

RECOMMENDATION ITU-R PN.1057

**PROBABILITY DISTRIBUTIONS RELEVANT TO
RADIOWAVE PROPAGATION MODELLING**

(1994)

The ITU Radiocommunication Assembly,

considering

- a) that the propagation of radio waves is mainly associated with a random medium which makes it necessary to analyse propagation phenomena by means of statistical methods;
- b) that, in most cases, it is possible to describe satisfactorily the variations in time and space of propagation parameters by known statistical distributions;
- c) that it is therefore important to know the fundamental properties of the probability distributions most commonly used in statistical propagation studies,

recommends

- 1. that the statistical information relevant to propagation modelling provided in Annex 1 be used in the planning of radiocommunication services and the prediction of system performance parameters.

ANNEX 1

Probability distributions relevant to radiowave propagation modelling**1. Introduction**

Experience has shown that information on the mean values of the signals received is not sufficient to characterize the performance of radiocommunication systems. The variations in time, space and frequency also have to be taken into consideration.

The dynamic behaviour of both wanted signals and interference plays a decisive role in the analysis of system reliability and in the choice of system parameters such as modulation type. It is essential to know the extent and rapidity of signal fluctuations in order to be able to specify such parameters as type of modulation, transmit power, protection ratio against interference, diversity measures, coding method, etc.

For the description of communication system performance it is often sufficient to observe the time series of signal fluctuation and characterize these fluctuations as a stochastic process. Modelling of signal fluctuations for the purpose of predicting radio system performance, however, requires also knowledge of the mechanisms of interaction of radio waves with the atmosphere (neutral atmosphere and the ionosphere).

The composition and physical state of the atmosphere is highly variable in space and time. Wave interaction modelling, therefore, requires extensive use of statistical methods to characterize various physical parameters describing the atmosphere as well as electrical parameters defining signal behaviour and the interaction processes via which these parameters are related.

In the following, some general information is given on the most important probability distributions. This may provide a common background to the statistical methods for propagation prediction used in the Recommendations of the ITU-R Study Groups.

2. Probability distributions

Stochastic processes are generally described either by a probability density function or by a cumulative distribution function. The probability density function, here denoted by $p(x)$ for the variable x , is such that the probability of x taking a value in the infinitesimal interval x to $x + dx$ is $p(x) dx$. The cumulative distribution function, denoted by $F(x)$, gives the probability that the variable takes a value less than x , i.e. the functions are related as follows:

$$p(x) = \frac{d}{dx} [F(x)]$$

or:

$$F(x) = \int_c^x p(t) dt$$

where c is the lowest limit of the values which t can take.

The following distributions are the most important:

- normal or Gaussian distribution,
- log-normal distribution,
- Rayleigh distribution,
- combined log-normal and Rayleigh distribution,
- Nakagami-Rice distribution (Nakagami n -distribution),
- gamma distribution and exponential distribution,
- Nakagami m -distribution,
- Pearson χ^2 distribution.

3. Gaussian or normal distribution

This distribution is applied to a continuous variable of any sign. The probability density is of the type:

$$p(x) = e^{-T(x)} \quad (1)$$

$T(x)$ being a non-negative second degree polynomial. If as parameters we use the mean, m , and the standard deviation, σ , then $p(x)$ is written in the usual way:

$$p(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp \left[-\frac{1}{2} \left(\frac{x-m}{\sigma} \right)^2 \right] \quad (2)$$

hence:

$$F(x) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^x \exp \left[-\frac{1}{2} \left(\frac{t-m}{\sigma} \right)^2 \right] dt = \frac{1}{2} \left[1 + \operatorname{erf} \left(\frac{x-m}{\sigma\sqrt{2}} \right) \right] \quad (3)$$

with:

$$\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} dt \quad (4)$$

The cumulative normal distribution $F(x)$ is generally tabulated in a short form, i.e. with m taken to be zero and σ equal to unity. Table 1 gives the correspondence between x and $F(x)$ for a number of round values of x or $F(x)$.

