of a second-order plant and a FORE was conducted in Chen *et al.* (2001) (see also Chen *et al.*, 2000b). The proofs are based on an explicit characterization of reset times which are proved to be equidistant under mild conditions. Using this fact, the authors prove asymptotic and BIBO stability of the reset system via the discrete-time model of the system that describes the system at reset times only. However, the same approach could not be used to analyze higher-order reset systems. Stability analysis of general reset systems can be found in Beker *et al.* (2004) (see also Hollot *et al.*, 2001; Chen *et al.*, 2000a)) where Lyapunov-based conditions for asymptotic stability were presented and computable conditions for quadratic stability based on linear matrix inequalities (LMIs) were given. Moreover, in Beker *et al.* (2004), BIBO stability of general reset systems was obtained as a consequence of quadratic stability and an internal model principle was proved for reference tracking and disturbance rejection.

In the last decade, perhaps triggered by the inspiring work of Beker *et al.* (2004), a significant renewed interest in Lyapunov-based analysis and synthesis for reset systems has been witnessed by the scientific community. This survey monograph reports on a research strand that started around 2005, motivated by the results in Beker *et al.* (2004), wherein some recent stability and performance analysis tools for hybrid dynamical systems have been brought to bear into the framework of reset control systems. While many alternative and relevant approaches

have been developed during the last decade (a selection of them is briefly overviewed in the following Section 1.5), we specifically concentrate here on a research strand that emerged from the hybrid Lyapunov theory proposed in Goebel *et al.* (2012). Some important differences between what is reported here and alternative approaches are discussed in the next sections, which explain the spirit of our approach to reset control. To this end, we need to somewhat come back to the very origins of reset control and take a close look at the analog circuit proposed by J.C. Clegg in 1958, which is discussed in the next section.

1.2 The Clegg integrator circuit

In 1958, J.C. Clegg published a paper (Clegg, 1958) where he proposed a modification to the existing analog control schemes to reduce the phase lag induced by a linear integrator. The relevance of Clegg's work was mostly targeted to analog control, because digital control systems were still non-existent in the late 1950s, nevertheless, follow-up works considered digital versions of the scheme proposed by Clegg. Let us here consider the analog device proposed by Clegg and derive the corresponding equations.

In the ideal case of using infinite gain operational amplifiers, it is well known that a linear integrator can be implemented using a resistor on the input path and a capacitor on the feedback path of the circuit, as represented in Figure 1.1. The corresponding input/output relation of the linear integrator can be written in the time domain as $\dot{x}_c = -\frac{1}{RC}e$, where we use x_c for the integrator output, to resemble the fact that the integrator state is the state of a feedback controller from the tracking error e . In this figure, R is the resistance on the input branch of the

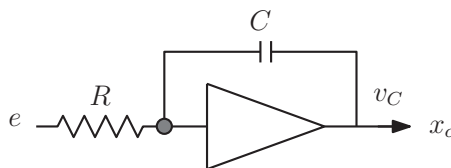


Figure 1.1: A linear analog integrator.

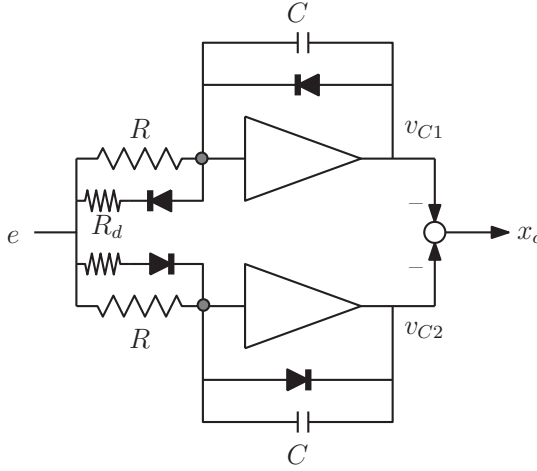


Figure 1.2: The “Clegg integrator”.

amplifier, C is the capacity in the feedback branch, and v_C is the voltage across the capacitor.

The modification proposed by Clegg corresponds to the scheme of Figure 1.2 (which is reported here from Clegg (1958) with a sign inversion at the output, for convenience of exposition), where we use the same notation as in Figure 1.1, with possibly different voltages v_{C1} and v_{C2} and extra diodes and resistors. We describe the Clegg integrator dynamics assuming that $R_d \ll R$ and disregarding the forward-bias voltage drop across the diodes, leading to the following ideal relationships between voltage and current across capacitors, resistors, and diodes:

$$\begin{aligned} i_C(t) &= C \frac{dv_C(t)}{dt} \\ i_R(t) &= v_R(t)/R \\ i_D(t) &= \begin{cases} +\infty, & \text{if } v_D(t) > 0, \\ 0 & \text{if } v_D(t) \leq 0. \end{cases} \end{aligned}$$

First note that by the infinite gain assumption of the operational amplifier, the input voltages (marked by gray dots on the figure) are always zero. Then, the two capacitors' voltages satisfy $v_{C1}(t) \leq 0$ and $v_{C2}(t) \geq 0$ for all times (otherwise the infinite current flowing in

the diodes would instantaneously discharge the capacitor). Moreover, when $e(t) < 0$, regardless of the preceding voltage stored in the upper capacitor, the current flowing in the two diodes and through the upper R_d will (almost) instantaneously impose $v_{C1}(t) = 0$. However, when $e(t) \geq 0$, the upper circuit will correspond to the linear integrator because the diodes will both be open (being subjected to a non-positive voltage). Similarly for the lower circuit, if $e(t) > 0$, we will have $v_{C2}(t) = 0$ and if $u(t) \leq 0$ the circuit will integrate. Since, as commented above, $v_{C1}(t) \leq 0$ and $v_{C2}(t) \geq 0$ for all times, given $x_c(t) := -v_{C1}(t) - v_{C2}(t)$, the integrating and reset conditions for both circuits can be written as the following hybrid dynamics

$$\begin{cases} \dot{x}_c = \frac{1}{RC}e, & \text{is allowed when } x_c e \geq 0, \\ x_c^+ = 0, & \text{is allowed when } x_c e \leq 0, \end{cases} \quad (1.1)$$

where we insist on the fact that the situation $e(t) = 0$ leads to an undetermined behavior of the circuit mostly dependent on the effect of unmodeled noise and uncertainties. In this model, we use the shorthand notation $\dot{x}$ for $\frac{d}{dt}x(t, j)$ and x^+ for $x(t, j + 1)$ which will be formally defined later on in the survey. The strategy that we prefer to adopt for handling such undetermined cases is to allow for multiple solutions to the dynamics (1.1), thereby considering in our model all the possible scenarios. See the discussion given later in Remark 1.1.

One way to understand the hybrid model (1.1) for the Clegg integrator is to call its first equation the “flow” equation and its second equation the “jump” equation. The two conditions at the right-hand side become then the “flow” condition and the “jump” condition. At any time, a solution to the hybrid system (1.1) may then flow or jump depending on whether its value at that time belongs to the so-called “jump set” (namely, the set of states for which the jump condition is true) or it belongs to the “flow set”. In case both conditions are true, then the solution will be free to choose whether flowing or jumping, thereby establishing a peculiar non-uniqueness feature.

Further insight into Equation (1.1) can be gained by observing that e and x_c can never have opposite signs. Indeed, if $e \geq 0$, then $v_{C2}^+ = 0$, and since $v_{C1} \leq 0$ for all times, $ex_c \geq 0$. Similarly for the case $e \leq 0$.

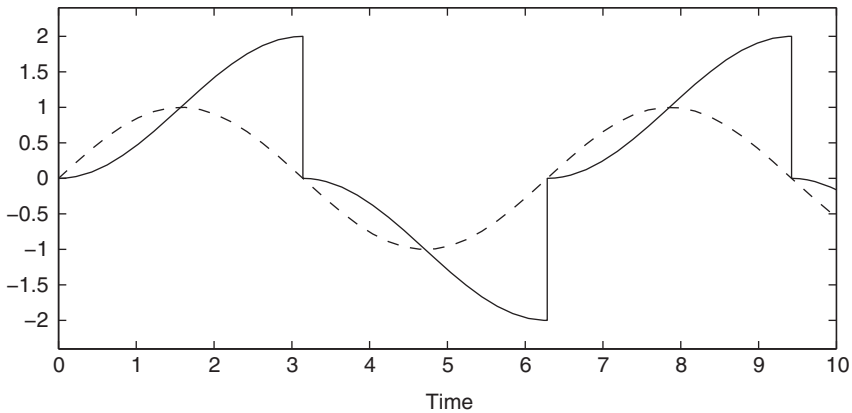


Figure 1.3: Response of the Clegg integrator (solid) to a sine input (dashed).

On the other hand, whenever $e \neq 0$, there will always be one circuit integrating (the upper one if $e > 0$ and the lower one if $e < 0$) and the other circuit will be forced to be at zero. For illustrative purposes, Figure 1.3 represents the Clegg integrator state (solid curve) when the input e is selected as a sine wave with unit frequency (dashed curve). Note that x_c and e always have the same sign, which is ensured by the fact that around times $t_1 = 3.2$ and $t_2 = 6.2$ the solution jumps. In particular, the upper condition of Equation (1.1) would not be satisfied if the solution did not jump, indeed violation of this condition forbids flowing.

