



GUIDE 98-3/Suppl.1

Uncertainty of measurement

Part 3:

**Guide to the expression of
uncertainty in measurement
(GUM:1995)**

Supplement 1:

**Propagation of distributions
using a Monte Carlo method**

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Foreword

ISO (the International Organization for Standardization) is a worldwide federation of national standards bodies (ISO member bodies). The work of preparing International Standards is normally carried out through ISO technical committees. Each member body interested in a subject for which a technical committee has been established has the right to be represented on that committee. International organizations, governmental and non-governmental, in liaison with ISO, also take part in the work. ISO collaborates closely with the International Electrotechnical Commission (IEC) on all matters of electrotechnical standardization.

International Standards are drafted in accordance with the rules given in the ISO/IEC Directives, Part 2.

Draft Guides adopted by the responsible Committee or Group are circulated to the member bodies for voting. Publication as a Guide requires approval by at least 75 % of the member bodies casting a vote.

Attention is drawn to the possibility that some of the elements of this document may be the subject of patent rights. ISO shall not be held responsible for identifying any or all such patent rights.

This first edition of Supplement 1 to ISO/IEC Guide 98-3 has been prepared by Working Group 1 of the JCGM, and has benefited from detailed reviews undertaken by member organizations of the JCGM and National Metrology Institutes. For further information, see the Introduction (0.2).

ISO/IEC Guide 98 consists of the following parts, under the general title *Uncertainty of measurement*:

- *Part 1: Introduction to the expression of uncertainty in measurement*
- *Part 3: Guide to the expression of uncertainty in measurement (GUM:1995)*

The following parts are planned:

- *Part 2: Concepts and basic principles*
- *Part 4: Role of measurement uncertainty in conformity assessment*
- *Part 5: Applications of the least-squares method*

ISO/IEC Guide 98-3 has one supplement.

- *Supplement 1: Propagation of distributions using a Monte Carlo method*

The following supplements to ISO/IEC Guide 98-3 are planned:

- *Supplement 2: Models with any number of output quantities*
- *Supplement 3: Modelling*

Note that in this document, GUM is used to refer to the industry-recognized publication, adopted as ISO/IEC Guide 98-3:2008. When a specific clause or subclause number is cited, the reference is to ISO/IEC Guide 98-3:2008.

Introduction

0.1 General

This Supplement to the *Guide to the expression of uncertainty in measurement (GUM)* is concerned with the propagation of probability distributions through a mathematical model of measurement [ISO/IEC Guide 98-3:2008, 3.1.6] as a basis for the evaluation of uncertainty of measurement, and its implementation by a Monte Carlo method. The treatment applies to a model having any number of input quantities, and a single output quantity.

The described Monte Carlo method is a practical alternative to the GUM uncertainty framework [ISO/IEC Guide 98-3:2008, 3.4.8]. It has value when

- a) linearization of the model provides an inadequate representation or
- b) the probability density function (PDF) for the output quantity departs appreciably from a Gaussian distribution or a scaled and shifted t -distribution, e.g. due to marked asymmetry.

In case a), the estimate of the output quantity and the associated standard uncertainty provided by the GUM uncertainty framework might be unreliable. In case b), unrealistic coverage intervals (a generalization of “expanded uncertainty” in the GUM uncertainty framework) might be the outcome.

The GUM [ISO/IEC Guide 98-3:2008, 3.4.8] “...provides a framework for assessing uncertainty ...”, based on the law of propagation of uncertainty [ISO/IEC Guide 98-3:2008, Clause 5] and the characterization of the output quantity by a Gaussian distribution or a scaled and shifted t -distribution [ISO/IEC Guide 98-3:2008, G.6.2, G.6.4]. Within that framework, the law of propagation of uncertainty provides a means for propagating uncertainties through the model. Specifically, it evaluates the standard uncertainty associated with an estimate of the output quantity, given

- 1) best estimates of the input quantities,
- 2) the standard uncertainties associated with these estimates, and, where appropriate,
- 3) degrees of freedom associated with these standard uncertainties, and
- 4) any non-zero covariances associated with pairs of these estimates.

Also within the framework, the PDF taken to characterize the output quantity is used to provide a coverage interval, for a stipulated coverage probability, for that quantity.

The best estimates, standard uncertainties, covariances and degrees of freedom summarize the information available concerning the input quantities. With the approach considered here, the available information is encoded in terms of PDFs for the input quantities. The approach operates with these PDFs in order to determine the PDF for the output quantity.

Whereas there are some limitations to the GUM uncertainty framework, the propagation of distributions will always provide a PDF for the output quantity that is consistent with the PDFs for the input quantities. This PDF for the output quantity describes the knowledge of that quantity, based on the knowledge of the input quantities, as described by the PDFs assigned to them. Once the PDF for the output quantity is available, that quantity can be summarized by its expectation, taken as an estimate of the quantity, and its standard deviation, taken as the standard uncertainty associated with the estimate. Further, the PDF can be used to obtain a coverage interval, corresponding to a stipulated coverage probability, for the output quantity.

The use of PDFs as described in this Supplement is generally consistent with the concepts underlying the GUM. The PDF for a quantity expresses the state of knowledge about the quantity, i.e. it quantifies the degree of belief about the values that can be assigned to the quantity based on the available information. The information usually consists of raw statistical data, results of measurement, or other relevant scientific statements, as well as professional judgement.

In order to construct a PDF for a quantity, on the basis of a series of indications, Bayes' theorem can be applied [27, 33]. When appropriate information is available concerning systematic effects, the principle of maximum entropy can be used to assign a suitable PDF [51, 56].

The propagation of distributions has wider application than the GUM uncertainty framework. It works with richer information than that conveyed by best estimates and the associated standard uncertainties (and degrees of freedom and covariances when appropriate).

Decimal sign: The decimal sign in the English text is the point on the line, and the comma on the line is the decimal sign in the French text. (See 4.12)

An historical perspective is given in Annex A.

NOTE 1 The GUM provides an approach when linearization is inadequate [ISO/IEC Guide 98-3:2008, 5.1.2 Note]. The approach has limitations: only the leading non-linear terms in the Taylor series expansion of the model are used, and the PDFs for the input quantities are regarded as Gaussian.

NOTE 2 Strictly, the GUM characterizes the variable $(Y - y)/u(y)$ by a t -distribution, where Y is the output quantity, y an estimate of Y , and $u(y)$ the standard uncertainty associated with y [ISO/IEC Guide 98-3:2008, G.3.1]. This characterization is also used in this Supplement. [The GUM in fact refers to the variable $(y - Y)/u(y)$.]

NOTE 3 A PDF for a quantity is not to be understood as a frequency density.

NOTE 4 "The evaluation of uncertainty is neither a routine task nor a purely mathematical one; it depends on detailed knowledge of the nature of the measurand and of the measurement method and procedure used. The quality and utility of the uncertainty quoted for the result of a measurement therefore ultimately depends on the understanding, critical analysis, and integrity of those who contribute to the assignment of its value." [17].

0.2 JCGM background information

In 1997, the Joint Committee for Guides in Metrology (JCGM), chaired by the Director of the Bureau International des Poids et Mesures (BIPM), was created by the seven international organizations that had originally in 1993 prepared the *Guide to the expression of uncertainty in measurement (GUM)* and the *International vocabulary of basic and general terms in metrology (VIM)*. The JCGM assumed responsibility for these two documents from the ISO Technical Advisory Group 4 (TAG4).

The Joint Committee is formed by the BIPM with the International Electrotechnical Commission (IEC), the International Federation of Clinical Chemistry and Laboratory Medicine (IFCC), the International Organization for Standardization (ISO), the International Union of Pure and Applied Chemistry (IUPAC), the International Union of Pure and Applied Physics (IUPAP), and the International Organization of Legal Metrology (OIML). A further organization joined these seven international organizations, namely, the International Laboratory Accreditation Cooperation (ILAC).

JCGM has two Working Groups. Working Group 1, "Expression of uncertainty in measurement", has the task to promote the use of the GUM and to prepare Supplements and other documents for its broad application. Working Group 2, "Working Group on International vocabulary of basic and general terms in metrology (VIM)", has the task to revise and promote the use of the VIM. For further information on the activity of the JCGM, see www.bipm.org.

Supplements such as this one are intended to give added value to the GUM by providing guidance on aspects of uncertainty evaluation that are not explicitly treated in the GUM. The guidance will, however, be as consistent as possible with the general probabilistic basis of the GUM.

Uncertainty of measurement

Part 3: Guide to the expression of uncertainty in measurement (GUM:1995)

Supplement 1: Propagation of distributions using a Monte Carlo method

1 Scope

This Supplement provides a general numerical approach, consistent with the broad principles of the GUM [ISO/IEC Guide 98-3:2008, G.1.5], for carrying out the calculations required as part of an evaluation of measurement uncertainty. The approach applies to arbitrary models having a single output quantity where the input quantities are characterized by any specified PDFs [ISO/IEC Guide 98-3:2008, G.1.4, G.5.3].

As in the GUM, this Supplement is primarily concerned with the expression of uncertainty in the measurement of a well-defined physical quantity—the measurand—that can be characterized by an essentially unique value [ISO/IEC Guide 98-3:2008, 1.2].

This Supplement also provides guidance in situations where the conditions for the GUM uncertainty framework [ISO/IEC Guide 98-3:2008, G.6.6] are not fulfilled, or it is unclear whether they are fulfilled. It can be used when it is difficult to apply the GUM uncertainty framework, because of the complexity of the model, for example. Guidance is given in a form suitable for computer implementation.

