



Recommendation ITU-R P.1057-5
(12/2017)

**Probability distributions relevant to
radiowave propagation modelling**

P Series
Radiowave propagation



Foreword

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Note: This ITU-R Recommendation was approved in English under the procedure detailed in Resolution ITU-R 1.

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RECOMMENDATION ITU-R P.1057-5

Probability distributions relevant to radiowave propagation modelling

(1994-2001-2007-2013-2015-2017)

Scope

This Recommendation describes the various probability distributions relevant to radiowave propagation modelling and prediction methods.

Keywords

Probability distributions, normal, Gaussian, log-normal, Rayleigh, Nakagami-Rice, gamma, exponential, Pearson

The ITU Radiocommunication Assembly,

considering

- a) that the propagation of radio waves is mainly associated with a random medium that makes it necessary to analyse propagation phenomena by means of statistical methods;
- b) that, in most cases, it is possible to describe satisfactorily the variations in time and space of propagation parameters by known statistical probability distributions;
- c) that it is important to know the fundamental properties of the probability distributions most commonly used in statistical propagation studies,

recommends

- 1 that the statistical information relevant to propagation modelling provided in Annex 1 should be used in the planning of radiocommunication services and the prediction of system performance parameters;
- 2 that the step-by-step procedure provided in Annex 2 should be used to approximate a complementary cumulative probability distribution by a log-normal complementary cumulative probability distribution.

Annex 1**Probability distributions relevant to radiowave propagation modelling****1 Introduction**

Experience has shown that information on the mean values of received signals is not sufficient to accurately characterize the performance of radiocommunication systems. The variations in time, space, and frequency should also be considered.

The dynamic behaviour of both wanted signals and interference plays a significant role in the analysis of system reliability and in the choice of system parameters such as modulation type. It is essential to know the probability distribution and rate of signal fluctuations in order to specify parameters such as modulation type, transmit power, protection ratio against interference, diversity measures, coding method, etc.

For the description of communication system performance, it is often sufficient to observe the time series of signal fluctuation and characterize these fluctuations as a stochastic process. Modelling of signal fluctuations for the purpose of predicting radio system performance requires knowledge of the mechanisms of the interaction of radio waves with the neutral atmosphere and ionosphere.

The composition and physical state of the atmosphere is highly variable in space and time. Wave interaction modelling, therefore, requires extensive use of statistical methods to characterize various physical parameters describing the atmosphere as well as electrical parameters defining signal behaviour and the interactive processes which these parameters are related.

In the following, some general information is given on the most important probability distributions. This may provide a common background to the statistical methods for propagation prediction used in the Recommendations of the Radiocommunication Study Groups.

2 Probability distributions

Probability distributions of stochastic processes are generally described either by a probability density function (PDF) or by a cumulative distribution function (CDF). The probability density function of the random variable X , denoted by $p(x)$, is the probability of X taking a value of x ; and the cumulative distribution function of the random variable X , denoted by $F(x)$, is the probability of X taking a value less than or equal to x . The PDF and CDF are related as follows:

$$p(x) = \frac{d}{dx}[F(x)] \quad (1a)$$

or:

$$F(x) = \int_c^x p(t) dt \quad (1b)$$

where c is the lower limit of integration.

The following probability distributions are the most important for the analysis of radiowave propagation:

- normal or Gaussian probability distribution;
- log-normal probability distribution;
- Rayleigh probability distribution;
- combined log-normal and Rayleigh probability distribution;
- Nakagami-Rice (Nakagami n) probability distribution;
- gamma probability distribution and exponential probability distribution;
- Nakagami m probability distribution;
- Pearson χ^2 probability distribution.

3 Normal probability distribution

The normal (Gaussian) probability distribution of a propagation random variable is usually encountered when a random variable is the sum of a large number of other random variables.

The normal (Gaussian) probability distribution is a continuous probability distribution in the interval from $x = -\infty$ to $+\infty$. The probability density function (PDF), $p(x)$, of a normal distribution is:

$$p(x) = k e^{-T(x)} \quad (2)$$

where $T(x)$ is a non-negative second degree polynomial of the form $\left(\frac{x-m}{\sigma}\right)^2$, where m and σ are the mean and standard deviation, respectively, of the normal probability distribution, and k is selected so $\int_{-\infty}^{\infty} p(x) dx = 1$. Then $p(x)$ is:

$$p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2\right) \quad (3)$$

The cumulative distribution function (CDF), $F(x)$, of a normal probability distribution is:

$$F(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^x \exp\left(-\frac{1}{2}\left(\frac{t-m}{\sigma}\right)^2\right) dt \quad (3a)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{x-m}{\sigma}} \exp\left(-\frac{1}{2}t^2\right) dt \quad (3b)$$

$$= \frac{1}{2} \left[1 + \operatorname{erf}\left(\frac{x-m}{\sigma\sqrt{2}}\right) \right] \quad (3c)$$

where:

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-t^2) dt \quad (3d)$$

The complementary cumulative distribution function (CCDF), $Q(x)$, of a normal probability distribution is:

$$Q(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \int_x^{\infty} \exp\left(-\frac{1}{2}\left(\frac{t-m}{\sigma}\right)^2\right) dt \quad (4a)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{\frac{x-m}{\sigma}}^{\infty} \exp\left(-\frac{1}{2}t^2\right) dt \quad (4b)$$

$$= \frac{1}{2} \operatorname{erfc}\left(\frac{x-m}{\sigma\sqrt{2}}\right) \quad (4c)$$

where:

$$\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^{\infty} \exp(-t^2) dt \quad (4d)$$

Note that $F(x) + Q(x) = 1$ and $\operatorname{erf}(x) + \operatorname{erfc}(x) = 1$.