One way to interpret the dynamics (1.1) is to regard it as a linear filter with a pole at the origin embedded with a special resetting rule dependent on the value of the input and output of the filter at each time. This interpretation is the starting point for the FORE generalization discussed in the next section. The interest of Clegg integrator will become clear in Section 2.2 when using Clegg integrator for a linear plant, and in Chapter 3 when designing hybrid loops for nonlinear control systems. As it will be explained in these parts (and in many other parts of this survey), the performance of Clegg integrators, and more generally of reset control is better to what could be done with classical linear or nonlinear controllers. The extra flexibility given by

the jump rule is exploited to improve the performance of the closed-loop system in the presence of reset controllers.

1.3 Modeling issues with reset control

The model (1.1) derived in the previous section for the hybrid dynamics can be easily generalized to the following dynamics, where the eigenvalue associated with the continuous dynamics is not necessarily at zero:

$$\begin{cases} \dot{x}_c = a_c x_c + b_c e, & \text{is allowed when } x_c e \geq 0, \\ x_c^+ = 0, & \text{is allowed when } x_c e \leq 0, \end{cases} \quad (1.2)$$

the model of the Clegg integrator corresponding to $a_c = 0$ and $b_c = \frac{1}{RC}$. The generalized dynamical system (1.2) was introduced in Horowitz and Rosenbaum (1975) and therein called First Order Reset Element (FORE). In Horowitz and Rosenbaum (1975) and follow-up works, this generalization was meant for stable filters (namely, $a_c \leq 0$), but it will be emphasized in this survey that this is not a necessary assumption and indeed unstable selections ($a_c > 0$) lead sometimes to desirable aggressive control actions.

It should be acknowledged that models (1.1) and (1.2) do not correspond to the models originally developed from the 1960s. Indeed, while Clegg's discussion in Clegg (1958) well referred to the dynamical behavior of his analog circuit as a circuit resetting to zero whenever input and output had opposite signs, his qualitative description incorporated the following observation:

“Whenever the input voltage e passes through zero from either direction, the output voltage is quickly dropped to zero.”

This sentence propagated into the follow-up work of Horowitz and co-authors, who never really wrote down equations but only described in words this behavior specifying resetting the controller state to zero at zero-crossings of the input. Much later in the 1990s, Hollot and co-authors (see, e.g., Chait and Hollot, 2002; Beker *et al.*, 2004, and references therein), and then also Baños and co-authors (see, e.g.,

Baños and Barreiro, 2011, and references therein) reported the following dynamical description of the Clegg mechanism:

$$\begin{cases} \dot{x}_c = a_c x_c + b_c e, & \text{if } e \neq 0, \\ x_c^+ = 0, & \text{if } e = 0. \end{cases} \quad (1.3)$$

The modified hybrid dynamics (1.3) was then used as the baseline hybrid reset control mechanism implemented in the control logic of modern digital control systems. Somehow the nice and intrinsic robustness properties of the analog circuit proposed by Clegg got lost along the route towards digitalization of modern feedback control. In particular, dynamics (1.3) no longer describes the behavior of the circuit in Figure 1.2 for $a_c = 0$ because solutions to Equation (1.3) starting from $x_c \neq 0$ and $e x_c < 0$ do not lead to an instantaneous reset to zero of the controller state x_c . In particular, dynamics (1.3) is associated with resetting in a so-called “thin set”, as opposed to dynamics (1.2) (and the behavior of Clegg’s circuit) where resets are enforced in half of the input–output space of the controller. This difference is well highlighted in Figure 1.4 where the e axis has been reversed in anticipation for negative error feedback interconnection of the FORE with a linear plant. In Figure 1.4 the sets enabling continuous flow of solutions are denoted by $\mathcal{F}$ (the “flow set”) and the sets enabling discrete jump of solutions are denoted by $\mathcal{J}$ (the “jump set”). Figure 1.4 also shows the possible different evolutions of

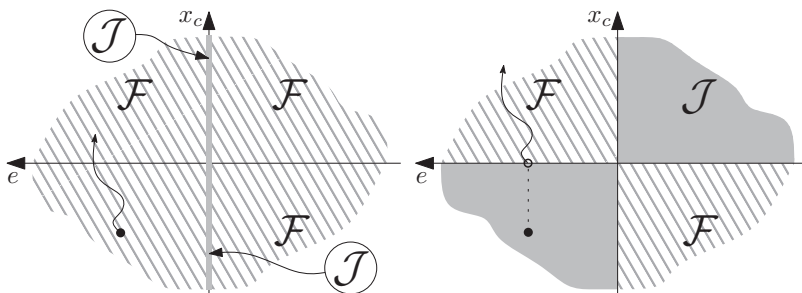


Figure 1.4: The jump (gray) and flow (striped) sets for the model (1.3) (left), and original model (1.2) (right) proposed by Clegg.

two solutions starting from the same initial conditions, for the two dynamics.

1.4 Using thin jump sets: Existence of solutions and robustness

The model (1.2) for the First Order Reset Element (FORE) was first introduced in Nešić *et al.* (2005) and Zaccarian *et al.* (2005), where the hybrid dynamical systems formalism of Goebel *et al.* (2009) and Goebel *et al.* (2012) has been employed for the first time for representing the peculiar evolutions of reset control systems whose state (notably, the controller state) may be integrated following a differential equation during the continuous-time flow phase, or may be reset to zero following a discrete update law at the jump times.

The useful features of model (1.2) have been first characterized in Nešić *et al.* (2005) and Zaccarian *et al.* (2005), and are worth summarizing here, with specific reference to the typical scenario of a Clegg integrator (or a more general FORE) interconnected in error feedback with a linear continuous-time plant $\mathcal{P}$, as represented in Figure 1.5. When focusing on stabilization only (that is, $r = 0$), the general dynamics arising from using this model may be well represented by using the notation in Goebel *et al.* (2009) and Goebel *et al.* (2012), and corresponds to the following closed loop involving the overall state $x := [x_p \ x_c]$, with x_p being the state of the plant $\mathcal{P}$:

$$\begin{cases} \dot{x} = Ax + Bd & \text{is allowed when } x \in \mathcal{F} \\ x^+ = Gx & \text{is allowed when } x \in \mathcal{J}, \end{cases} \quad (1.4)$$

where A , G and B are suitable constant matrices and the jump and flow sets correspond to the following symmetric cones, defined on the

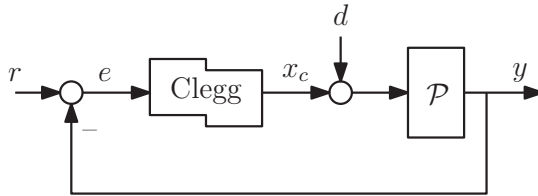


Figure 1.5: A Clegg integrator in error feedback interconnection with a linear plant.

basis of an output equation $y = C_y x$, and already shown at the right of Figure 1.4 (remember that we are looking at a negative feedback interconnection that motivates reversing the horizontal axis)

$$\begin{aligned}\mathcal{F} &:= \left\{ (x_p, x_c) : \begin{bmatrix} x_c \\ y \end{bmatrix}^\top \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_c \\ y \end{bmatrix} \leq 0 \right\}, \\ \mathcal{J} &:= \left\{ (x_p, x_c) : \begin{bmatrix} x_c \\ y \end{bmatrix}^\top \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_c \\ y \end{bmatrix} \geq 0 \right\}.\end{aligned}\tag{1.5}$$

The overall dynamics (1.4), (1.5) falls into the larger class of homogeneous hybrid systems (see, e.g., Goebel and Teel, 2010; Tuna and Teel, 2006), and will be the modeling framework adopted in this survey. We emphasize that the jump and flow sets defined in Equation (1.5) are closed. This condition is necessary for the theoretical developments in Goebel *et al.* (2012, Ch. 7) to apply. Those results (which have also been used in Nešić *et al.* (2005) and Zaccarian *et al.* (2005) and later works) allow us to establish existence of solutions from any initial conditions (therefore, some type of well posedness of the hybrid dynamics) in addition to suitable robustness properties of asymptotic stability of the origin for the error dynamics of the closed loop represented in Figure 1.5.

Remark 1.1. (Non-uniqueness of solutions) Note that asking that the sets $\mathcal{F}$ and $\mathcal{J}$ be closed implies that there are some regions of the state space belonging to both sets. Therefore, solutions may jump or flow in these regions, so that the solutions to the arising reset linear systems may be nonunique. Nonuniqueness becomes a necessary notion when wanting to establish robust results for the reset system, as a matter of fact, when the feedback system is affected by (arbitrarily small) noise, the state could be pushed in several different directions and different solutions may correspond to different noise selections. These and other robustness issues are addressed and solved in the hybrid framework that we adopt here and in the stability results that we will rely on in this survey.

We emphasize now that the desirable existence and robustness properties highlighted above for the adopted modeling framework are

not guaranteed, in general, whenever relying on the alternative model (1.3). When using that model, and using the notation of some recent papers, the closed loop in Figure 1.5 may be represented by the equations

$$\begin{cases} \dot{x} = Ax + Bd, & \text{if } x \notin \mathcal{M} \\ x^+ = Gx, & \text{if } x \in \mathcal{M}, \end{cases} \quad (1.6)$$

where $\mathcal{M} := \{x : C_y x = 0, \text{ and } (I - G)x \neq 0\}$ (recall that we defined $y = C_y x$).