This Supplement can be used to provide (a representation of) the PDF for the output quantity from which

- a) an estimate of the output quantity,
- b) the standard uncertainty associated with this estimate, and
- c) a coverage interval for that quantity, corresponding to a specified coverage probability

can be obtained.

Given (i) the model relating the input quantities and the output quantity and (ii) the PDFs characterizing the input quantities, there is a unique PDF for the output quantity. Generally, the latter PDF cannot be determined analytically. Therefore, the objective of the approach described here is to determine a), b), and c) above to a prescribed numerical tolerance, without making unquantified approximations.

For a prescribed coverage probability, this Supplement can be used to provide any required coverage interval, including the probabilistically symmetric coverage interval and the shortest coverage interval.

This Supplement applies to input quantities that are independent, where each such quantity is assigned an appropriate PDF, or not independent, i.e. when some or all of these quantities are assigned a joint PDF.

Typical of the uncertainty evaluation problems to which this Supplement can be applied include those in which

- the contributory uncertainties are not of approximately the same magnitude [ISO/IEC Guide 98-3:2008, G.2.2],

- it is difficult or inconvenient to provide the partial derivatives of the model, as needed by the law of propagation of uncertainty [ISO/IEC Guide 98-3:2008, Clause 5],
- the PDF for the output quantity is not a Gaussian distribution or a scaled and shifted t -distribution [ISO/IEC Guide 98-3:2008, G.6.5],
- an estimate of the output quantity and the associated standard uncertainty are approximately of the same magnitude [ISO/IEC Guide 98-3:2008, G.2.1],
- the models are arbitrarily complicated [ISO/IEC Guide 98-3:2008, G.1.5], and
- the PDFs for the input quantities are asymmetric [ISO/IEC Guide 98-3:2008, G.5.3].

A validation procedure is provided to check whether the GUM uncertainty framework is applicable. The GUM uncertainty framework remains the primary approach to uncertainty evaluation in circumstances where it is demonstrably applicable.

It is usually sufficient to report measurement uncertainty to one or perhaps two significant decimal digits. Guidance is provided on carrying out the calculation to give reasonable assurance that in terms of the information provided the reported decimal digits are correct.

Detailed examples illustrate the guidance provided.

This document is a Supplement to the GUM and is to be used in conjunction with it. Other approaches generally consistent with the GUM may alternatively be used. The audience of this Supplement is that of the GUM.

NOTE 1 This Supplement does not consider models that do not define the output quantity uniquely (for example, involving the solution of a quadratic equation, without specifying which root is to be taken).

NOTE 2 This Supplement does not consider the case where a prior PDF for the output quantity is available, but the treatment here can be adapted to cover this case [\[16\]](#).

2 Normative references

The following referenced documents are indispensable for the application of this document. For dated references, only the edition cited applies. For undated references, the latest edition of the referenced document (including any amendments) applies.

ISO/IEC Guide 98-3:2008, *Uncertainty of measurement — Part 3: Guide to the expression of uncertainty in measurement (GUM:1995)*

ISO/IEC Guide 99:2007, *International vocabulary of metrology — Basic and general concepts and associated terms (VIM)*

3 Terms and definitions

For the purposes of this document, the terms and definitions of the ISO/IEC Guide 98-3 and the ISO/IEC Guide 99 apply unless otherwise indicated. Some of the most relevant definitions, adapted where necessary from these documents (see [4.2](#)), are given below. Further definitions are given, including definitions taken or adapted from other sources, that are important for this Supplement.

A glossary of principal symbols is given in Annex [G](#).

3.1 probability distribution

⟨random variable⟩ function giving the probability that a random variable takes any given value or belongs to a given set of values

NOTE The probability on the whole set of values of the random variable equals 1.

[Adapted from ISO 3534-1:1993, 1.3; ISO/IEC Guide 98-3:2008, C.2.3]

NOTE 1 A probability distribution is termed univariate when it relates to a single (scalar) random variable, and multivariate when it relates to a vector of random variables. A multivariate probability distribution is also described as a joint distribution.

NOTE 2 A probability distribution can take the form of a distribution function or a probability density function.

3.2

distribution function

function giving, for every value ξ , the probability that the random variable X be less than or equal to ξ :

$$G_X(\xi) = \Pr(X \leq \xi)$$

[Adapted from ISO 3534-1:1993, 1.4; ISO/IEC Guide 98-3:2008, C.2.4]

3.3

probability density function

derivative, when it exists, of the distribution function

$$g_X(\xi) = dG_X(\xi)/d\xi$$

NOTE $g_X(\xi) d\xi$ is the “probability element”

$$g_X(\xi) d\xi = \Pr(\xi < X < \xi + d\xi)$$

[Adapted from ISO 3534-1:1993, 1.5; ISO/IEC Guide 98-3:2008, C.2.5]

3.4

normal distribution

probability distribution of a continuous random variable X having the probability density function

$$g_X(\xi) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{\xi - \mu}{\sigma}\right)^2\right)$$

for $-\infty < \xi < +\infty$

NOTE μ is the expectation and σ is the standard deviation of X .

[Adapted from ISO 3534-1:1993, 1.37; ISO/IEC Guide 98-3:2008, C.2.14]

NOTE The normal distribution is also known as a Gaussian distribution.

3.5

t-distribution

probability distribution of a continuous random variable X having the probability density function

$$g_X(\xi) = \frac{\Gamma((\nu+1)/2)}{\sqrt{\pi\nu}\Gamma(\nu/2)} \left(1 + \frac{\xi^2}{\nu}\right)^{-(\nu+1)/2}$$

for $-\infty < \xi < +\infty$, with parameter ν , a positive integer, the degrees of freedom of the distribution, where

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt, \quad z > 0$$

is the gamma function

3.6 expectation
property of a random variable, which, for a continuous random variable X characterized by a PDF $g_X(\xi)$, is given by

$$E(X) = \int_{-\infty}^{\infty} \xi g_X(\xi) d\xi$$

NOTE 1 Not all random variables have an expectation.

NOTE 2 The expectation of the random variable $Z = F(X)$, for a given function $F(X)$, is

$$E(Z) = E[F(X)] = \int_{-\infty}^{\infty} F(\xi) g_X(\xi) d\xi$$

3.7 variance
property of a random variable, which, for a continuous random variable X characterized by a PDF $g_X(\xi)$, is given by

$$V(X) = \int_{-\infty}^{\infty} (\xi - E(X))^2 g_X(\xi) d\xi$$

NOTE Not all random variables have a variance.

3.8 standard deviation
positive square root $[V(X)]^{1/2}$ of the variance

3.9 moment of order r
expectation of the r th power of a random variable, namely

$$E(X^r) = \int_{-\infty}^{\infty} \xi^r g_X(\xi) d\xi$$

NOTE 1 The central moment of order r is the expectation of the random variable $Z = [X - E(X)]^r$.

NOTE 2 The expectation $E(X)$ is the first moment. The variance $V(X)$ is the central moment of order 2.

3.10 covariance
property of a pair of random variables, which, for two continuous random variables X_1 and X_2 characterized by a joint (multivariate) PDF $g_{\mathbf{X}}(\xi)$, where $\mathbf{X} = (X_1, X_2)^T$ and $\xi = (\xi_1, \xi_2)^T$, is given by

$$\text{Cov}(X_1, X_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} [\xi_1 - E(X_1)][\xi_2 - E(X_2)] g_{\mathbf{X}}(\xi) d\xi_1 d\xi_2$$

NOTE Not all pairs of random variables have a covariance.

3.11 uncertainty matrix
matrix of dimension $N \times N$, containing on its diagonal the squares of the standard uncertainties associated with estimates of the components of an N -dimensional vector quantity, and in its off-diagonal positions the covariances associated with pairs of estimates

NOTE 1 An uncertainty matrix U_x of dimension $N \times N$ associated with the vector estimate $\mathbf{x}$ of a vector quantity $\mathbf{X}$ has the representation

$$U_x = \begin{bmatrix} u(x_1, x_1) & \cdots & u(x_1, x_N) \\ \vdots & \ddots & \vdots \\ u(x_N, x_1) & \cdots & u(x_N, x_N) \end{bmatrix}$$

where $u(x_i, x_i) = u^2(x_i)$ is the variance (squared standard uncertainty) associated with x_i and $u(x_i, x_j)$ is the covariance associated with x_i and x_j . $u(x_i, x_j) = 0$ if elements X_i and X_j of $\mathbf{X}$ are uncorrelated.

NOTE 2 Covariances are also known as mutual uncertainties.

NOTE 3 An uncertainty matrix is also known as a covariance matrix or variance-covariance matrix.

3.12

coverage interval

interval containing the value of a quantity with a stated probability, based on the information available

NOTE 1 A coverage interval is sometimes known as a credible interval or a Bayesian interval.

NOTE 2 Generally there is more than one coverage interval for a stated probability.

NOTE 3 A coverage interval should not be termed 'confidence interval' to avoid confusion with the statistical concept [ISO/IEC Guide 98-3:2008, 6.2.2].

NOTE 4 This definition differs from that in the ISO/IEC Guide 99:2007, since the term 'true value' has not been used in this Supplement, for reasons given in the GUM [ISO/IEC Guide 98-3:2008, E.5].

3.13

coverage probability

probability that the value of a quantity is contained within a specified coverage interval

NOTE The coverage probability is sometimes termed "level of confidence" [ISO/IEC Guide 98-3:2008, 6.2.2].