TABLE 1

x	$1 - F(x)$	x	$1 - F(x)$
0	$5.000\ 00 \times 10^{-1}$	1.281 55	10^{-1}
1	$1.586\ 55 \times 10^{-1}$	2.326 35	10^{-2}
2	$2.275\ 01 \times 10^{-2}$	3.090 23	10^{-3}
3	$1.349\ 90 \times 10^{-3}$	3.719 01	10^{-4}
4	$3.167\ 12 \times 10^{-5}$	4.264 89	10^{-5}
5	$2.866\ 52 \times 10^{-7}$	4.753 42	10^{-6}
6	$9.865\ 87 \times 10^{-10}$	5.199 34	10^{-7}
		5.612 00	10^{-8}

For the purpose of practical calculations, $F(x)$ can be represented by approximate functions, for example the following which is valid for positive x with a relative error less than 2.8×10^{-3} :

$$1 - F(x) = \frac{\exp(-x^2/2)}{\sqrt{2\pi} \left(0.661 x + 0.339 \sqrt{x^2 + 5.51} \right)} \quad (5)$$

A Gaussian distribution is mainly encountered when values of the quantity considered result from the additive effect of numerous random causes, each of them of relatively slight importance.

In propagation most of the physical quantities involved (power, voltage, fading time, etc.) are essentially positive quantities and cannot therefore be represented directly by a Gaussian distribution. On the other hand this distribution is used in two important cases:

- to represent the fluctuations of a quantity around its mean value (scintillation);
- to represent the logarithm of a quantity. We then obtain the log-normal distribution which is studied later.

Diagrams in which one of the coordinates is a so-called Gaussian coordinate are available commercially, i.e. the graduation is such that a Gaussian distribution is represented by a straight line. These diagrams are very frequently used even for the representation of non-Gaussian distributions.

4. Log-normal distribution

This is the distribution of a positive variable whose logarithm has a Gaussian distribution. It is possible therefore to write directly the probability density and the cumulative density:

$$p(x) = \frac{1}{\sigma \sqrt{2\pi}} \frac{1}{x} \exp \left[-\frac{1}{2} \left(\frac{\ln x - m}{\sigma} \right)^2 \right] \quad (6)$$