Model (1.6), which is based on Equation (1.3), is actually used in a large number of results that can be found in the literature (see Bupp *et al.* (2000), Chait and Hollot (2002), Beker *et al.* (2004), Baños and Barreiro (2011), Barreiro *et al.* (2014), and Ghaffari *et al.* (2014) just to cite a few) but is associated to some subtle issues related to existence of solutions. In particular, the following observation was already made in Nešić *et al.* (2005) regarding (Beker *et al.*, 2004, Theorem 1), which establishes asymptotic stability of the origin under suitable Lyapunov conditions. Consider however the reset system (1.6) with $d = 0$ and $A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & 1 & -1 \end{bmatrix}$, $G = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $C_y = [1 \ 0 \ 0]$. Then it is not clear how to define solutions for an initial condition satisfying $C_y x_0 = 0$, $(I - G)x_0 = 0$. Indeed, in that case $x_0 \notin \mathcal{M}$ and the reset is not possible at the initial time, which means that the dynamics can only be governed by the flow equation (1.6) for small $t \geq 0$. Moreover, integrating the differential equation (1.6) from the same initial condition yields $C_y x(t) = 0$ for all t and $(I - G)x(t) = [0 \ 0 \ x_3(t)]^\top$, which is initially zero but is nonzero for all small t (thus $x(t) \in \mathcal{M}$ for $t > 0$ and thus flowing from the initial condition is not possible). Note that the conditions of Beker *et al.* (2004, Theorem 1) hold for this example by simply selecting $V(x) = |x|^2$, which yields $\dot{V} = -2V$ and $\Delta V \leq 0$. However, the established stability conditions hold for a system that does not guarantee existence of solutions from some initial conditions.

Due to the reasons above, and due to the lack of guarantee of robustness of asymptotic stability, we will restrict our attention to the formalism in dynamics (1.4), (1.5) and will establish robust properties via the Lyapunov tools of Goebel *et al.* (2012).

Remark 1.2. It should be further emphasized that in the model (1.6) resets are only possible on the hyperplane $C_y x = 0$ (as long as some flow has occurred since the last reset), whereas in our model (1.4), (1.5) resets are enforced on a sector $\mathcal{J}$. As a consequence, solutions to model (1.6) flow also in regions of the state space where our model does not allow flowing solutions. The consequence of this fact is that when wanting to use Lyapunov tools to prove asymptotic or exponential stability of the origin, using Equation (1.6) it is necessary to impose a “decrease along flows” condition in almost all the state space, which then implies, by continuity, that this condition holds everywhere. Instead, with our model, and using the Lyapunov tools of Goebel *et al.* (2012), we only need to impose the “decrease along flows” condition in half of the state space (well understood from Figure 1.4) and this leads to less conservative conditions. In particular, even for just a Clegg integrator connected to an integrator plant $\mathcal{P} = \frac{1}{s}$ (this is the system considered in Beker *et al.* (2001a), corresponding to $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$) the model (1.6) cannot lead to a strict Lyapunov function proving asymptotic stability of the origin. We will show in the next chapters that desirable stability properties can be shown with model (1.4), (1.5) even in cases when the continuous-time linear dynamics associated with matrix A is exponentially unstable, because the stabilization is obtained by way of the resetting mechanism. This peculiar feature of stabilizing a plant by way of exponentially diverging inputs, that are eventually reset to some value leads to desirable and aggressive control actions, is well illustrated by the simulations and experimental studies reported in the last part of this survey.

1.5 An overview of recent reset systems results

In the previous section we clarified that this survey monograph is focused on robust reset systems, arising from the use of model (1.1) and the arising closed-loop representation (1.4), (1.5).

This modeling framework has been used in a range of recent papers to address various analysis and design questions for reset systems. Those papers constitute the basis for the results in this survey and are shortly described below. The first paper where the modeling framework has

been used is Zaccarian *et al.* (2005), where the main observations of this chapter were first pointed out, together with Nešić *et al.* (2005), where Lyapunov like conditions for $\mathcal{L}_2$ stability and exponential stability of reset systems were proposed (later improved and revised in Nešić *et al.* (2008b)). Among other things, the conditions proposed in these papers involved locally Lipschitz Lyapunov functions as opposed to the continuously differentiable ones considered in Beker *et al.* (2004). This allowed us to consider piecewise quadratic Lyapunov functions in verifying exponential or $\mathcal{L}_2$ stability of reset systems (see Zaccarian *et al.* (2011), which is a revised and improved version of Zaccarian *et al.* (2005)). In later years, further developments of this field provided some explicit Lyapunov constructions for a class of planar reset systems, reported in Zaccarian *et al.* (2006), while the properties of reset set-point stabilizers and necessary and sufficient conditions for exponential and $\mathcal{L}_2$ stability have been reported first in Zaccarian *et al.* (2007) and Nešić *et al.* (2008a), and then revised and better illustrated, together with several properties of homogeneous hybrid systems, in Nešić *et al.* (2011). This modeling framework has also been used to provide LMI-based approaches for the $\mathcal{H}_2$ performance analysis and $\mathcal{L}_2$ performance analysis of reset control systems, respectively, in Witvoet *et al.* (2007) and Aangenent *et al.* (2008) (see also Aangenent *et al.*, 2010). More recently, higher-dimensional generalizations of these reset controllers were initially investigated in Prieur *et al.* (2010), later improved in Prieur *et al.* (2013). That specific generalization was actually focusing on a full state feedback architecture and was therefore generalized, in the context of linear plants, to the case of output feedback and Luenberger observers in Fichera *et al.* (2013b), where suitable dwell-time logics were introduced to avoid Zeno phenomena. The arising LMI-based conditions, finally led to a solution of the $\mathcal{H}_\infty$ design problem in Fichera *et al.* (2016) (which is a revised version of the preliminary work in Fichera *et al.*, 2012a). Parallel to these works, Loquen *et al.* (2007) addressed the presence of input saturation in reset systems while Tarbouriech *et al.* (2011) and Fichera *et al.* (2013a) suggested anti-windup actions to manage input saturation effects. Finally, Loquen *et al.* (2008) studied stability of reset systems in the presence of nonzero reference signals.

Even though this survey monograph concentrates on the above-mentioned bulk of literature, there are several recent additional results in the literature, following somewhat different routes, but indicating that reset control is yet an active research field attracting much scientific interest. One route is devoted to strategies where the reset actions are triggered at fixed time instants, often periodic. Stability and $\mathcal{L}_2$ gain properties have been addressed in Heemels *et al.* (2016) for a class of hybrid systems that exhibit linear flow dynamics, periodic time-triggered jumps and arbitrary nonlinear (possibly discontinuous) jump maps, making a link with the lifted system approach from sampled-data control theory. Discrete-time triggering conditions have been provided in Guo *et al.* (2012) in view of a computer-based implementation, for which the triggering condition is replaced by a discrete-time counterpart using a sampled triggering signal. In that paper, the reset controller is designed in three steps involving the baseline controller design, the reset matrix design and the triggering condition tuning. Nearly-periodic situations have been addressed in Hetel *et al.* (2013), considering uncertain intervals between to reset instants, and manipulating the time condition rather than the state condition for the resetting rule.

Recently, van Loon *et al.* (2017) proposed frequency-domain tools for stability analysis of reset control system, with the objective to attract interest of industrials in reset control strategies. Actually, little has been done until now in the direction of frequency-domain tools allowing to derive graphically verifiable stability conditions based on measured frequency response data, with no need to manipulate a parametric model. First attempts in that direction are contained in Holot *et al.* (2001), in which results regarding BIBO stability were provided for a second-order linear continuous-time closed-loop dynamics. In Zhao and Hua (2017), a Generalized First Order Reset Element (GFORE) has been introduced to better fit for implementation on a physical analog device. The main idea was to establish reset conditions depending on input and output data, rather than less accessible information about the plant states.

Reset control systems have also been extensively studied in the presence of time delays. Stability analysis independent of the delay was proposed in Baños and Barreiro (2009) using Lyapunov–Krasovskii

functionals. On the other hand, stability analysis dependent of the delay was addressed in Barreiro and Baños (2010) and Davó *et al.* (2017), using an impulsive delay dynamical systems framework to manage resets as impulsive events. Moreover, using a delay-dependent approach Zhao and Wang (2014) considered piecewise Lyapunov functions to prove stability of the reset observer. Stability analysis of time-delay reset systems is also addressed in an application to networked control system involving time-varying network-induced delays in Baños *et al.* (2014a), but considering discrete-time reset system descriptions. Finally, reset logics for improving the performance of high-gain observers have been studied in Andrieu *et al.* (2016), the idea being to use reset to project trajectories approaching a region of the state-space where high-gain peaking occurs to another region free of peaking phenomena.

A different research route corresponds to the study of reset observers (not addressed in this survey due to their different nature), as reported in Paesa *et al.* (2011) and Paesa *et al.* (2012) and Andrieu *et al.* (2016) with the idea to exploit resetting of some observer states to improve the observer settling time and overshoot/peaking performance.

The main stream of studies relative to reset control systems has been considering interconnection of reset compensators with linear plants. There exist however a few works that explored the interconnection of reset control systems with nonlinear plants. A passivity-based approach has been proposed in Carrasco *et al.* (2010), considering a linear compensator with reset action interconnected with a nonlinear plant. A key advantage of the passivity-based approach is that it allows to derive stability conditions to be checked on the linear compensator without considering the reset action. Passivation-based arguments were also used in Forni *et al.* (2011) using a passivity property of the continuous-time part of the reset controller associated with a suitable non-increase condition for the storage function at jumps to prove stability. Stability analysis for a class of Lipschitz nonlinear systems involving some resetting action at fixed time instants was also performed in Rios *et al.* (2017). LMI-based stability conditions were therein proposed thanks to the use of a 2D vector Lyapunov function issued from the impulsive system representation.

2

A First Planar Closed Loop

2.1 Hybrid representation for solutions

Before even addressing the problem of characterizing useful properties of the solutions to reset linear systems (such as the Clegg integrator (1.1)), it is mandatory to adopt a mathematical representation for such solutions, which is capable of guaranteeing fundamental properties, such as existence of solutions for all times and perhaps robustness of the stability results.