3.14

length of a coverage interval

largest value minus smallest value in a coverage interval

3.15

probabilistically symmetric coverage interval

coverage interval for a quantity such that the probability that the quantity is less than the smallest value in the interval is equal to the probability that the quantity is greater than the largest value in the interval

3.16

shortest coverage interval

coverage interval for a quantity with the shortest length among all coverage intervals for that quantity having the same coverage probability

3.17

propagation of distributions

method used to determine the probability distribution for an output quantity from the probability distributions assigned to the input quantities on which the output quantity depends

NOTE The method may be analytical or numerical, exact or approximate.

3.18

GUM uncertainty framework

application of the law of propagation of uncertainty and the characterization of the output quantity by a Gaussian distribution or a scaled and shifted t -distribution in order to provide a coverage interval

3.19**Monte Carlo method**

method for the propagation of distributions by performing random sampling from probability distributions

3.20**numerical tolerance**

semi-width of the shortest interval containing all numbers that can correctly be expressed to a specified number of significant decimal digits

EXAMPLE All numbers greater than 1.75 and less than 1.85 can be expressed to two significant decimal digits as 1.8. The numerical tolerance is $(1.85 - 1.75)/2 = 0.05$.

NOTE For the calculation of numerical tolerance associated with a numerical value, see [7.9.2](#).

4 Conventions and notation

For the purposes of this Supplement, the following conventions and notation are adopted.

4.1 A mathematical model of a measurement [ISO/IEC Guide 98-3:2008, 4.1] of a single (scalar) quantity can be expressed as a functional relationship f :

$$Y = f(\mathbf{X}) \quad (1)$$

where Y is a scalar output quantity and $\mathbf{X}$ represents the N input quantities $(X_1, \dots, X_N)^T$. Each X_i is regarded as a random variable with possible values ξ_i and expectation x_i . Y is a random variable with possible values η and expectation y .

NOTE 1 The same symbol is used for a physical quantity and the random variable that represents that quantity (cf. [ISO/IEC Guide 98-3:2008, 4.1.1 Note 1]).

NOTE 2 Most models of measurement can be expressed in the form of Equation (1). A more general form is

$$h(Y, \mathbf{X}) = 0$$

which implicitly relates $\mathbf{X}$ and Y . In any case, to apply the described Monte Carlo method, it is only necessary that Y can be formed corresponding to any meaningful $\mathbf{X}$.

4.2 This Supplement departs from the symbols often used for 'PDF' and 'distribution function' [24]. The GUM uses the generic symbol f to refer to a model and a PDF. Little confusion arises in the GUM as a consequence of this usage. The situation in this Supplement is different. The concepts of model, PDF, and distribution function are central to following and implementing the guidance provided. Therefore, in place of the symbols f and F to denote a PDF and a distribution function, respectively, the symbols g and G are used. These symbols are indexed appropriately to denote the quantity concerned. The symbol f is reserved for the model.

NOTE The definitions in Clause 3 that relate to PDFs and distributions are adapted accordingly.

4.3 In this Supplement, a PDF is assigned to a quantity, which may be a single, scalar quantity X or a vector quantity $\mathbf{X}$. In the scalar case, the PDF for X is denoted by $g_X(\xi)$, where ξ is a variable describing the possible values of X . This X is considered as a random variable with expectation $E(X)$ and variance $V(X)$ ([3.6](#), [3.7](#)).

4.4 In the vector case, the PDF for $\mathbf{X}$ is denoted by $g_{\mathbf{X}}(\xi)$, where $\xi = (\xi_1, \dots, \xi_N)^T$ is a vector variable describing the possible values of the vector quantity $\mathbf{X}$. This $\mathbf{X}$ is considered as a random vector variable with (vector) expectation $E(\mathbf{X})$ and covariance matrix $V(\mathbf{X})$.

4.5 A PDF for more than one input quantity is often called joint even if all the input quantities are independent.

4.6 When the elements X_i of $\mathbf{X}$ are independent, the PDF for X_i is denoted by $g_{X_i}(\xi_i)$.

4.7 The PDF for Y is denoted by $g_Y(\eta)$ and the distribution function for Y by $G_Y(\eta)$.

4.8 In the body of this Supplement, a quantity is generally denoted by an upper case letter and the expectation of the quantity or an estimate of the quantity by the corresponding lower case letter. For example, the expectation or an estimate of a quantity Y would be denoted by y . Such a notation is largely inappropriate for physical quantities, because of the established use of specific symbols, e.g. T for temperature and t for time. Therefore, in some of the examples (Clause 9), a different notation is used. There, a quantity is denoted by its conventional symbol and its expectation or an estimate of it by that symbol hatted. For instance, the quantity representing the deviation of the length of a gauge block being calibrated from nominal length (9.5) is denoted by δL and an estimate of δL by $\widehat{\delta L}$.

NOTE A hatted symbol is generally used in the statistical literature to denote an estimate.

4.9 In this Supplement, the term “law of propagation of uncertainty” applies to the use of a first-order Taylor series approximation to the model. The term is qualified accordingly when a higher-order approximation is used.

4.10 The subscript “c” [ISO/IEC Guide 98-3:2008, 5.1.1] for the combined standard uncertainty is redundant in this Supplement. The standard uncertainty associated with an estimate y of an output quantity Y can therefore be written as $u(y)$, but the use of $u_c(y)$ remains acceptable if it is helpful to emphasize the fact that it represents a combined standard uncertainty. The qualifier “combined” in this context is also regarded as superfluous and may be omitted: the presence of “ y ” in “ $u(y)$ ” already indicates the estimate with which the standard uncertainty is associated. Moreover, when the results of one or more uncertainty evaluations become inputs to a subsequent uncertainty evaluation, the use of the subscript “c” and the qualifier “combined” are then inappropriate.

4.11 The terms “coverage interval” and “coverage probability” are used throughout this Supplement. The GUM uses the term “level of confidence” as a synonym for coverage probability, drawing a distinction between “level of confidence” and “confidence level” [ISO/IEC Guide 98-3:2008, 6.2.2], because the latter has a specific definition in statistics. Since, in some languages, the translation from English of these two terms yields the same expression, the use of these terms is avoided here.

4.12 According to Resolution 10 of the 22nd CGPM (2003) “... the symbol for the decimal marker shall be either the point on the line or the comma on the line ...”.

Exceptionally, for the decimal sign in this Guide 98 series, it has been decided to adopt the point on the line in the English texts and the comma on the line in the French texts.

4.13 Unless otherwise qualified, numbers are expressed in a manner that indicates the number of meaningful significant decimal digits.

EXAMPLE The numbers 0.060, 0.60, 6.0 and 60 are expressed to two significant decimal digits. The numbers 0.06, 0.6, 6 and 6×10^1 are expressed to one significant decimal digit. It would be incorrect to express 6×10^1 as 60, since two significant decimal digits would be implied.

4.14 Some symbols have more than one meaning in this Supplement. See Annex G. The context clarifies the usage.

4.15 The following abbreviations are used in this Supplement:

CGPM	Conférence Générale des Poids et Mesures
IEEE	Institute of Electrical and Electronic Engineers
GUF	GUM uncertainty framework
JCGM	Joint Committee for Guides in Metrology
GUM	Guide to the expression of uncertainty in measurement
MCM	Monte Carlo method
PDF	probability density function
VIM	International vocabulary of basic and general terms in metrology

5 Basic principles

5.1 Main stages of uncertainty evaluation

5.1.1 The main stages of uncertainty evaluation constitute formulation, propagation, and summarizing:

a) Formulation:

- 1) define the output quantity Y , the quantity intended to be measured (the measurand);
- 2) determine the input quantities $\mathbf{X} = (X_1, \dots, X_N)^T$ upon which Y depends;
- 3) develop a model relating Y and $\mathbf{X}$;
- 4) on the basis of available knowledge, assign PDFs—Gaussian (normal), rectangular (uniform), etc.—to the X_i . Assign instead a joint PDF to those X_i that are not independent;

b) Propagation:

propagate the PDFs for the X_i through the model to obtain the PDF for Y ;

c) Summarizing:

use the PDF for Y to obtain

- 1) the expectation of Y , taken as an estimate y of the quantity,
- 2) the standard deviation of Y , taken as the standard uncertainty $u(y)$ associated with y [ISO/IEC Guide 98-3:2008, E.3.2], and
- 3) a coverage interval containing Y with a specified probability (the coverage probability).

NOTE 1 The expectation may not be appropriate for all applications (cf. [ISO/IEC Guide 98-3:2008, 4.1.4]).

NOTE 2 The quantities described by some distributions, such as the Cauchy distribution, have no expectation or standard deviation. A coverage interval for the output quantity can always be obtained, however.

5.1.2 The GUM uncertainty framework does not explicitly refer to the assignment of PDFs to the input quantities. However [ISO/IEC Guide 98-3:2008, 3.3.5], "... a Type A standard uncertainty is obtained from a probability density function ... derived from an observed frequency distribution ..., while a Type B standard uncertainty is obtained from an assumed probability density function based on the degree of belief that an event will occur Both approaches employ recognized interpretations of probability."

NOTE The use of probability distributions in a Type B evaluation of uncertainty is a feature of Bayesian inference [21, 27]. Research continues [22] on the boundaries of validity for the assignment of degrees of freedom to a standard uncertainty based on the Welch-Satterthwaite formula.