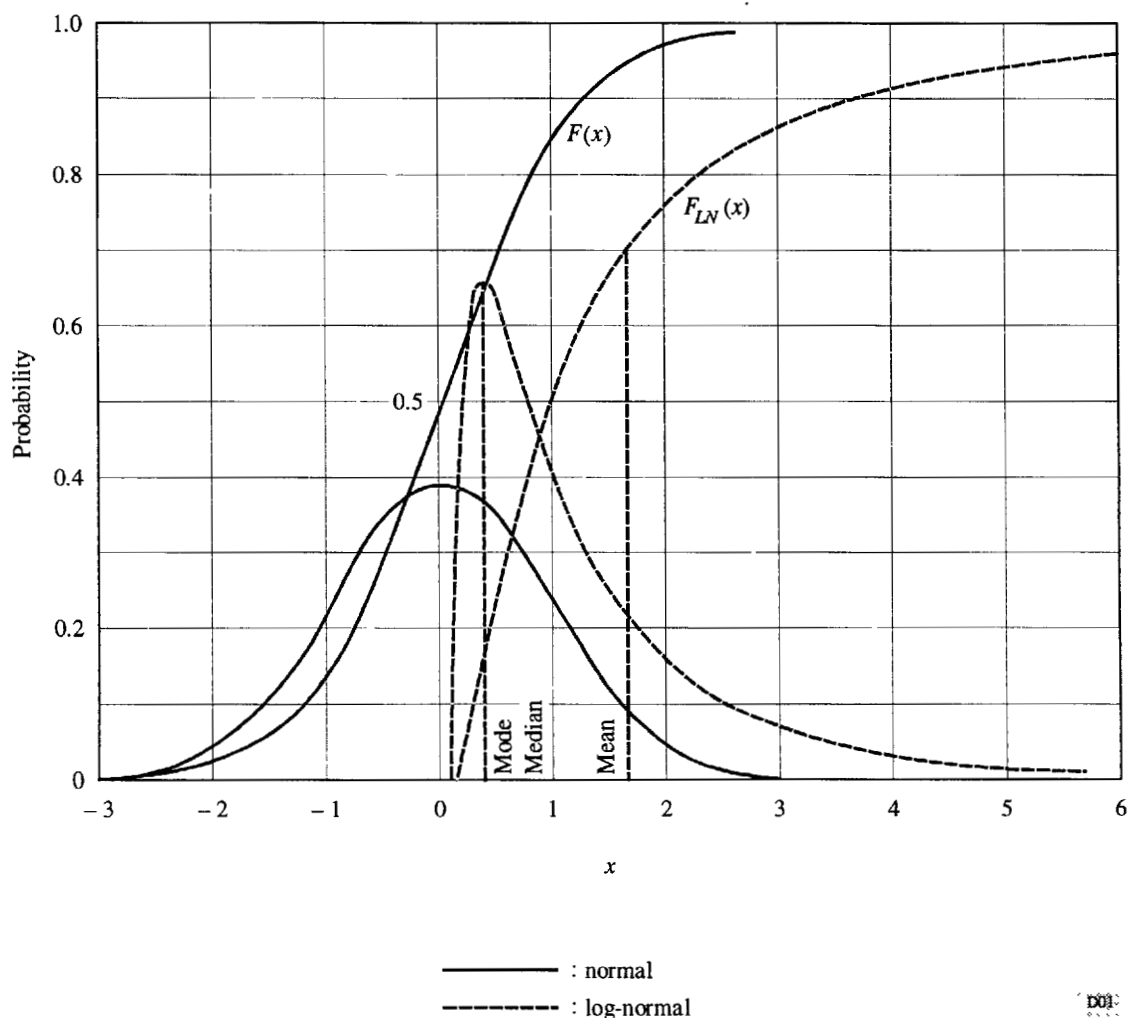
$$F(x) = \frac{1}{\sigma \sqrt{2\pi}} \int_0^x \frac{1}{t} \exp \left[-\frac{1}{2} \left(\frac{\ln t - m}{\sigma} \right)^2 \right] dt = \frac{1}{2} \left[1 + \operatorname{erf} \left(\frac{\ln x - m}{\sigma \sqrt{2}} \right) \right] \quad (7)$$

However, in these relations m and σ are the mean and the standard deviation not of the variable x but of the logarithm of this variable. The characteristic quantities of the variable x can be derived without difficulty. We find:

- most probable value: $\exp(m - \sigma^2)$
- median value: $\exp(m)$
- mean value: $\exp\left(m + \frac{\sigma^2}{2}\right)$
- root mean square value: $\exp(m + \sigma^2)$
- standard deviation: $\exp\left(m + \frac{\sigma^2}{2}\right) \sqrt{\exp(\sigma^2) - 1}$

Unlike the Gaussian distribution, a log-normal distribution is extremely asymmetrical. In particular, the mean value, the median value and the most probable value (often called the mode) are not identical (see Fig. 1).

FIGURE 1
Normal and log-normal distributions



The log-normal distribution is very often found in connection with propagation, mainly for quantities associated either with a power or field-strength level or a time. Power or field-strength levels are generally only expressed in decibels so that it is more usual to refer to a normal distribution of levels. In the case of time (for example fading durations), the log-normal distribution is used explicitly because the natural variable is the second or the minute and not their logarithm.

Since the reciprocal of a variable with a log-normal distribution also has a log-normal distribution, this distribution is sometimes found in the case of rates (reciprocals of time). For example, it is used to represent rainfall rate distribution at least for low and medium rainfall rates.

In comparison with a Gaussian distribution, it can be considered that a log-normal distribution means that the numerical values of the variable are the result of the action of numerous causes of slight individual importance which are multiplicative.

5. Rayleigh distribution

The Rayleigh distribution applies to a non-limited positive continuous variable. The probability density and the cumulative distribution are given by:

$$p(x) = \frac{x}{q^2} \exp\left(-\frac{x^2}{2q^2}\right) \quad (8)$$

$$F(x) = 1 - \exp\left(-\frac{x^2}{2q^2}\right) \quad (9)$$

Figure 2 represents these functions $p(x)$ and $F(x)$.

FIGURE 2
Rayleigh distribution