A first step in this direction corresponds to suitably defining the domain of solutions for the reset system. In particular, the domain of solutions x_c to the Clegg integrator of Figure 1.2 will be taken to be the Cartesian product of two domains: one, denoted by t , keeping track of the elapsed ordinary continuous time and the other one, denoted by j , related to the number of jumps that occurred since the initial hybrid time $(t_0, j_0) = (0, 0)$. This type of mathematical framework for hybrid systems was introduced in Goebel *et al.* (2004) and has been first used in Nešić *et al.* (2005) and Zaccarian *et al.* (2005) in the context of reset control systems.

According to the notion of solutions commented above, the Clegg integrator dynamics (1.1) can be written as

$$\begin{cases} \dot{x}_c(t, j) = \frac{1}{RC}e(t), & \text{is allowed when } x_c(t, j)e(t) \geq 0, \\ x_c(t, j+1) = 0, & \text{is allowed when } x_c(t, j)e(t) \leq 0, \end{cases} \quad (2.1)$$

where $x_c(t, j)$ represents the state of the Clegg integrator at time t , also considering that j resets have occurred before time t . Note that with this notation, if t_j represents the time at which the j -th reset occurred, then $x_c(t_j, j-1)$ represents the integrator value before the jump and $x_c(t_j, j)$ represents the integrator value after the jump. With this idea in mind, one realizes that the domain of a solution is a suitable subset of $\mathbb{R}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ wherein continuous and discrete-time directions (t, j) never evolve backwards. Upon deciding whether jumping or flowing, each solution will then select its suitable domain.

In particular, according to the hybrid systems framework of Goebel *et al.* (2009) and Goebel *et al.* (2012), and as represented at the left of Figure 2.1, a hybrid time domain is defined as a subset of $[0, \infty) \times \mathbb{Z}_{\geq 0}$, given as a union of finitely or infinitely many intervals $[t_i, t_{i+1}] \times \{i\}$ where the numbers $0 = t_0, t_1, \dots$, form a finite or infinite nondecreasing sequence and $t_1, t_2, \dots$ are called “jump times”. The last interval is allowed to be of the form $[t_i, T)$ with T finite or $T = +\infty$. As an example, Figure 2.1 represents the same response shown in Figure 1.3

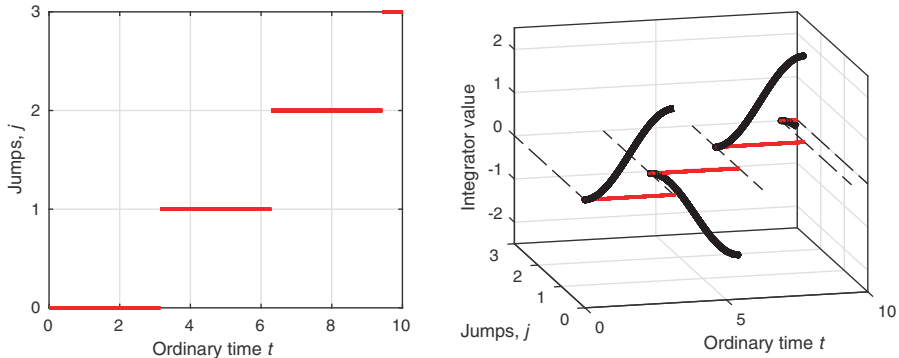


Figure 2.1: Response of Figure 1.3 in a hybrid domain.

with this notion of solution. The dashed horizontal lines in the right subfigure represent the jump times t_j (here corresponding to $t_1 = 2.2$, $t_2 = 6.15$ and $t_3 = 9.6$) and the red lines in both plots are the hybrid domain of the solution x_c . Throughout the rest of this monograph we will always use the shorthand notation $\dot{x}$ for $\dot{x}(t, j)$ and x^+ for $x(t, j + 1)$.

Following a similar generalization, we realize that we may represent dynamics (1.4) as a hybrid dynamical system, under the assumption that d is a locally integrable signal (even though in most cases a piecewise continuous selection should be enough). Under these mild assumptions on d , to guarantee existence of solutions (see Goebel *et al.*, 2009; Goebel *et al.*, 2012), it is sufficient to assume that $\mathcal{F}$ and $\mathcal{J}$ are closed sets (this is the case, for example, if one selects them as the symmetric cones in Equation (1.5)). Then a solution $x(\cdot, \cdot)$ of the hybrid system (1.4), or its shortcut representation given by

$$\dot{x} = Ax + B_d d, \quad x \in \mathcal{F} \quad (2.2a)$$

$$x^+ = Gx, \quad x \in \mathcal{J} \quad (2.2b)$$

is a function defined on the hybrid time domain $\text{dom}(x)$ such that:

(i) for all j and almost¹ all t such that $(t, j) \in \text{dom}(x)$, we have

$$x(t, j) \in \mathcal{F}, \text{ and } \dot{x}(t, j) = Ax(t, j) + B_d d(t),$$

(ii) for all $(t, j) \in \text{dom}(x)$ such that $(t, j + 1) \in \text{dom}(x)$ we have

$$x(t, j) \in \mathcal{J} \text{ and } x(t, j + 1) = Gx(t, j).$$

Note that, following the well-established notation in Goebel *et al.* (2012), we dropped the sentence “is allowed when” from Equations (1.4) to obtain the compact notation in Equation (2.2).

We may now be more precise about some notation that has been already used in the previous paragraphs. We call Equation (2.2a) the “flow” equation and Equation (2.2b) the “jump” equation. The two conditions at the right are the “flow” condition and the “jump” condition. The solution to the hybrid system will then flow or jump depending on whether its value at that time belongs to the so-called “jump set”

¹By “almost all t ” we mean all t except for some isolated points (that is, a set of zero measure in the Lebesgue sense).

(namely, the set of states and inputs for which the jump condition is true) or it belongs to the “flow set” or even both. Note that if an initial state belongs to both sets then both flows and jumps are allowed, which may lead to non-unique solutions.

While the example treated in this chapter has linear flow and jump maps, therefore is well described by the homogeneous dynamics (2.2), the solution concept introduced above also applies to a nonlinear generalization of the hybrid dynamics, corresponding to

$$\begin{cases} \dot{x} = f(x, d), & x \in \mathcal{F} \\ x^+ = g(x), & x \in \mathcal{J} \end{cases} \quad (2.3)$$

for some selection of the closed sets $\mathcal{F}$ and $\mathcal{J}$. This type of generalization will be useful in Chapter 3.

2.2 A motivating example

To introduce and motivate the structure of this survey, it is useful to consider the simplest possible example of reset control system, which has been the subject of several studies, including the one in Beker *et al.* (2001a): the case of a Clegg integrator controlling a linear plant corresponding to a single integrator, in negative error feedback. This is essentially the control scheme of Figure 1.5 with $\mathcal{P} = \frac{1}{s}$. In state-space form, which is necessary for suitably describing the nonlinear reset phenomenon, the plant dynamics is described by the differential equation $\dot{y} = x_c$, where x_c is the output of the Clegg integrator in Equation (2.1). Considering the simplest possible case of $RC = 1$ in the Clegg dynamics, for a constant reference r , the feedback interconnection is well described in error coordinates $x := (x_p, x_c) := (y - r, x_c)$ as

$$\begin{cases} \dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x, & x \in \mathcal{F} = \{(x_p, x_c) : x_p x_c \leq 0\} \\ x^+ = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} x, & x \in \mathcal{J} = \{(x_p, x_c) : x_p x_c \geq 0\}, \end{cases} \quad (2.4)$$

where the flow and jump sets $\mathcal{F}$ and $\mathcal{J}$ exactly correspond to the two half planes represented at the left of Figure 2.3 (see also the right of

Figure 1.4). The flow equation in hybrid dynamics (2.4) represents the integrator nature of both plant and controller, and the minus sign in Equation (2,1) entry of matrix A is motivated by the negative feedback interconnection. The jump equation, instead, captures the fact that x_p does not change across jumps, whereas x_c is reset to zero.

The solutions of the closed loop (2.4) exhibit a peculiar behavior. Whenever they evolve in the second and fourth quadrants, they move along circles (because the continuous dynamics is governed by two complex conjugate eigenvalues at $\pm i$). It is quite straightforward to conclude that these flowing solutions spin in the clockwise direction in the (x_p, x_c) plane as shown at the left of Figure 2.3. When they reach the vertical axis, since they cannot flow outside $\mathcal{F}$, they are forced to jump and this jump instantaneously resets them to zero. From zero, solutions may jump or flow, but regardless of their favorite (discrete or continuous) evolution, they are forced to remain in zero, which is an equilibrium, despite the peculiar non-uniqueness of solutions characterizing it.