5.1.3 The steps in the formulation stage are carried out by the metrologist, perhaps with expert support. Guidance on the assignment of PDFs (step 4) of stage a) in 5.1.1) is given in this Supplement for some common cases (6.4). The propagation and summarizing stages, b) and c) in 5.1.1, for which detailed guidance is provided here, require no further metrological information, and in principle can be carried out to any required numerical tolerance for the problem specified in the formulation stage.

NOTE Once the formulation stage a) in 5.1.1 has been carried out, the PDF for the output quantity is completely specified mathematically, but generally the calculation of the expectation, standard deviation and coverage intervals requires numerical methods that involve a degree of approximation.

5.2 Propagation of distributions

In this Supplement a generally efficient approach for determining (a numerical approximation to) the distribution function

$$G_Y(\eta) = \int_{-\infty}^{\eta} g_Y(z) \, dz$$

for Y is considered. It is based on applying a Monte Carlo method (MCM) as an implementation of the propagation of distributions (5.9).

NOTE A formal definition [9] for the PDF for Y is

$$g_Y(\eta) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} g_X(\xi) \delta(\eta - f(\xi)) \, d\xi_N \cdots d\xi_1$$

where $\delta(\cdot)$ denotes the Dirac delta function. This multiple integral cannot generally be evaluated analytically. A numerical integration rule can be applied to provide an approximation to $g_Y(\eta)$, but this is not an efficient approach.

5.3 Obtaining summary information

5.3.1 An estimate y of Y is the expectation $E(Y)$. The standard uncertainty $u(y)$ associated with y is given by the standard deviation of Y , the positive square root of the variance $V(Y)$ of Y .

5.3.2 A coverage interval for Y can be determined from $G_Y(\eta)$. Let α denote any numerical value between zero and $1 - p$, where p is the required coverage probability. The endpoints of a $100p$ % coverage interval for Y are $G_Y^{-1}(\alpha)$ and $G_Y^{-1}(p + \alpha)$, i.e. the α - and $(p + \alpha)$ -quantiles of $G_Y(\eta)$.

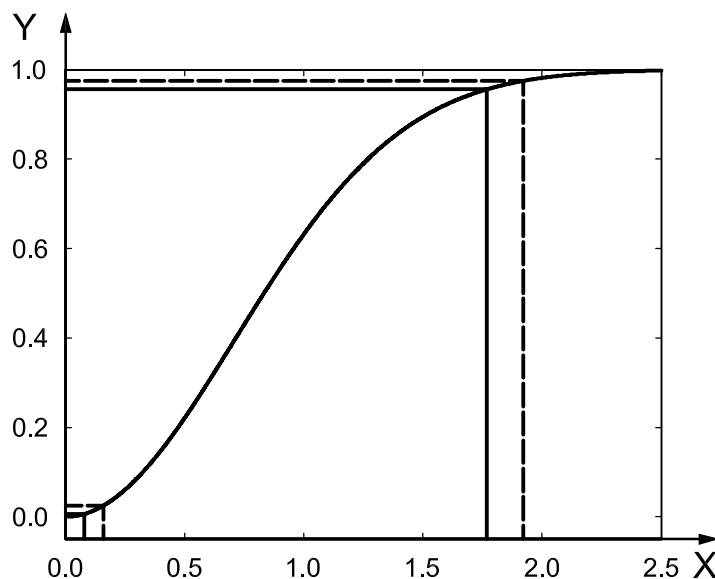
5.3.3 The choice $\alpha = (1 - p)/2$ gives the coverage interval defined by the $(1 - p)/2$ - and $(1 + p)/2$ -quantiles, providing a probabilistically symmetric $100p$ % coverage interval.

NOTE When the PDF for Y is symmetric about the estimate y , the coverage interval obtained would be identical to $y \pm U_p$, where the expanded uncertainty [ISO/IEC Guide 98-3:2008, 2.3.5] U_p is given by the product of the standard uncertainty $u(y)$ and the coverage factor that is appropriate for that PDF. This PDF is generally not known analytically.

5.3.4 A numerical value of α different from $(1 - p)/2$ may be more appropriate if the PDF is asymmetric. The shortest $100p$ % coverage interval can be used in this case. It has the property that, for a unimodal (single-peaked) PDF, it contains the mode, the most probable value of Y . It is given by the numerical value of α satisfying $g_Y(G_Y^{-1}(\alpha)) = g_Y(G_Y^{-1}(p + \alpha))$, if $g_Y(\eta)$ is unimodal, and in general by the numerical value of α such that $G_Y^{-1}(p + \alpha) - G_Y^{-1}(\alpha)$ is a minimum.

5.3.5 The probabilistically symmetric $100p$ % coverage interval and the shortest $100p$ % coverage interval are identical for a symmetric PDF, such as the Gaussian and scaled and shifted t -distribution used within the GUM uncertainty framework. Therefore, in comparing the GUM uncertainty framework with other approaches, either of these intervals can be used.

5.3.6 Figure 1 shows the distribution function $G_Y(\eta)$ corresponding to an asymmetric PDF. Broken vertical lines mark the endpoints of the probabilistically symmetric 95 % coverage interval and broken horizontal lines mark the corresponding probability points, viz. 0.025 and 0.975. Continuous lines mark the endpoints of the shortest 95 % coverage interval and the corresponding probability points, which are 0.006 and 0.956 in this case. The lengths of the 95 % coverage intervals in the two cases are 1.76 unit and 1.69 unit, respectively.

**Key**X Output quantity Y/unit

Y Probability

"Unit" denotes any unit.

Figure 1 — A distribution function $G_Y(\eta)$ corresponding to an asymmetric PDF and the probabilistically symmetric and shortest 95 % coverage intervals (5.3.6)

5.4 Implementations of the propagation of distributions

5.4.1 The propagation of distributions can be implemented in several ways:

- analytical methods, i.e. methods that provide a mathematical representation of the PDF for Y ;
- uncertainty propagation based on replacing the model by a first-order Taylor series approximation [ISO/IEC Guide 98-3:2008, 5.1.2] — the law of propagation of uncertainty;
- as b), except that contributions derived from higher-order terms in the Taylor series approximation are included [ISO/IEC Guide 98-3:2008, 5.1.2 Note];
- numerical methods [ISO/IEC Guide 98-3:2008, G.1.5] that implement the propagation of distributions, specifically using MCM (5.9).

NOTE 1 Analytical methods are ideal in that they do not introduce any approximation. They are applicable in simple cases only, however. A treatment and examples are available [8, 13]. These methods are not considered further in this Supplement, apart from in the examples (Clause 9) for comparison purposes.

NOTE 2 MCM as considered here is regarded as a means for providing a numerical representation of the distribution for the output quantity, rather than a simulation method *per se*. In the context of the propagation stage of uncertainty evaluation, the problem to be solved is deterministic, there being no random physical process to be simulated.

5.4.2 Approaches to uncertainty evaluation other than the GUM uncertainty framework are permitted by the GUM [ISO/IEC Guide 98-3:2008, G.1.5]. The approach advocated in this Supplement, based on the propagation of distributions, is general. For linear or linearized models and input quantities for which the PDFs are Gaussian, the approach yields results consistent with the GUM uncertainty framework. However, in cases where the conditions for the GUM uncertainty framework to be applied (5.7 and 5.8) do not hold, the approach of this Supplement can generally be expected to lead to a valid uncertainty statement.

5.4.3 An appropriate method has to be chosen for the propagation stage. If it can be demonstrated that the conditions necessary for the GUM uncertainty framework to give valid results hold, then that approach can be used. If there are indications that the GUM uncertainty framework is likely to be invalid, then another approach should be employed. A third situation can arise in which it is difficult to assess whether or not the GUM uncertainty framework will be valid. In all three cases, MCM provides a practical (alternative) method. In the first case, MCM may sometimes be easier to apply due to difficulties in calculating sensitivity coefficients [ISO/IEC Guide 98-3:2008, 5.1.3], for example. In the second, MCM can generally be expected to give valid results, since it does not make approximating assumptions. In the third, MCM can be applied either to determine the results directly or to assess the quality of those provided by the GUM uncertainty framework.

5.4.4 The propagation of the PDFs $g_{X_i}(\xi_i)$, $i = 1, \dots, N$, for the input quantities X_i through the model to provide the PDF $g_Y(\eta)$ for the output quantity Y is illustrated in Figure 2 for $N = 3$ independent X_i . Figure 2 may be compared to Figure 3 for the law of propagation of uncertainty. In Figure 2, the $g_{X_i}(\xi_i)$, $i = 1, 2, 3$, are Gaussian, triangular, and Gaussian, respectively. $g_Y(\eta)$ is indicated as being asymmetric, as generally arises for non-linear models or asymmetric $g_{X_i}(\xi_i)$.

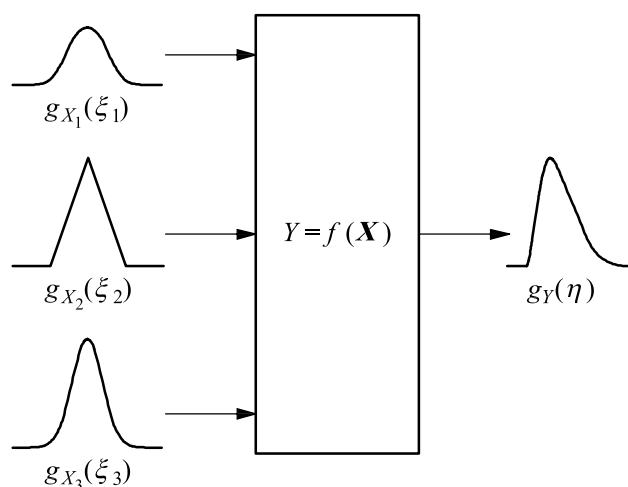


Figure 2 — Illustration of the propagation of distributions for $N = 3$ independent input quantities (5.4.4)

5.4.5 In practice, only for simple cases can the propagation of distributions be implemented without making approximations. The GUM uncertainty framework implements one approximate method, and MCM another. For a small but important subset of problems, the GUM uncertainty framework is exact. MCM is never exact, but is more valid than the GUM uncertainty framework for a large class of problems.