The inverse cumulative distribution function $x = F^{-1}(p)$ is the value of x such that $F(x) = p$; and the inverse complementary cumulative distribution function $x = Q^{-1}(p)$ is the value of x such that $Q(x) = p$.

The solid lines in Fig. 1 represent the functions $p(x)$ and $F(x)$ with $m = 0$ and $\sigma = 1$, and Table 1 shows the correspondence between x and $1 - F(x)$ for various example values of x or $1 - F(x)$.

TABLE 1

x	$1 - F(x)$	x	$1 - F(x)$
0	0.5	1.282	10^{-1}
1	0.1587	2.326	10^{-2}
2	0.02275	3.090	10^{-3}
3	1.350×10^{-3}	3.719	10^{-4}
4	3.167×10^{-5}	4.265	10^{-5}
5	2.867×10^{-7}	4.753	10^{-6}
6	9.866×10^{-10}	5.199	10^{-7}
		5.612	10^{-8}

For practical calculations, the following simple approximation for $Q(x) = 1 - F(x)$ is valid for any positive x and has a relative approximation error of less than 2.8×10^{-3} :

$$Q(x) = 1 - F(x) = \frac{\exp\left(-\frac{x^2}{2}\right)}{\sqrt{2\pi}(0.661x + 0.339\sqrt{x^2 + 5.51})} \quad (5)$$

Most modern mathematical software packages include the $F(x)$, $Q(x)$, $\text{erf}(x)$, and $\text{erfc}(x)$ functions.

In propagation, most of the physical quantities involved (power, voltage, fading time, etc.) are essentially positive quantities and cannot be represented directly by a normal probability distribution. The normal probability distribution is used in two important cases:

- to represent the fluctuations of a random variable around its mean value (e.g. scintillation fades and enhancements);
- to represent the fluctuations of the logarithm of a random variable, in which case the variable has a log-normal probability distribution (see § 4).

Diagrams in which one of the coordinates is a so-called normal coordinate, where a normal cumulative probability distribution is represented by a straight line, are available commercially. These diagrams are frequently used even for the representation of non-normal probability distributions.

4 Log-normal probability distribution

The log-normal probability distribution is the probability distribution of a positive random variable X whose natural logarithm has a normal probability distribution. The probability density function, $p(x)$, and the cumulative distribution function, $F(x)$, are:

$$p(x) = \frac{1}{\sigma\sqrt{2\pi}} \frac{1}{x} \exp\left[-\frac{1}{2}\left(\frac{\ln x - m}{\sigma}\right)^2\right] \quad (6)$$

$$F(x) = \frac{1}{\sigma\sqrt{2\pi}} \int_0^x \frac{1}{t} \exp\left[-\frac{1}{2}\left(\frac{\ln t - m}{\sigma}\right)^2\right] dt = \frac{1}{2} \left[1 + \text{erf}\left(\frac{\ln x - m}{\sigma\sqrt{2}}\right) \right] \quad (7)$$

where m and σ are the mean and the standard deviation of the logarithm of X (i.e. not the mean and the standard deviation of X).

The log-normal probability distribution is very often found in propagation probability distributions associated with power and field-strength. Since power and field-strength are generally expressed in decibels, their probability distributions are sometimes incorrectly referred to as normal rather than log-normal. In the case of probability distributions vs. time (e.g. fade duration in seconds), the log-normal terminology is always used explicitly because the natural dependent variable is time rather than the logarithm of time.

Since the reciprocal of a variable with a log-normal probability distribution also has a log-normal probability distribution, this probability distribution is sometimes found in the case of the probability distribution of the rate of change (e.g. fading rate in dB/s or rainfall rate in mm/hr).

In comparison with a normal probability distribution, a log-normal probability distribution is usually encountered when values of the random variable of interest results from the product of other approximately equally weighted random variables.

Unlike a normal probability distribution, a log-normal probability distribution is extremely asymmetrical. In particular, the mean value, the median value, and the most probable value (often called the mode) are not identical (see the dashed lines in Fig. 1).

The characteristic values of the random variable X are:

- most probable value: $\exp (m - \sigma^2)$;
- median value: $\exp (m)$;
- mean value: $\exp \left(m + \frac{\sigma^2}{2} \right)$;
- root mean square value: $\exp (m + \sigma^2)$;
- standard deviation: $\exp \left(m + \frac{\sigma^2}{2} \right) \sqrt{\exp (\sigma^2) - 1}$.