Figure 2.2 shows (see the bold line) a possible solution of Equation (2.4) starting from $(x_p(0,0), x_c(0,0)) = (1,0)$ and compares it to the solution of a purely continuous-time dynamics $\dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x$ where the controller is never reset to zero. The domain of the hybrid solution is projected on the t direction to simplify the comparison to the continuous solution. The specific blue solution that we show here decides to flow

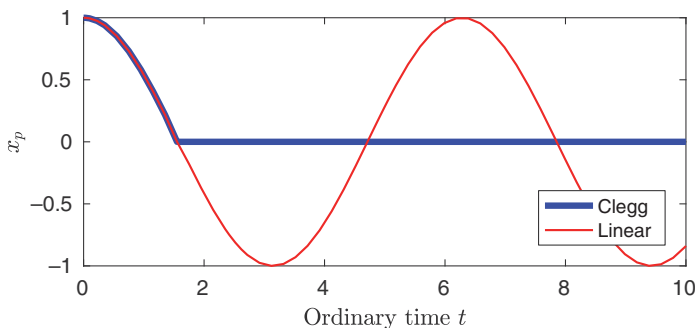


Figure 2.2: Plant output response for the motivating example, starting from $x_c(0,0) = 0$ and $x_p(0,0) = 1$.

indefinitely so that it can be regarded as an eventually continuous solution (it only performs one jump). While the blue solution is unique in the first portion of its time domain, until the jump time $t_1 = \pi$ (this is the time when x_p reaches zero), its evolution is only one among infinitely many others after this time, because from $x(t_1, 1) = (0, 0)$, solutions may jump or flow at each one of the following hybrid time. As observed in Figure 2.2, the flexibility offered by Clegg integrator allows to improve the performance, that is, it allows the plant output to reach 0 in finite time without any overshoot. Such a non-overshooting feature is associated in Beker *et al.* (2001a) to intrinsic limitations of linear control, which are overcome by the nonlinear reset architecture.

The left of Figure 2.3 shows the phase plane representation of the solution in Figure 2.2 and of all the possible other ones, because once they reach the origin they all remain there in the future (continuous or discrete, or both) evolution. The right of Figure 2.3 shows the x_p component of one of the possible solutions, together with its time domain. That specific solution is eventually discrete and performs an infinite number of jumps never flowing again after reaching the origin.

Despite the non-uniqueness of solutions at the origin, it is rather intuitive to imagine that the following property of solutions should hold for the closed loop (2.4). The origin is *globally exponentially stable* for

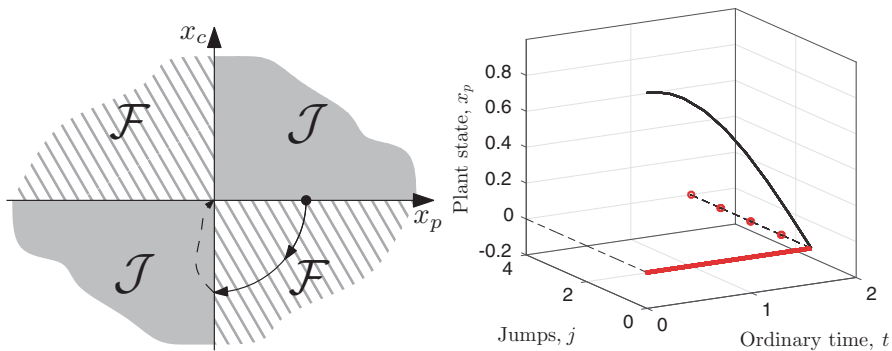


Figure 2.3: Phase-plane and hybrid representation of a solution of hybrid dynamics (2.4).

system (2.4), in the sense that there exist constants $m, \ell > 0$ such that given any initial condition $x(0, 0) = (x_{p0}, x_{c0}) \in \mathbb{R}^2$ all solutions x satisfy

$$|x(t, j)| \leq me^{-\ell(t+j)}|x(0, 0)|, \text{ for all } (t, j) \in \text{dom}(x). \quad (2.5)$$

When using hybrid tools to address reset control systems, it should be recognized that these control systems are generally designed for a continuous-time plant. Therefore, the continuous-time evolution of the solution is somewhat more important due to its physical meaning. Because of this reason, in this survey we will soon concentrate on alternative t -related definitions of exponential convergence corresponding to modifying the bound above as follows:

$$|x(t, j)| \leq me^{-\ell t}|x(0, 0)|, \text{ for all } (t, j) \in \text{dom}(x). \quad (2.6)$$

Clearly this bound is weaker than the previous one, but well represents certain stability and performance specifications that emerge in reset control systems.

A further reason emphasizing the suitability of t -related bounds is that we will eventually augment reset control systems with so-called dwell-time. Roughly speaking, for systems with mixed discrete-time and continuous-time dynamics, a dwell-time is a duration of time without any jump. A solution with such a dwell-time experiences jumps that do not accumulate. A dwell-time may result from a temporal regularization constraints (see Chapter 4 and the subsequent ones) whose goal is to rule out solutions that stop flowing (such as the one represented at the right of Figure 2.3). With a dwell-time logic in place, it is possible to show that there exists $\rho > 0$ such that all hybrid times (t, j) in the domain of a solution satisfy $t \geq \rho(j - 1)$. Due to this fact, the hybrid exponential bound (2.5) and its t -restriction (2.6) become equivalent. Following the paradigm behind transforming (2.5) into (2.6) we will discuss in this survey t -decay rates and t - $\mathcal{L}_2$ gains for reset control systems with temporal regularization.

While the global exponential stability defined above is a relevant property to be pursued in the presence of flow and jump dynamics governed by linear equations (as in Equation (2.2)), we will address in this survey also situations with nonlinear dynamics where it will be

useful to characterize the milder property of *global asymptotic stability*, comprising the next two properties:

1. *Lyapunov stability*: For each $\varepsilon > 0$ there exists $\delta > 0$ such that the following implication holds for all solutions x :

$$|x(0, 0)| \leq \delta \quad \Rightarrow \quad |x(t, j)| \leq \varepsilon, \quad \forall (t, j) \in \text{dom } x;$$

2. *global attractivity*: All solutions are bounded and solutions whose domain is unbounded (namely, solutions evolving forever) converge to zero.

We emphasize that the global convergence property defined above comprises the fact that, with hybrid dynamics, it is not necessarily true that solutions evolve for arbitrarily large hybrid times (or evolve forever). Those solutions evolving forever are called “complete” and are well characterized by the fact of having an unbounded domain. Clearly, asymptotic convergence can only be stated for those solutions, but global attractivity also requires that all other solutions be bounded.

2.3 An overview of this survey

The tools introduced in this monograph will illustrate the methods for proving the above exponential stability property for this specific example and also for some suitable generalizations of it. Below we discuss a point of view about the organization of the monograph, stemming from certain considerations involving the elementary closed loop described in the previous section.

The survey is divided in three parts, the first one being an introductory part completed by this chapter. The second part deals with non-planar reset systems and comprises four chapters covering several reset rules that may be used to augment high-order controllers for plants of any order. The second part of the book reports on a nontrivial generalization of the basic mechanisms emerging in Clegg integrators and FOREs. We come back to those simpler ideas addressing a very simple class of SISO control systems with scalar plants and a scalar FORE in the third part of the survey. Despite the simplicity of such

planar reset systems, interesting conclusions can be drawn and some experimental validations of the advantages arising from the use of reset controllers are also reported. We provide extra comments on the contents of each chapter in the remaining part of this section.

Chapter 3. The reset mechanism well represented in Figure 2.3 can be generalized to nonlinear continuous-time control systems with reset loops, is also the source for the design method later presented in Chapter 6. In particular, notice that a (Lyapunov-like) function $V(x)$ non-increasing along flowing solutions of Equation (2.4), corresponds to $V(x_p, x_c) = \|[x_c^p]\|^2$. For this function, one can check that the minimizer $\phi(x_p) = \operatorname{argmin}_{x_c} V(x_p, x_c) = 0$, and it is possible to introduce the function $V_p(x_p) := V(x_p, \phi(x_p)) = |x_p|^2$, corresponding to the future value of $V(x^+)$ if one decided to impose a jump of x_c to $\phi(x_p)$. Then the selection of $\mathcal{F}$ and $\mathcal{J}$ is well understood in Equation (2.4), as the sets from which (respectively) the function V_p is nonincreasing and nondecreasing along solutions. A strict version of this reset mechanism is proposed in Chapter 6.

Chapter 4. The illustration in Figure 2.3 of the possible solutions to Equation (2.4) reveals that one has to expect solutions possibly performing an infinite number of jumps at the origin, and stopping to flow, thereby having a bounded domain in the t direction (Zeno phenomenon). This fact may be problematic when wanting to prioritize flowing within a description of a continuous-time plant controlled by a reset controller. Moreover, it creates problems in higher-order FORE control systems, where the origin of the (e, x_c) plane may correspond to a whole subspace of the state space. Then Zeno solutions may emerge, which do not even converge to the origin. Due to this fact, and motivated by the need of using continuous-time Luenberger observers in the reset laws of the preceding chapter, we introduce and illustrate the concept of temporal regularization, or dwell-time mechanisms, in Chapter 4.

Chapter 5. Once the dwell-time mechanism has been introduced, all solutions will be forced to flow indefinitely (namely they have an unbounded domain in the t direction) and are characterized by isolated jumps. This fact is a motivation for introducing suitable continuous-time gains and signal norms and take into account the effect of disturbances acting on reset closed loops. For that scenario, Chapter 5 discusses

ways to construct quadratic and nonquadratic Lyapunov functions that strictly decrease along solutions to (a temporally regularized version of) dynamics (2.4) and its higher-order generalizations. In that context, the specific dynamics (2.4) constitutes an interesting example because it can be proven (see the discussion in Section 5.4.3 for details) that no quadratic Lyapunov function exists that decreases along those solutions to Equation (2.4), whereas nonquadratic ones can be constructed and lead to interesting tight performance estimates.

Chapter 6. All of the material presented up to this point of the survey corresponds to stability and performance analysis tools. Chapter 6 combines the ideas of Chapters 3 and 5 for the design of a reset $\mathcal{H}_\infty$ plant-order controller for a linear plant.