5.5 Reporting the results

5.5.1 The following items would typically be reported following the use of the propagation of distributions:

- an estimate y of the output quantity Y ;
- the standard uncertainty $u(y)$ associated with y ;
- the stipulated coverage probability $100p$ % (e.g. 95 %);
- the endpoints of the selected $100p$ % coverage interval (e.g. 95 % coverage interval) for Y ;
- any other relevant information, such as whether the coverage interval is a probabilistically symmetric coverage interval or a shortest coverage interval.

5.5.2 y , $u(y)$ and the endpoints of a 100% coverage interval for Y should be reported to a number of decimal digits such that the least significant decimal digit is in the same position with respect to the decimal point as that for $u(y)$ [ISO/IEC Guide 98-3:2008, 7.2.6]. One or two significant decimal digits would usually be adequate to represent $u(y)$.

NOTE 1 Each reported numerical value would typically be obtained by rounding a numerical value expressed to a greater number of significant decimal digits.

NOTE 2 A factor influencing the choice of one or two significant decimal digits is the leading significant decimal digit of $u(y)$. If this digit is 1 or 2, the deviation of the reported numerical value of $u(y)$ from its numerical value before rounding is large relative to the latter numerical value. If the leading significant decimal digit is 9, the deviation is relatively smaller.

NOTE 3 If the results are to be used within further calculations, consideration should be given to whether additional decimal digits should be retained.

EXAMPLE Reported results corresponding to declaring two significant decimal digits in $u(y)$, for a case in which the coverage interval is asymmetric with respect to y , are

$y = 1.024 \text{ V}$, $u(y) = 0.028 \text{ V}$,
shortest 95 % coverage interval = [0.983, 1.088] V.

The same results reported to one significant decimal digit in $u(y)$ would be

$y = 1.02 \text{ V}$, $u(y) = 0.03 \text{ V}$,
shortest 95 % coverage interval = [0.98, 1.09] V.

5.6 GUM uncertainty framework

5.6.1 The GUM provides general guidance on many aspects of the stages of uncertainty evaluation presented in 5.1.1. It also provides the GUM uncertainty framework for the propagation and summarizing stages of uncertainty evaluation. The GUM uncertainty framework has been adopted by many organizations, is widely used, and has been implemented in standards and guides on measurement uncertainty and also in software.

5.6.2 The GUM uncertainty framework comprises the following stages. Each model input quantity X_i is summarized by its expectation and standard deviation, as given by the PDF for that quantity [ISO/IEC Guide 98-3:2008, 4.1.6]. The expectation is taken as the best estimate x_i of X_i and the standard deviation as the standard uncertainty $u(x_i)$ associated with x_i . This information is propagated, using the law of propagation of uncertainty [ISO/IEC Guide 98-3:2008, 5.1.2], through a first- or higher-order Taylor series approximation to the model to provide

- a) an estimate y of the output quantity Y and
- b) the standard uncertainty $u(y)$ associated with y .

The estimate y is given by evaluating the model at the x_i . A coverage interval for Y is provided based on taking the PDF for Y as Gaussian or, if the degrees of freedom associated with $u(y)$ is finite [ISO/IEC Guide 98-3:2008, Annex G], as a scaled and shifted t -distribution.

NOTE The summaries of the X_i also include, where appropriate, the degrees of freedom associated with the $u(x_i)$ [ISO/IEC Guide 98-3:2008, 4.2.6]. They also include, where appropriate, covariances associated with pairs of x_i [ISO/IEC Guide 98-3:2008, 5.2.5].

5.6.3 The propagation and summarizing stages of the GUM uncertainty framework (stages **b**) and **c**) in 5.1.1) constitute the following computational steps. Also see Figure 3, which illustrates the law of propagation of uncertainty for a model having $N = 3$ independent input quantities $\mathbf{X} = (X_1, X_2, X_3)^T$, which are estimated by x_i with associated standard uncertainties $u(x_i)$, $i = 1, 2, 3$. The output quantity Y is estimated by y , with associated standard uncertainty $u(y)$.

- a) Obtain from the PDFs for the input quantities $\mathbf{X} = (X_1, \dots, X_N)^T$ the expectations $\mathbf{x} = (x_1, \dots, x_N)^T$ and the standard deviations (standard uncertainties) $\mathbf{u}(\mathbf{x}) = [u(x_1), \dots, u(x_N)]^T$. Use instead the joint PDF for $\mathbf{X}$ if pairs of the X_i are not independent (in which case they have non-zero covariance).
- b) Set the degrees of freedom (infinite or finite) associated with each $u(x_i)$.
- c) For each pair i, j for which X_i and X_j are not independent, obtain from the joint PDF for X_i and X_j the covariance (mutual uncertainty) $u(x_i, x_j)$ associated with x_i and x_j .
- d) Form the partial derivatives of first order of $f(\mathbf{X})$ with respect to $\mathbf{X}$.
- e) Calculate y , the model evaluated at $\mathbf{X}$ equal to $\mathbf{x}$.
- f) Calculate the model sensitivity coefficients [ISO/IEC Guide 98-3:2008, 5.1.3] as the above partial derivatives evaluated at $\mathbf{x}$.
- g) Calculate the standard uncertainty $u(y)$ by combining $\mathbf{u}(\mathbf{x})$, the $u(x_i, x_j)$, and the model sensitivity coefficients [ISO/IEC Guide 98-3:2008, Formulae (10), (13)].
- h) Calculate ν_{eff} , the effective degrees of freedom associated with $u(y)$, using the Welch-Satterthwaite formula [ISO/IEC Guide 98-3:2008, Formula (G.2 b)].
- i) Calculate the expanded uncertainty U_p , and hence a coverage interval (for a stipulated coverage probability p) for Y , regarded as a random variable, by forming the appropriate multiple of $u(y)$ through taking the probability distribution of $(Y - y)/u(y)$ as a standard Gaussian distribution ($\nu_{\text{eff}} = \infty$) or t -distribution ($\nu_{\text{eff}} < \infty$).

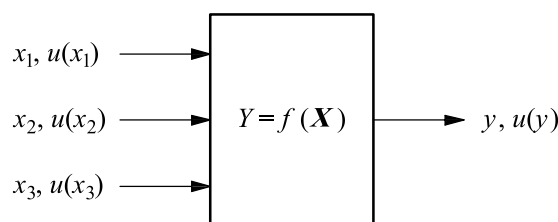


Figure 3 — Illustration of the law of propagation of uncertainty for $N = 3$ independent input quantities (5.4.4 and 5.6.3)

5.7 Conditions for valid application of the GUM uncertainty framework for linear models

5.7.1 No condition is necessary for the valid application of the law of propagation of uncertainty to linear models (models that are linear in the X_i).

5.7.2 A coverage interval can be determined, in terms of the information provided in the GUM, under the following conditions:

- a) the Welch-Satterthwaite formula is adequate for calculating the effective degrees of freedom associated with $u(y)$ [ISO/IEC Guide 98-3:2008, G.4.1], when one or more of the $u(x_i)$ has an associated degree of freedom that is finite;
- b) the X_i are independent when the degrees of freedom associated with the $u(x_i)$ are finite;
- c) the PDF for Y can adequately be approximated by a Gaussian distribution or a scaled and shifted t -distribution.

NOTE 1 Condition [a\)](#) is required in order that Y can be characterized by an appropriate scaled and shifted t -distribution.

NOTE 2 Condition [b\)](#) is required because the GUM does not treat X_i that are not independent in conjunction with finite degrees of freedom.

NOTE 3 Condition [c\)](#) is satisfied when each X_i is assigned a Gaussian distribution. It is also satisfied when the conditions for the central limit theorem [ISO/IEC Guide 98-3:2008, G.2] hold.

NOTE 4 The GUM uncertainty framework may not validly be applicable when there is an X_i whose assigned distribution is non-Gaussian and the corresponding contribution to $u(y)$ is dominant.

5.7.3 When the conditions in [5.7.2](#) hold, the results from the application of the GUM uncertainty framework can be expected to be valid for linear models. These conditions apply in many circumstances.

5.8 Conditions for valid application of the GUM uncertainty framework for non-linear models

5.8.1 The law of propagation of uncertainty can validly be applied for non-linear models under the following conditions:

- a) f is continuously differentiable with respect to the elements X_i of $\mathbf{X}$ in the neighbourhood of the best estimates x_i of the X_i ;
- b) condition [a\)](#) applies for all derivatives up to the appropriate order;
- c) the X_i involved in significant higher-order terms of a Taylor series approximation to $f(\mathbf{X})$ are independent;
- d) the PDFs assigned to X_i involved in higher-order terms of a Taylor series approximation to $f(\mathbf{X})$ are Gaussian;
- e) higher-order terms that are not included in the Taylor series approximation to $f(\mathbf{X})$ are negligible.

NOTE 1 Condition [a\)](#) is necessary for the applicability of the law of propagation of uncertainty based on a first-order Taylor series approximation to $f(\mathbf{X})$ when the non-linearity of f is insignificant [ISO/IEC Guide 98-3:2008, 5.1.2].

