

Efficient Re-Analysis Methodology for Probabilistic Vibration of Large-Scale Structures

Geng Zhang
Oakland University

Efstratios Nikolaidis
The University of Toledo

Zissimos P. Mourelatos
Oakland University

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ABSTRACT

It is challenging to perform probabilistic analysis and design of large-scale structures because probabilistic analysis requires repeated finite element analyses of large models and each analysis is expensive. This paper presents a methodology for probabilistic analysis and reliability based design optimization of large scale structures that consists of two re-analysis methods; one for estimating the deterministic vibratory response and another for estimating the probability of the response exceeding a certain level. The deterministic re-analysis method can analyze efficiently large-scale finite element models consisting of tens or hundreds of thousand degrees of freedom and large numbers of design variables that vary in a wide range. The probabilistic re-analysis method calculates very efficiently the system reliability for many probability distributions of the design variables by performing a single Monte Carlo simulation.

The methodology is demonstrated on probabilistic vibration analysis and reliability-based design optimization of a realistic vehicle model. It is shown that computational cost of the proposed re-analysis method for a single reliability analysis is about $1/20^{\text{th}}$ of the cost of the same analysis using NASTRAN. Moreover, the probabilistic re-analysis approach enables a designer to perform reliability based design optimization of the vehicle at a cost almost equal to that of a single reliability analysis. Without using the probabilistic reanalysis approach, it would be impractical to perform reliability based design optimization of the vehicle.

1. INTRODUCTION

1.1 PROBABILISTIC ANALYSIS AND DESIGN OF STRUCTURES

Continuous efforts of the aerospace, automotive and shipbuilding industries to minimize the structural weight of vehicles while maintaining an acceptable level of safety compel engineers to develop reliable and efficient design optimization techniques. In many structural design situations the operating conditions, material properties and geometry are uncertain. Optimizing the design in the face of uncertainty requires quantification and control of the risk of failure. For this purpose, these industries are trying to incorporate risk-based analysis and design in product development.

Probabilistic methods are being increasingly accepted by the industry [1-8]. These methods use probability theory and statistics to model uncertainty and to determine the probability of failure of a design. Then a designer can perform optimization to find the design with minimum weight for a given allowable probability of failure. A designer faces many challenges when applying reliability analysis and design to real-world problems. Finding the probability of failure of a given design requires repeating the structural analysis for different sets of the values of the random variables, which may be computationally expensive, especially when using finite element analysis [1-3]. Also, the computational expense may be grossly compounded when calculating the probability of failure of different designs during the search for an optimum reliability-based design.

Two methods are commonly used to calculate the reliability of a design; Monte-Carlo simulation techniques [1-3, 8-11] and analytical reliability approximate methods (e.g., the First Order Reliability Method, the Second

Order Reliability Method and the Advanced Mean Value Method). The latter methods find the Most Probable Point (MPP) for the design, which is an optimization problem [1-3, 12-15].

Attempts have been made to reduce the cost of reliability based analysis and design. For example, some researchers have developed efficient surrogate models (e.g., response surface polynomials and neural networks) to approximate the structural response and then used these approximate models to perform reliability-based design [16-21]. Others approximated the reliability function itself [22, 23]. Yet, these approximate models can be computationally very expensive to develop or may be highly inaccurate [24], especially when there are many random variables (e.g., more than 10).

An efficient probabilistic re-analysis approach (PRA) has been proposed to estimate the failure probability of a design very efficiently for many different probability distributions of the random variables using results of a single Monte Carlo simulation. PRA is based on the concept of importance sampling (IS). PRA also estimates the sensitivity derivatives of the failure probability to the distribution parameters of the random variables. PRA can be used in reliability-based design optimization (RBDO) problems where the design variables are random.

Another way to reduce the computational effort of probabilistic analysis and design is by constructing *physics-based approximations* of the deterministic response. These approximations use basis functions obtained by analyzing representative designs, instead of generic basis functions, such as polynomials or exponential functions. Because of the superior quality of the basis functions, physics based approximations are accurate over a wide range of the random variables and still they are more efficient than structural analysis. This paper focuses on the use of two physics-based approximations; parametric reduced basis models and the combined approximations method for probabilistic analysis of complex structures. It also integrates these methods with PRA in order to develop an efficient methodology for probabilistic analysis and RBDO of realistic structures.