Chapter 7. We conclude this overview paper by commenting on a somewhat parallel research direction coming back to the basic SISO reset mechanism described by Equation (2.4) and its FORE generalizations stemming from Equation (1.2), to derive tight necessary and sufficient stability conditions of planar reset closed loop, under a slightly modified dynamics that avoids the problem associated with the fact that solutions jumping from the horizontal line in the left plot of Figure 2.3 (this one belongs to $\mathcal{J}$ as one can inspect from the right of Figure 1.4) do not cause any change in x , and therefore of any possible Lyapunov function. Such a modification simplifies the analysis and leads to interesting conditions. Chapters 8 and 9 also discuss some interesting experimental results obtained using this FORE control system paradigm.

Part II

Non-planar Reset Systems

3

Nonlinear Extensions of Clegg's Integrator

3.1 A generalization of the FORE reset rule

This part is inspired by Prieur *et al.* (2013) and exploits basic ideas on systems with mixed discrete/continuous dynamics, and Lyapunov theory for such hybrid dynamical systems. The basic idea consists in starting from an existing closed loop and to reset the controller as soon as the Lyapunov function is no more decreasing (or at least is decreasing sufficiently slowly with respect to the nominal decreasing of the Lyapunov function along the closed loop). This paradigm is useful not only for limiting the input updating of a control plant (when our approach is based on a modification of a given controller), but it may be also fruitful to update control inputs when the original closed loop is unstable. This latter case may even lead to better performance than considering only stabilized closed loop. Let us now proceed with more specific statement, and consider the following dynamical model for the plant:

$$\dot{x}_p = f_p(x_p, u), \quad (3.1)$$

with x_p in $\mathbb{R}^{n_p}$, in feedback interconnection with a (not necessarily stabilizing) dynamic controller:

$$\dot{x}_c = f_c(x_c, x_p), \quad u = h_c(x_c, x_p), \quad (3.2)$$

with x_c in $\mathbb{R}^{n_c}$. Then defining the closed-loop functions $\bar{f}_p(x_p, x_c) = f_p(x_p, h_c(x_c, x_p))$ and $\bar{f}_c(x_c, x_p) = f_c(x_c, x_p)$, the interconnection between Equations (3.1) and (3.2) can be described in a compact way as:

$$\dot{x} = f(x) := (\bar{f}_p(x_p, x_c), \bar{f}_c(x_c, x_p)), \quad (3.3)$$

where $x := (x_p, x_c)$, $\bar{f}_p : \mathbb{R}^{n_p} \times \mathbb{R}^{n_c} \rightarrow \mathbb{R}^{n_p}$ and $\bar{f}_c : \mathbb{R}^{n_c} \times \mathbb{R}^{n_p} \rightarrow \mathbb{R}^{n_c}$. We assume that f_p , f_c and h_c are such that $\bar{f}_p$ and $\bar{f}_c$ are continuous functions satisfying $\bar{f}_p(0, 0) = 0$ and $\bar{f}_c(0, 0) = 0$.

The general objective of this chapter is to design a suitable reset rule, or jump law, for controller (3.2), and to design a partition of the state space $\mathbb{R}^n$ (where $n = n_p + n_c$) in two subsets, called flow and jump sets. The state x_c of controller (3.2) endowed with such an additional logic is then instantaneously reset according to the jump law whenever the state belongs to the jump set. This extended scheme, which is allowed to flow according to Equation (3.3) only if the state belongs to the flow set, defines a hybrid system. More specifically, the proposed hybrid augmentation is designed to guarantee the decrease of one or both of two Lyapunov-like functions: one of them $V : \mathbb{R}^{n_p} \times \mathbb{R}^{n_c} \rightarrow \mathbb{R}_{\geq 0}$, defined on the whole state space, and the other one $V_p : \mathbb{R}^{n_p} \rightarrow \mathbb{R}_{\geq 0}$ defined only in the plant state subspace. The functions V and V_p can be selected to capture some closed-loop stability and performance property (see the developments in Sections 3.5.1 and 3.5.2 where maximized decay rate and output overshoot reduction are tackled, respectively). Functions V and V_p are linked to each other by a function $\phi : \mathbb{R}^{n_p} \rightarrow \mathbb{R}^{n_c}$ such that for all (x_p, x_c) in $\mathbb{R}^{n_p} \times \mathbb{R}^{n_c}$

$$V(x_p, \phi(x_p)) \leq V(x_p, x_c), \quad (3.4)$$

and, in particular, by the relation

$$V_p(x_p) = V(x_p, \phi(x_p)), \quad \forall x_p \in \mathbb{R}^{n_p}. \quad (3.5)$$

Within the above scenario, we will design flow and jump sets and jump rules such that the arising hybrid systems guarantee non-increase of V or V_p , or both of them in two relevant cases:

- (V) (addressed in Section 3.2.1) where the function $V(\cdot, \cdot)$ is given and satisfies suitable conditions guaranteeing the existence¹ of $\phi(\cdot)$ from which $V_p(\cdot)$ can be derived according to Equation (3.5);
- (V_p) (addressed in Section 3.2.2) where $V_p(\cdot)$ and $\phi(\cdot)$ are given (their existence resembles an asymptotic controllability assumption), from which $V(\cdot, \cdot)$ satisfying Equations (3.4) and (3.5) will be constructed.

Section 3.4 deals with the special case where system (3.3) is linear. In this case, quadratic versions of V and V_p can be constructed under reasonably weak properties required for the closed-loop dynamics. The particularization to the linear case allows to strengthen the nonlinear results by exploiting the homogeneity property of hybrid systems acting on cones and obeying linear flow and jump rules. Finally, as a last contribution of this part, we will show how to design V_p in item (V_p) to augment linear continuous-time control systems with hybrid loops that reduce the overshoot of a scalar plant output. In comparison to previous works, the aim is to design hybrid strategies to guarantee some asymptotic stability property by enforcing that suitable Lyapunov-like functions are not increasing along the hybrid solutions.

The arising hybrid closed loop resembles the so-called impulsive systems, considered in Haddad *et al.* (2001). However, the objectives of Haddad *et al.* (2001) and of the present part are different. Indeed an inverse optimal control involving a hybrid nonlinear-non-quadratic performance functional is developed by Haddad *et al.* (2001), whereas here we provide a design method of a hybrid loop (namely, the jump map and the jump/flow sets) to ensure asymptotic stability and non-increase of suitable scalar functions. Our results are also linked to the self-triggered control literature (see Anta and Tabuada, 2010) for stability

¹Here, to keep the discussion simple, it is assumed that V is continuously differentiable and that there exists $\phi(x_p) \in \operatorname{argmin}_{x_c \in \mathbb{R}^{n_c}} V(x_p, x_c)$, which implies Equation (3.4).

analysis of networked control systems, where it is necessary to reduce the number of times when the state is measured by the controller and the actuators are updated. The most important difference between the results mentioned above and our contribution is that in those works the resetting value for the state is uniquely associated with the transmission of a measurement sample, whereas in our results it depends on the Lyapunov-like functions that should not increase along solutions.

3.2 Hybrid stabilization with reset loops: Nonlinear case

3.2.1 Constructing V_p from V

In this section we consider the closed-loop nonlinear system (3.3) and a function V of the closed-loop state to address item (V) defined Section 3.1. To this aim, we make the following assumption on the function V .

Assumption 3.1. The function $V : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$ is continuously differentiable and such that there exists a continuous differentiable function $\phi : \mathbb{R}^{n_p} \rightarrow \mathbb{R}^{n_c}$ such that

$$\phi(x_p) \in \operatorname{argmin}_{x_c \in \mathbb{R}^{n_c}} V(x_p, x_c). \quad (3.6)$$

Moreover, there exists a class $\mathcal{K}$ function α (namely, a continuous function that is zero at zero and strictly increasing) such that, for all x_p in $\mathbb{R}^{n_p}$, $x_p \neq 0$,

$$\langle \nabla_p V(x_p, \phi(x_p)), \bar{f}_p(x_p, \phi(x_p)) \rangle < -\alpha(V(x_p, \phi(x_p))), \quad (3.7)$$

where $\nabla_p V$ denotes the gradient of V with respect to its first argument.

Remark 3.1. In Assumption 3.1 we do not impose that Equation (3.3) is globally asymptotically stable, because Equation (3.7) requires the function V to be decreasing only in the subset of the state space defined by $(x_p, x_c) = (x_p, \phi(x_p))$. Nevertheless, if system (3.3) is globally asymptotically stable, then there exists a function $V : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$ and a class $\mathcal{K}$ function α such that $\langle \nabla V(x), f(x) \rangle < -\alpha(V(x))$ for all $x \neq 0$, $x = (x_p, x_c) \in \mathbb{R}^n$, which implies Equation (3.7). Moreover, note that in Assumption 3.1, it is not required that $\operatorname{argmin}_{x_c \in \mathbb{R}^{n_c}} V(x_p, x_c)$ is a

single valued map, but only that a continuous differentiable selection of this map does exist. For example, with $V(x_p, x_c) = x_p^4 + x_c^4 - x_p^2 x_c^2$, we have $\operatorname{argmin}_{x_c \in \mathbb{R}^{n_c}} V(x_p, x_c) = \{x_p, -x_p\}$ which is not a singleton even though Assumption 3.1 can be satisfied, e.g., with $\phi(x_p) = x_p$.