NOTE 2 Condition [b\)](#) is necessary for the application of the law of propagation of uncertainty based on a higher-order Taylor series approximation to $f(\mathbf{X})$ [ISO/IEC Guide 98-3:2008, 5.1.2]. An expression for the most important terms of next highest order to be included are given in the GUM [ISO/IEC Guide 98-3:2008, 5.1.2 Note].

NOTE 3 Condition [c\)](#) relates to the statement in the GUM [ISO/IEC Guide 98-3:2008, 5.1.2 Note] concerning significant model non-linearity in the case of independent X_i . The GUM does not consider X_i that are not independent in this context.

NOTE 4 Condition [d\)](#) constitutes a correction to the statement in the GUM [ISO/IEC Guide 98-3:2008, 5.1.2 Note] that the version of the law of propagation of uncertainty using higher-order terms is based on the symmetry of the PDFs for the X_i [[19](#), [27](#)].

NOTE 5 If the analytical determination of the higher derivatives, required when the non-linearity of the model is significant, is difficult or error-prone, suitable software for automatic differentiation can be used. Alternatively, these derivatives can be approximated numerically using finite differences [[5](#)]. (The GUM provides a finite-difference formula for partial derivatives of first order [ISO/IEC Guide 98-3:2008, 5.1.3 Note 2].) Care should be taken, however, because of the effects of subtractive cancellation when forming differences between numerically close model values.

5.8.2 A coverage interval can be determined, in terms of the information provided in the GUM, when conditions [a\)](#), [b\)](#) and [c\)](#) in [5.7.2](#) apply, with the exception that the content of Note [3](#) in that subclause is replaced by “Condition [c\)](#) is required in order that coverage intervals can be determined from these distributions.”

5.8.3 When the conditions in [5.8.1](#) and [5.8.2](#) hold, the results from the application of the GUM uncertainty framework can be expected to be valid for non-linear models. These conditions apply in many circumstances.

5.9 Monte Carlo approach to the propagation and summarizing stages

5.9.1 MCM provides a general approach to obtain an approximate numerical representation $\mathbf{G}$, say, of the distribution function $G_Y(\eta)$ for Y [32, page 75]. The heart of the approach is repeated sampling from the PDFs for the X_i and the evaluation of the model in each case.

5.9.2 Since $G_Y(\eta)$ encodes all the information known about Y , any property of Y such as expectation, variance and coverage intervals can be approximated using $\mathbf{G}$. The quality of these calculated results improves as the number of times the PDFs are sampled increases.

5.9.3 Expectations and variances (and higher moments) can be determined directly from the set of model values obtained. The determination of coverage intervals requires these model values to be ordered.

5.9.4 If y_r , for $r = 1, \dots, M$, represent M model values sampled independently from a probability distribution for Y , then the expectation $E(Y)$ and variance $V(Y)$ can be approximated using the y_r . In general, the moments of Y [including $E(Y)$ and $V(Y)$] are approximated by those of the sampled model values. Let M_{y_0} denote the number of y_r that are no greater than y_0 , any prescribed number. The probability $\Pr(Y \leq y_0)$ is approximated by M_{y_0}/M . In this way, the y_r provide a step function (histogram-like) approximation to the distribution function $G_Y(\eta)$.

5.9.5 Each y_r is obtained by sampling at random from each of the PDFs for the X_i and evaluating the model at the sampled values so obtained. $\mathbf{G}$, the primary output from MCM, constitutes the y_r arranged in strictly increasing order.

NOTE It is remotely possible that equalities exist amongst the y_r , in which case suitable minute perturbations made to the y_r would enable the y_r to be arranged in strictly increasing order. See 7.5.1.

5.9.6 MCM as an implementation of the propagation of distributions is shown diagrammatically in Figure 4 for M provided in advance (see 7.9 otherwise). MCM can be stated as a step-by-step procedure:

- a) select the number M of Monte Carlo trials to be made. See 7.2;
- b) generate M vectors, by sampling from the assigned PDFs, as realizations of the (set of N) input quantities X_i . See 7.3;
- c) for each such vector, form the corresponding model value of Y , yielding M model values. See 7.4;
- d) sort these M model values into strictly increasing order, using the sorted model values to provide $\mathbf{G}$. See 7.5;
- e) use $\mathbf{G}$ to form an estimate y of Y and the standard uncertainty $u(y)$ associated with y . See 7.6;
- f) use $\mathbf{G}$ to form an appropriate coverage interval for Y , for a stipulated coverage probability p . See 7.7.

NOTE 1 Subclause 6.4 and Annex C provide information on sampling from probability distributions.

NOTE 2 Mathematically, the average of the M model values is a realization of a random variable with expectation $E(Y)$ and variance $V(Y)/M$. Thus, the closeness of agreement between this average and $E(Y)$ can be expected to be proportional to $M^{-1/2}$.

NOTE 3 Step e) can equally be carried out by using the M model values of Y unsorted. It is necessary to sort these model values to determine the coverage interval in step f).

5.9.7 The effectiveness of MCM to determine y , $u(y)$ and a coverage interval for Y depends on the use of an adequately large value of M [step a) in 5.9.6]. Guidance on obtaining such a value and generally on implementing MCM is available [7]. Also see 7.2 and 7.9.

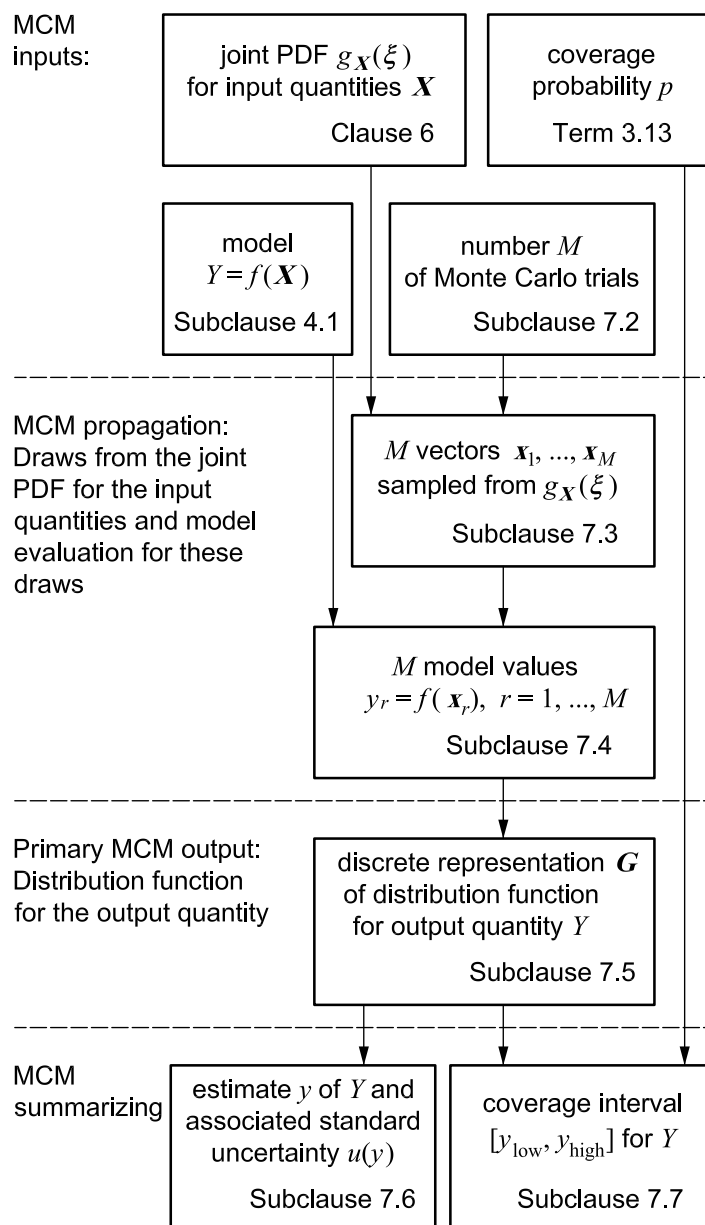


Figure 4 — The propagation and summarizing stages of uncertainty evaluation using MCM to implement the propagation of distributions (5.9.6 and 7.1)

5.10 Conditions for the valid application of the described Monte Carlo method

5.10.1 The propagation of distributions implemented using MCM can validly be applied, and the required summary information subsequently determined, using the approach provided in this Supplement, under the following conditions:

- f is continuous with respect to the elements X_i of X in the neighbourhood of the best estimates x_i of the X_i ;
- the distribution function for Y is continuous and strictly increasing;
- the PDF for Y is
 - continuous over the interval for which this PDF is strictly positive,

- 2) unimodal (single-peaked), and
- 3) strictly increasing (or zero) to the left of the mode and strictly decreasing (or zero) to the right of the mode;
- d) $E(Y)$ and $V(Y)$ exist;
- e) a sufficiently large value of M is used.

NOTE 1 Regarding condition [a\)](#), no condition on the derivatives of f is required.

NOTE 2 Conditions [a\)](#) and [b\)](#) are necessary to ensure that the inverse of the distribution function is unique and hence coverage intervals can be determined. Only condition [a\)](#) is needed if a coverage interval is not required.

NOTE 3 Condition [c\)](#) is needed only if the shortest coverage interval is to be determined. In that case, the condition is necessary to ensure that the shortest coverage interval corresponding to a stipulated coverage probability is unique. The mode may occur at an endpoint of the interval over which this PDF is strictly positive, in which case one of the two conditions in [3\)](#) is vacuous.