A natural way to stabilize the closed-loop system (3.3) is to flow when one (or both) of V and V_p is strictly decreasing and to reset the x_c -component of the state to the value $\phi(x_p)$ (where strict decrease is guaranteed by Equation (3.7)) when the function is not decreasing. This leads to the following hybrid system

$$\begin{cases} \dot{x} = f(x), & x \in \hat{\mathcal{F}}, \\ x^+ = (x_p^+, x_c^+) = (x_p, \phi(x_p)), & x \in \hat{\mathcal{J}}, \end{cases} \quad (3.8)$$

where $\hat{\mathcal{F}} \subset \mathbb{R}^n$ and $\hat{\mathcal{J}} \subset \mathbb{R}^n$ are suitable closed subsets of the state space such that $\hat{\mathcal{F}} \cup \hat{\mathcal{J}} = \mathbb{R}^n$. In particular, $\hat{\mathcal{F}}$ and $\hat{\mathcal{J}}$ are defined by suitably combining the following two pairs of sets arising, respectively, from the knowledge of V and V_p :

$$\begin{aligned} \mathcal{F} &= \{x \in \mathbb{R}^n, \langle \nabla V(x), f(x) \rangle \leq -\bar{\alpha}(V(x))\} \\ \mathcal{J} &= \{x \in \mathbb{R}^n, \langle \nabla V(x), f(x) \rangle \geq -\bar{\alpha}(V(x))\} \end{aligned} \quad (3.9)$$

$$\begin{aligned} \bar{\mathcal{F}} &= \{x \in \mathbb{R}^n, \langle \nabla V_p(x_p), \bar{f}_p(x_p, x_c) \rangle \leq -\bar{\alpha}(V_p(x_p))\} \\ \bar{\mathcal{J}} &= \{x \in \mathbb{R}^n, \langle \nabla V_p(x_p), \bar{f}_p(x_p, x_c) \rangle \geq -\bar{\alpha}(V_p(x_p))\} \end{aligned} \quad (3.10)$$

where $\bar{\alpha}$ is any class $\mathcal{K}$ function such that $\bar{\alpha}(s) \leq \alpha(s)$ for all $s \geq 0$ (this will be denoted next by the shortcut notation $\bar{\alpha} \leq \alpha$). The sets introduced in Equations (3.9) and (3.10) arise from either focusing on the whole state x through the function V or focusing only on the state of the plant x_p through function V_p . The two selections correspond to different transient behaviors and it is unclear what method is more advantageous from a practical viewpoint. Nevertheless, an interesting observation is that forcing V_p to decrease along solutions comprises, roughly speaking, that no overshoots of x_p (as measured by V_p) can be induced by the transient evolutions of the closed loop (through possible interplays between x_p and x_c). The next theorem establishes useful properties when using one method or the other, or even a combination of the two.

Theorem 3.1. Consider system (3.3) and suppose Assumption 3.1 holds. Then for any class $\mathcal{K}$ function $\bar{\alpha}$ satisfying $\bar{\alpha} \leq \alpha$ the following holds.

1. If V is positive definite and radially unbounded, then the hybrid system (3.8), (3.9) with $\hat{\mathcal{F}} = \mathcal{F}$ and $\hat{\mathcal{J}} = \mathcal{J}$ is globally asymptotically stable and V is non-increasing along solutions.
2. If V_p , defined in Equation (3.5), is positive definite and radially unbounded, then the hybrid system (3.8), (3.10) with $\hat{\mathcal{F}} = \bar{\mathcal{F}}$ and $\hat{\mathcal{J}} = \bar{\mathcal{J}}$ is such that the plant state x_p converges to zero, and V_p is non-increasing along solutions.
3. If V is positive definite and radially unbounded, then the hybrid system (3.8), (3.9), (3.10) with $\hat{\mathcal{F}} = \mathcal{F} \cap \bar{\mathcal{F}}$ and $\hat{\mathcal{J}} = \mathcal{J} \cup \bar{\mathcal{J}}$ is globally asymptotically stable and both V and V_p are non-increasing along solutions.

Remark 3.2. The three items of Theorem 3.1 address, respectively, the three goals of guaranteeing suitable stability properties together with non-increase of V , V_p and both of them. In particular, the flow and jump sets of the hybrid system (3.8) are defined in Theorem 3.1 from the knowledge of the Lyapunov-like function V . Note also that the hybrid system (3.8), (3.10) with $\hat{\mathcal{F}} = \bar{\mathcal{F}}$ and $\hat{\mathcal{J}} = \bar{\mathcal{J}}$ characterized at item 2 may be unstable. In other words, with this selection of the flow and jump sets, the plant state converges to the origin, although the state of the controller may diverge (which may complicate the implementation of the associate controller). In the linear case, this item will be strengthened under a detectability assumption (see Theorem 3.5 reported later in this chapter).

Remark 3.3. It can be checked that, due to the assumption on the regularity of $\bar{f}_p$ and $\bar{f}_c$, as specified before and after Equation (3.3), and on the regularity of ϕ as required in Assumption 3.1, the hybrid system satisfies certain conditions (called *hybrid basic conditions* in Goebel and Teel (2006) and Prieur *et al.* (2007)). Then the hybrid system (3.8) inherits several good structural properties, such as simple viability conditions ensuring existence of solutions (Goebel and Teel, 2006, Prop. 6.10), closedness of the solutions set (namely converging sequences of

solutions converge to a solution) and robustness of asymptotic stability. Moreover, instead of the assumption that the function α is of class $\mathcal{K}$ in Theorem 3.1, it may be assumed a LaSalle-type assumption. More precisely, using the LaSalle invariance property for hybrid systems (see e.g., Goebel *et al.*, 2009, Theorem 23, page 64), this assumption may be replaced by: for any solution x of Equation (3.3), if $\langle \nabla V(x), f(x) \rangle \equiv 0$, then $x \equiv 0$.

Remark 3.4. If the closed-loop system (3.3) is globally asymptotically stable, then selecting V and α satisfying Assumption 3.1, the sets in Equation (3.9) correspond to $\mathcal{F} = \mathbb{R}^n$ and $\mathcal{J} = \emptyset$, so that the reset systems characterized in items 2 and 3 of Theorem 3.1 coincide. Therefore selecting the jump and flow sets as in Equation (3.10) a non-increasing property for both V and V_p , together with GAS is obtained. Note however that as long as Assumption 3.1 is satisfied, the flow set is never empty because Equation (3.7) guarantees that after any jump the state belongs to the interior of the flow set.

3.2.2 Constructing V from V_p

In this section we consider the closed-loop nonlinear system (3.3) and a function V_p of the plant state to address item (V_p) defined in Section 3.1. To this aim, we make the following assumption on the function V_p .

Assumption 3.2. The function $V_p : \mathbb{R}^{n_p} \rightarrow \mathbb{R}_{\geq 0}$ is continuously differentiable and radially unbounded and there exists a continuously differentiable function $\phi : \mathbb{R}^{n_p} \rightarrow \mathbb{R}^{n_c}$, and a class $\mathcal{K}$ function α such that, for all x_p in $\mathbb{R}^{n_p}$, $x_p \neq 0$,

$$\langle \nabla V_p(x_p), \bar{f}_p(x_p, \phi(x_p)) \rangle < -\alpha(V_p(x_p)). \quad (3.11)$$

Note that, when the x_p -equation of the closed-loop system (3.3) is affine with respect to the input x_c , this condition is related to the asymptotic controllability to the origin (Artstein, 1983). In this case, a control law ϕ can be computed from a Control Lyapunov Function V_p and from the so-called *universal formulas* (see Freeman and Kokotović, 1996; Lin and Sontag, 1991).

Assumption 3.2 is sufficient to construct a function V satisfying Assumption 3.1 so that the design strategy of the previous section can be

employed. In particular, let M be any symmetric positive semidefinite matrix² in $\mathbb{R}^{n_c \times n_c}$ and define $V : \mathbb{R}^{n_p} \times \mathbb{R}^{n_c} \rightarrow \mathbb{R}_{\geq 0}$ for all (x_p, x_c) in $\mathbb{R}^{n_p} \times \mathbb{R}^{n_c}$,

$$V(x) = V_p(x_p) + (x_c - \phi(x_p))^T M (x_c - \phi(x_p)). \quad (3.12)$$

Note that V is continuously differentiable, and radially unbounded. Moreover, if $\phi(0) = 0$ and $M > 0$, then it is a positive definite function because, for each x_p , it is the sum of two positive definite terms, the first one strictly positive when $x_p \neq 0$ and the second one strictly positive when $x_p = 0$. The following theorem is a straightforward application of Theorem 3.1 in light of the V_p and ϕ given in Assumption 3.2 and of the V in Equation (3.12). A nonlinear example illustrating the potential behind the nonlinear design of Theorem 3.1 is given in Section 3.3 below (see also Prieur *et al.*, 2010).

Theorem 3.2. Consider system (3.3) and suppose Assumption 3.2 holds. Given any symmetric positive semidefinite matrix $M \in \mathbb{R}^{n_c \times n_c}$, the corresponding function V in Equation (3.12), and any class $\mathcal{K}$ function $\bar{\alpha}$ satisfying $\bar{\alpha} \leq \alpha$, the three items of Theorem 3.1 hold.

3.3 A nonlinear control system stabilized by adding a hybrid loop

Let us illustrate the use of Theorem 3.2 on a nonlinear system satisfying Assumption 3.2. To do that, after introducing the nonlinear system under consideration in this section, we will check Assumption 3.2, and by applying Theorem 3.2, we will see on simulations that the second item of Theorem 3.1 applies. To simulate the hybrid systems we use the simulator presented in Goebel *et al.* (2009, pages 78–81).