NOTE 4 Condition [d\)](#) is needed for (stochastic) convergence of MCM as the number M of trials ([7.2](#)) increases.

NOTE 5 Condition [e\)](#) is necessary to ensure that the summarizing information is reliable. See [8.2](#).

5.10.2 When the conditions in [5.10.1](#) hold, the results from the application of the propagation of distributions implemented in terms of MCM can be expected to be valid. These conditions are less restrictive than those ([5.7](#) and [5.8](#)) for the application of the GUM uncertainty framework.

5.11 Comparison of the GUM uncertainty framework and the described Monte Carlo method

5.11.1 The intention of this subclause is to compare the principles on which the GUM uncertainty framework and MCM as an implementation of the propagation of distributions are based. This subclause also provides some motivation for the use of MCM in circumstances where it is questionable whether the application of the GUM uncertainty framework is valid.

5.11.2 For the purposes of comparing the GUM uncertainty framework and MCM, it is helpful to review the considerations in the GUM regarding Type A and Type B evaluations of uncertainty. For Type A evaluation, the GUM provides guidance on obtaining a best estimate of a quantity and the associated standard uncertainty from the average and the associated standard deviation of a set of indications of the quantity, obtained independently. For Type B evaluation, prior knowledge concerning the quantity is used to characterize the quantity by a PDF, from which a best estimate of the quantity and the standard uncertainty associated with that estimate are determined. The GUM states that both types of evaluation are based on probability distributions [ISO/IEC Guide 98-3:2008, 3.3.4], and that both approaches employ recognized interpretations of probability [ISO/IEC Guide 98-3:2008, 3.3.5]. The GUM considers PDFs as underpinning uncertainty evaluation: in the context of the law of propagation of uncertainty, it refers explicitly to input and output quantities as being describable or characterized by probability distributions [ISO/IEC Guide 98-3:2008, G.6.6]. Also see [5.1.2](#).

5.11.3 The GUM uncertainty framework does not explicitly determine a PDF for the output quantity. However, the probability distribution used by that framework to characterize the output quantity is sometimes referred to in this Supplement as “provided by” or “resulting from” the GUM uncertainty framework.

5.11.4 This Supplement attempts to provide an approach that is as consistent with the GUM as possible, especially relating to the use of PDFs for all quantities, but departs from it in a clearly identified way where appropriate. These departures are:

- a) PDFs are explicitly assigned to all input quantities X_i (rather than associating standard uncertainties with estimates x_i of X_i) based on information concerning these quantities. The classification into Type A and Type B evaluations of uncertainty is not needed;
- b) sensitivity coefficients [ISO/IEC Guide 98-3:2008, 5.1.3] are not an inherent part of the approach, and hence the calculation or numerical approximation of the partial derivatives of the model with respect to the X_i is not required. Approximations to sensitivity coefficients can, however, be provided that

correspond to taking all higher-order terms in the Taylor series expansion of the model into account (Annex B);

- c) a numerical representation of the distribution function for Y is obtained that is defined completely by the model and the PDFs for the X_i , and not restricted to a Gaussian distribution or scaled and shifted t -distribution;
- d) since the PDF for Y is not in general symmetric, a coverage interval for Y is not necessarily centred on the estimate of Y . Consideration therefore needs to be given to the choice of coverage interval corresponding to a specified coverage probability.

5.11.5 Since the GUM uncertainty framework explicitly uses only best estimates x_i and the associated uncertainties (and covariances and degrees of freedom where appropriate), it is restricted in the information it can provide about Y . Essentially it is limited to providing an estimate y of Y and the standard uncertainty $u(y)$ associated with y , and perhaps the related (effective) degrees of freedom. y and $u(y)$ will be valid for a model that is linear in X . Any other information about Y , e.g. coverage intervals, is derived using additional assumptions, e.g. that the distribution for Y is Gaussian or a scaled and shifted t -distribution.

5.11.6 Some features of MCM are

- a) reduction in the analysis effort required for complicated or non-linear models, especially since the partial derivatives of first- or higher-order used in providing sensitivity coefficients for the law of propagation of uncertainty are not needed,
- b) generally improved estimate of Y for non-linear models (cf. [ISO/IEC Guide 98-3:2008, 4.1.4]),
- c) improved standard uncertainty associated with the estimate of Y for non-linear models, especially when the X_i are assigned non-Gaussian (e.g. asymmetric) PDFs, without the need to provide derivatives of higher order [ISO/IEC Guide 98-3:2008, 5.1.2 Note],
- d) provision of a coverage interval corresponding to a stipulated coverage probability when the PDF for Y cannot adequately be approximated by a Gaussian distribution or a scaled and shifted t -distribution, i.e. when the central limit theorem does not apply [ISO/IEC Guide 98-3:2008, G.2.1, G.6.6]. Such an inadequate approximation can arise when (1) the PDF assigned to a dominant X_i is not a Gaussian distribution or a scaled and shifted t -distribution, (2) the model is non-linear, or (3) the approximation error incurred in using the Welch-Satterthwaite formula for effective degrees of freedom is not negligible, and
- e) a coverage factor [ISO/IEC Guide 98-3:2008, 2.3.6] is not required when determining a coverage interval.

6 Probability density functions for the input quantities

6.1 General

6.1.1 This clause gives guidance on the assignment, in some common circumstances, of PDFs to the input quantities X_i in the formulation stage of uncertainty evaluation. Such an assignment can be based on Bayes' theorem [20] or the principle of maximum entropy [8, 26, 51, 56].

NOTE In some circumstances, another approach for assigning a PDF may be useful. In any case, as in any scientific discipline, the reason for the decision should be recorded.

6.1.2 Generally, a joint PDF $g_{\mathbf{X}}(\xi)$ is assigned to the input quantities $\mathbf{X} = (X_1, \dots, X_N)^T$. See 6.4.8.4 Note 2.

6.1.3 When the X_i are independent, PDFs $g_{X_i}(\xi_i)$ are assigned individually based on an analysis of a series of indications (Type A evaluation of uncertainty) or based on scientific judgement using information [50] such as historical data, calibrations, and expert judgement (Type B evaluation of uncertainty) [ISO/IEC Guide 98-3:2008, 3.3.5].

6.1.4 When some of the X_i are mutually independent, PDFs are assigned individually to them and a joint PDF to the remainder.

NOTE It may be possible to remove some or all dependencies by re-expressing relevant input quantities in terms of more fundamental independent input quantities on which the original input quantities depend [ISO/IEC Guide 98-3:2008, F1.2.4, H.1.2]. Such changes can simplify both the application of the law of propagation of uncertainty and the propagation of distributions. Details and examples are available [15].

6.1.5 Information relevant to the assignment of PDFs to the X_i is contained in the GUM [ISO/IEC Guide 98-3:2008, 4.3].

6.1.6 Comprehensive guidance on the assignment of PDFs individually or jointly to the X_i is beyond the scope of this Supplement. Such assigned PDFs encode the knowledge and expertise of the metrologist who formulates the model and who is ultimately responsible for the quality of the final results.

6.1.7 A standard text on probability distributions is Evans, Hastings and Peacock [18].

6.2 Bayes' theorem

6.2.1 Suppose that information about an input quantity X consists of a series of indications regarded as realizations of independent, identically distributed random variables characterized by a specified PDF, but with unknown expectation and variance. Bayes' theorem can be used to calculate a PDF for X , where X is taken to be equal to the unknown average of these random variables. Calculation proceeds in two steps. First, a non-informative joint prior (pre-data) PDF is assigned to the unknown expectation and variance. Using Bayes' theorem, this joint prior PDF is then updated, based on the information supplied by the series of indications, to yield a joint posterior (post-data) PDF for the two unknown parameters. The desired posterior PDF for the unknown average is then calculated as a marginal PDF by integrating over the possible values of the unknown variance (6.4.9.2).

6.2.2 With the use of Bayes' theorem, the updating is carried out by forming the product of a likelihood function and the prior PDF [20]. The likelihood function is the product of functions, one function for each indication and identical in form, e.g. to a Gaussian PDF with expectation equal to the indication and variance formally equal to the unknown variance. The posterior PDF is then determined by integrating this product over all possible values of the variance and normalizing the resulting expression.

NOTE 1 In some cases (e.g. as in 6.4.11), the random variables, of which the indications are regarded as realizations, are characterized by a PDF with only one parameter. In such cases, a non-informative prior PDF is assigned to the unknown expectation of the random variables, and the posterior distribution for X is given directly by Bayes' theorem, without the need for marginalization.

NOTE 2 Bayes' theorem can also be applied in other circumstances, e.g. when the expectation and standard deviation are unknown and equal.

6.3 Principle of maximum entropy

6.3.1 When using the principle of maximum entropy, introduced by Jaynes [25], a unique PDF is selected among all possible PDFs having specified properties, e.g. specified central moments of different orders or specified intervals for which the PDF is non-zero. This method is particularly useful for assigning PDFs to quantities for which a series of indications is not available or to quantities that have not explicitly been measured at all.

6.3.2 In applying the principle of maximum entropy, to obtain a PDF $g_X(\xi)$ that adequately characterizes incomplete knowledge about a quantity X according to the information available, the functional

$$S[g] = - \int g_X(\xi) \ln g_X(\xi) d\xi$$

the "information entropy", introduced by Shannon [48], is maximized under constraints given by the information.