Consider the following plant in $\mathbb{R}^2$

$$\begin{cases} \dot{x}_1 = x_2 + x_1^2 \\ \dot{x}_2 = u + x_1^2 \end{cases} \quad (3.13)$$

where (x_1, x_2) is the plant state and u stands for the control variable in $\mathbb{R}$. By using a backstepping method (see Krstić *et al.*, 1995), the

²Note that the matrix M may be a function of x . This extra degree of freedom could be used to perform convenient selections of V .

following controller $u : \mathbb{R}^2 \rightarrow \mathbb{R}$ defined, for all (x_1, x_2) in $\mathbb{R}^2$, by

$$u = \phi(x_1, x_2) = -2x_1 - 2x_2 - 3x_1^2 - 2x_1(x_2 + x_1^2) \quad (3.14)$$

and the positive definite function $V_p : \mathbb{R}^2 \rightarrow \mathbb{R}_{\geq 0}$ defined, for all (x_1, x_2) in $\mathbb{R}^2$, by

$$V_p(x_1, x_2) = \frac{1}{2}x_1^2 + \frac{1}{2}(x_2 + x_1 + x_1^2)^2$$

may be computed. Assumption 3.2 holds with $\alpha(s) = \frac{s}{2}$, for all $s \geq 0$. To check the attractivity property, let us consider the initial condition $(x_1(0,0), x_2(0,0)) = (10, 10)$ and let us numerically compute the solution to system (3.13) in closed loop with the controller (3.14). The time evolution of the x_1 and x_2 -variables, and of the controller u are depicted on Figure 3.1.

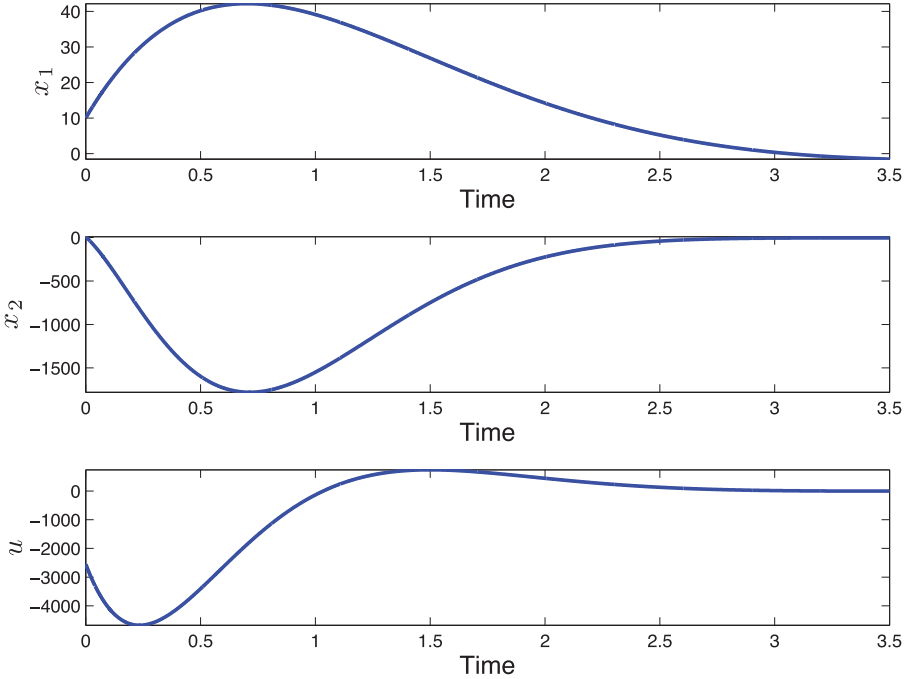


Figure 3.1: Time evolution of x_1 (top), of x_2 (middle) for system (3.13) in closed-loop with the controller (3.14) (down).

Now let us consider the system (3.13) in closed loop with the controller $u = x_3$, where $\dot{x}_3 = x_3 - x_1^3$. This nonlinear closed-loop system is unstable (the initial condition $(x_1(0,0), x_2(0,0), x_3(0,0)) = (10, 10, 10)$ gives a diverging solution). Let us apply Theorem 3.2 with $\bar{\alpha}(s) = 2 \cdot 1; 0^{-3}\alpha(s) = 10^{-3}s$ because, based on the above discussion, functions V_p and ϕ satisfy Assumption 3.2. Consider then the hybrid system (3.8), (3.10) with $\hat{\mathcal{F}} = \bar{\mathcal{F}}$ and $\hat{\mathcal{J}} = \bar{\mathcal{J}}$, that is, with $x_p = (x_1, x_2)$, $x_c = x_3$ and $x = (x_p, x_c)$,

$$\begin{cases} \dot{x} = \begin{pmatrix} x_2 + x_1^2 \\ -2x_1 - 2x_2 - 3x_1^2 - 2x_1(x_2 + x_1^2) + x_1^2(x) \\ x_3 - x_1^3 \end{pmatrix}, & x \in \bar{\mathcal{F}}, \\ x^+ = (x_p, \phi(x_p)), & x \in \bar{\mathcal{J}}, \end{cases}$$

where

$$\begin{aligned} \bar{\mathcal{F}} &= \{x \in \mathbb{R}^3, \langle \nabla V_p(x_p), \bar{f}_p(x_p, x_c) \rangle \leq -10^{-3}V_p(x_p)\} \\ \bar{\mathcal{J}} &= \{x \in \mathbb{R}^3, \langle \nabla V_p(x_p), \bar{f}_p(x_p, x_c) \rangle \geq -10^{-3}V_p(x_p)\} \end{aligned}$$

and with $\phi(x_p)$ defined in Equation (3.14).

The shape of the flow set of this hybrid system, intersected with the plane $x_3 = 0$ is given in Figure 3.2 (due to Equation (3.10), the jump set is the closure of the complement). Let us numerically compute the solution starting from the initial condition $(x_1(0,0), x_2(0,0), x_3(0,0)) = (10, 10, 10)$. We check on Figure 3.3 that the plant state x_p -variable is globally asymptotically stable. Moreover, we note that the x_3 -variable converges also to 0 and has some jumps (when it is reset to the value of the controller (3.14)). The number of jumps depends on the size of $\mathcal{J}$ and thus on the choice of the function $\bar{\alpha}$. By comparing Figures 3.1 and 3.3 we note that the speed of convergence is improved using the hybrid system (3.8), (3.10) with $\hat{\mathcal{F}} = \bar{\mathcal{F}}$ and $\hat{\mathcal{J}} = \bar{\mathcal{J}}$.

The solutions issuing from different initial conditions are drawn on Figure 3.4 in the space (x_1, x_2) . Figure 3.5 represents, for the same solutions, the evolution of function V_p in the continuous-time direction. The flow set is the convex hull of the set of the blue points on Figure 3.4. The vertical lines are the jumps of the x_3 -variable. It can be checked that they converge to the equilibrium (this is the center of the back ball on both figures).

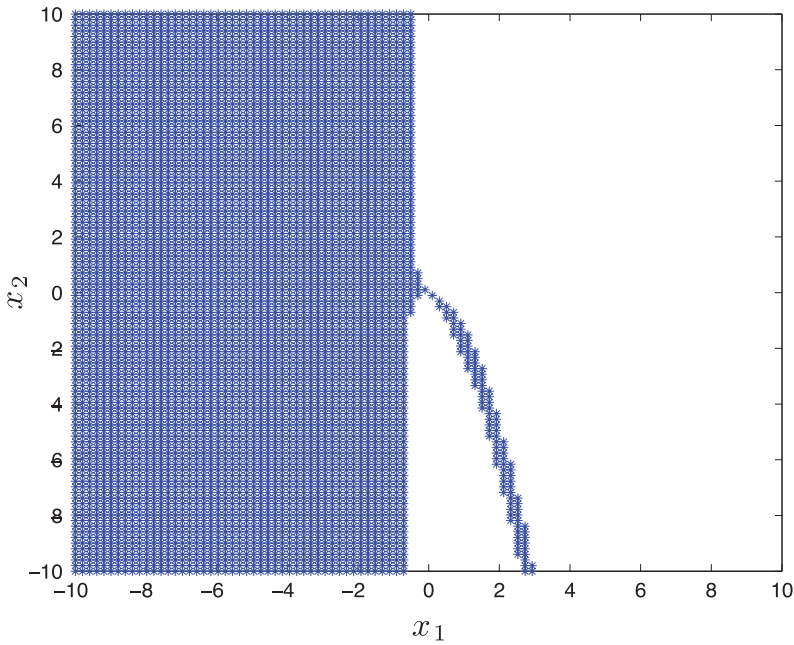


Figure 3.2: Flow set, shaded in blue at the plane corresponding to $x_3 = 0$ for system (3.8), (3.10) with $\hat{\mathcal{F}} = \bar{\mathcal{F}}$ and $\hat{\mathcal{J}} = \bar{\mathcal{J}}$ in the box $[-10, 10] \times [-10, 10]$.

3.4 Hybrid stabilization with reset loops: Linear case

When focusing on linear dynamics, the two items (V) and (V_p) defined in Section 3.1 are addressed here. Consider the following linear plant:

$$\dot{x}_p = A_p x_p + B_p u, \quad (3.15)$$

with $x_p \in \mathbb{R}^{n_p}$, in feedback interconnection with a (not necessarily stabilizing) linear dynamic controller:

$$\dot{x}_c = A_c x_c + B_c x_p, \quad u = C_c x_c + D_c x_p, \quad (3.16)$$

with $x_c \in \mathbb{R}^{n_c}$. The closed loop is described by the following linear system

$$\dot{x} = Ax := \begin{bmatrix} \bar{A}_p & \bar{B}_p \\ \bar{B}_c & \bar{A}_c \end{bmatrix} x, \quad (3.17)$$

where $x = (x_p, x_c)$ and $\bar{A}_p$, $\bar{B}_p$, $\bar{A}_c$, and $\bar{B}_c$ are matrices of appropriate dimensions uniquely based on the matrices in Equations (3.15) and