6.4 Probability density function assignment for some common circumstances

6.4.1 General

Subclauses 6.4.2 to 6.4.11 provide assignments of PDFs to quantities based on various types of information regarding those quantities. Given for each PDF $g_X(\xi)$ are

- a) formulae for the expectation and variance of X , and
- b) the manner in which sampling from $g_X(\xi)$ can be undertaken.

Table 1 facilitates the use of these subclauses and also illustrates the corresponding PDFs.

NOTE These illustrations of the PDFs are not drawn to scale. The multivariate Gaussian PDF is not illustrated.

6.4.2 Rectangular distributions

6.4.2.1 If the only available information regarding a quantity X is a lower limit a and an upper limit b with $a < b$, then, according to the principle of maximum entropy, a rectangular distribution $R(a, b)$ over the interval $[a, b]$ would be assigned to X .

6.4.2.2 The PDF for X is

$$g_X(\xi) = \begin{cases} 1/(b-a), & a \leq \xi \leq b, \\ 0, & \text{otherwise.} \end{cases}$$

6.4.2.3 X has expectation and variance

$$E(X) = \frac{a+b}{2}, \quad V(X) = \frac{(b-a)^2}{12} \quad (2)$$

6.4.2.4 To sample from $R(a, b)$, make a draw r from the standard rectangular distribution $R(0, 1)$ (C.3.3), and form

$$\xi = a + (b-a)r$$

6.4.3 Rectangular distributions with inexactly prescribed limits

6.4.3.1 A quantity X is known to lie between limits A and B with $A < B$, where the midpoint $(A+B)/2$ of the interval defined by these limits is fixed and the length $B-A$ of the interval is not known exactly. A is known to lie in the interval $a \pm d$ and B in $b \pm d$, where a, b and d , with $d > 0$ and $a+d < b-d$, are specified. If no other information is available concerning X , A and B , the principle of maximum entropy can be applied to assign to X a "curvilinear trapezoid" (a rectangular distribution with inexactly prescribed limits).

6.4.3.2 The PDF for X is

$$g_X(\xi) = \frac{1}{4d} \begin{cases} \ln((w+d)/(x-\xi)), & a-d \leq \xi \leq a+d, \\ \ln((w+d)/(w-d)), & a+d < \xi < b-d, \\ \ln((w+d)/(\xi-x)), & b-d \leq \xi \leq b+d, \\ 0, & \text{otherwise,} \end{cases} \quad (3)$$

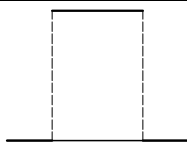
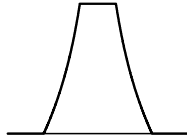
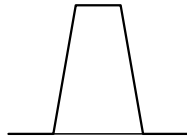
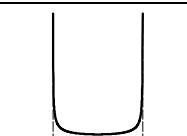
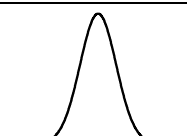
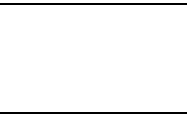
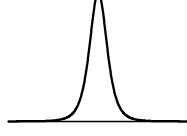
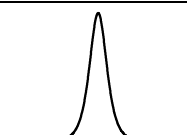
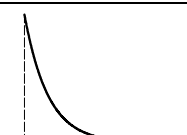
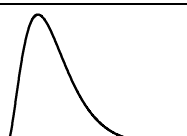
where $x = (a+b)/2$ and $w = (b-a)/2$ are, respectively, the midpoint and semi-width of the interval $[a, b]$ [ISO/IEC Guide 98-3:2008, 4.3.9 Note 2]. This PDF is trapezoidal-like, but has flanks that are not straight lines.

NOTE Formula (3) can be expressed as

$$g_X(\xi) = \frac{1}{4d} \max \left(\ln \frac{w+d}{\max(|\xi-x|, w-d)}, 0 \right)$$

for computer implementation.

Table 1 — Available information and the PDF assigned on the basis of that information
(6.4.1 and C.1.2)

Available information	Assigned PDF and illustration (not to scale)	Subclause
Lower and upper limits a, b	Rectangular: $R(a, b)$	 6.4.2
Inexact lower and upper limits $a \pm d, b \pm d$	Curvilinear trapezoid: $CTrap(a, b, d)$	 6.4.3
Sum of two quantities assigned rectangular distributions with lower and upper limits a_1, b_1 and a_2, b_2	Trapezoidal: $Trap(a, b, \beta)$ with $a = a_1 + a_2$, $b = b_1 + b_2$, $\beta = (b_1 - a_1) - (b_2 - a_2) / (b - a)$	 6.4.4
Sum of two quantities assigned rectangular distributions with lower and upper limits a_1, b_1 and a_2, b_2 and the same semi-width ($b_1 - a_1 = b_2 - a_2$)	Triangular: $T(a, b)$ with $a = a_1 + a_2, b = b_1 + b_2$	 6.4.5
Sinusoidal cycling between lower and upper limits a, b	Arc sine (U-shaped): $U(a, b)$	 6.4.6
Best estimate x and associated standard uncertainty $u(x)$	Gaussian: $N(x, u^2(x))$	 6.4.7
Best estimate x of vector quantity and associated uncertainty matrix U_x	Multivariate Gaussian: $N(x, U_x)$	 6.4.8
Series of indications $x_1, \dots, x_n$ sampled independently from a quantity having a Gaussian distribution, with unknown expectation and unknown variance	Scaled and shifted t : $t_{n-1}(\bar{x}, s^2/n)$ with $\bar{x} = \sum_{i=1}^n x_i/n$, $s^2 = \sum_{i=1}^n (x_i - \bar{x})^2 / (n-1)$	 6.4.9.2
Best estimate x , expanded uncertainty U_p , coverage factor k_p and effective degrees of freedom ν_{eff}	Scaled and shifted t : $t_{\nu_{\text{eff}}}(x, (U_p/k_p)^2)$	 6.4.9.7
Best estimate x of non-negative quantity	Exponential: $Ex(1/x)$	 6.4.10
Number q of objects counted	Gamma: $G(q+1, 1)$	 6.4.11

6.4.3.3 X has expectation and variance

$$E(X) = \frac{a+b}{2}, \quad V(X) = \frac{(b-a)^2}{12} + \frac{d^2}{9} \quad (4)$$

NOTE 1 The variance in Expression (4) is always greater than the variance holding for exact limits in Expression (2), i.e. when $d = 0$.

NOTE 2 The GUM treats the information about X in 6.4.3.1 by assigning a degrees of freedom to the standard uncertainty associated with the best estimate of X [ISO/IEC Guide 98-3:2008, G.4.2].

6.4.3.4 To sample from $\text{CTrap}(a, b, d)$, make two draws r_1 and r_2 independently from the standard rectangular distribution $R(0, 1)$ (C.3.3), and form

$$a_s = (a - d) + 2dr_1, \quad b_s = (a + b) - a_s$$

and

$$\xi = a_s + (b_s - a_s)r_2$$

NOTE a_s is a draw from the rectangular distribution with limits $a \pm d$. b_s is then formed to ensure that the midpoint of a_s and b_s is the prescribed value $x = (a + b)/2$.

EXAMPLE A certificate states that a voltage X lies in the interval $10.0 \text{ V} \pm 0.1 \text{ V}$. No other information is available concerning X , except that it is believed that the magnitude of the interval endpoints is the result of rounding correctly some numerical value (3.20). On this basis, that numerical value lies between 0.05 V and 0.15 V , since the numerical value of every point in the interval $(0.05, 0.15)$ rounded to one significant decimal digit is 0.1 . The location of the interval can therefore be regarded as fixed, whereas its width is inexact. The best estimate of X is $x = 10.0 \text{ V}$ and, using Expression (4) based on $a = 9.9 \text{ V}$, $b = 10.1 \text{ V}$ and $d = 0.05 \text{ V}$, the associated standard uncertainty $u(x)$ is given by

$$u^2(x) = \frac{(0.2)^2}{12} + \frac{(0.05)^2}{9} = 0.003 \ 6$$

Hence $u(x) = (0.003 \ 6)^{1/2} = 0.060 \text{ V}$, which can be compared with $0.2/\sqrt{12} = 0.058 \text{ V}$ in the case of exact limits, given by replacing d by zero. The use of exact limits in this case gives a numerical value for $u(x)$ that is 4 % smaller than that for inexact limits. The relevance of such a difference needs to be considered in the context of the application.

6.4.4 Trapezoidal distributions

6.4.4.1 The assignment of a symmetric trapezoidal distribution to a quantity is discussed in the GUM [ISO/IEC Guide 98-3:2008, 4.3.9]. Suppose a quantity X is defined as the sum of two independent quantities X_1 and X_2 . Suppose, for $i = 1$ and $i = 2$, X_i is assigned a rectangular distribution $R(a_i, b_i)$ with lower limit a_i and upper limit b_i . Then the distribution for X is a symmetric trapezoidal distribution $\text{Trap}(a, b, \beta)$ with lower limit a , upper limit b , and a parameter β equal to the ratio of the semi-width of the top of the trapezoid to that of the base. The parameters of this trapezoidal distribution are related to those of the rectangular distributions by

$$a = a_1 + a_2, \quad b = b_1 + b_2, \quad \beta = \frac{\lambda_1}{\lambda_2} \quad (5)$$

where

$$\lambda_1 = \frac{|(b_1 - a_1) - (b_2 - a_2)|}{2}, \quad \lambda_2 = \frac{b - a}{2} \quad (6)$$

and

$$0 \leq \lambda_1 \leq \lambda_2$$