1.2 APPROXIMATION METHODS FOR DETERMINISTIC STRUCTURAL ANALYSIS

Finite element analysis (FEA) is a well-established numerical simulation method for structural dynamics. It serves as the main computational tool for NVH analysis in the low-frequency range. As a result of developments in numerical methods, as well as advances in computer software and hardware, FEA can now handle much more complex models far more efficiently than even a few years ago. However, the demand for computational capabilities increases in step with, or even beyond the

pace of these improvements. For example, automotive companies are constructing increasingly detailed models with millions of degrees of freedom (DOF) to study vibro-acoustic problems in higher frequency ranges. Although these tasks can be performed with FEA, the computational costs are enormous, even for high-end workstations with the most advanced software.

There are also other factors that increase the computational costs. When design changes are involved, the FEA analysis must be repeated many times in order to obtain the optimum design. Furthermore in probabilistic analysis where parameter uncertainties are present, the FEA analysis must be repeated for a large number of sample points. In such cases, the computational cost is even higher, if not prohibitive.

Re-analysis methods are intended to analyze efficiently structures that are modified due to various changes. The objective is to evaluate the structural response for such changes without solving the complete set of modified analysis equations. Several reviews have been published on re-analysis methods [25-27]. These methods are usually based on local and global approximations. Local approximations are very efficient but they are effective only for small structural changes. Global approximations are preferable for large changes, but they are usually computationally expensive especially for cases with many design parameters. The well-known Rayleigh-Ritz re-analysis procedure [28, 29] belongs to the category of local approximation methods. The mode shapes of a nominal design can be used to form a Ritz basis for predicting the response in a small parametric zone around the nominal design point. However, the Rayleigh-Ritz re-analysis procedure can not capture relatively large design changes.

A parametric reduced-order modeling (PROM) method, developed by Balmes [30, 31], expands on the Rayleigh-Ritz method, by using the mode shapes from a few selected design points to predict the response throughout the design space. The PROM method still belongs to the category of local approximation methods, but it can handle relatively larger parameter changes because it uses multiple design points. An improved component-based PROM method has been introduced by Zhang et. al. [32, 33] for design changes at the component level. The method was implemented in NASTRAN DMAP (Direct Matrix Abstraction Program), and its efficiency and accuracy were demonstrated using a medium size, finite-element vehicle model.

The PROM method using a 'parametric' approach (see section 2.2), has been successfully applied to design optimization and probabilistic analysis of vehicle structures. However, the 'parametric' approach is only applicable to problems where the mass and stiffness matrices can be approximated by a polynomial function of the design parameters and their powers.

The Combined Approximations (CA) method, which has been developed recently by Kirsch [34-36], combines the strengths of both local and global approximations, being accurate even for large design changes. The CA method is a combination of binomial series (local) approximations (also called Neumann expansion approximations) and reduced basis (global) approximations. Originally, the CA method was developed for linear static re-analysis. However, the method has been recently extended to eigen-problem re-analysis [37-41]. Accurate results and significant computational savings have been reported. A comparative study [42] has shown that the method is suitable for large design changes.

All reported studies on the CA method [34-41], have used relatively simple frame or truss systems for static or dynamics analysis with a relatively small number of DOF and/or modes. For these problems, the computational efficiency was improved by a factor of 5 to 10. Such an improvement is beneficial for a single design change evaluation or even for gradient-based design optimization, where only a limited number of re-analyses (e.g. less than 50) is performed. However, the computational efficiency of the CA method may not be adequate in simulation-based (e.g. Monte-Carlo) probabilistic dynamic analysis of large finite-element models, where re-analysis must be performed hundreds or thousands of times in order to estimate accurately the reliability of a design. The main reason that the CA method is still expensive for such problems is the considerable cost for re-calculating the basis vectors used to approximate the response.

A large number of modes must be calculated and used in a dynamic analysis of a large finite-element model with a high modal density, even if a reduced-order modeling approach is used (see section 2.1). In such a case, the implicit assumption of the CA method that the cost of solving a linear system is dominated by the cost of matrix decomposition of the stiffness matrix is no longer valid (see section 3.1). As a result, the computational savings from using the CA method may not be substantial. For this reason, a modified combined approximations (MCA) method is proposed in this paper and is integrated with the PROM method for improving accuracy and computational efficiency by using basis vectors that are of higher quality than those used by the CA method. The MCA method approximates efficiently the modes of a number of designs. These modes are subsequently used by the PROM method for probabilistic analysis. Examples involving real-world complex structures have demonstrated that the computational savings can be substantial for problems with a large number of design parameters.

The paper is organized as follows. In section 2, PROM and CA methods are reviewed after a brief introduction to reduced order modeling for dynamic systems. In section 3, the modified combined approximations (MCA) method is presented and compared to the CA method and the PROM method. Then an efficient re-analysis

method is formulated by integrating the MCA and Kriging interpolation methods [43] into the PROM method. Section 4 presents the PRA approach and its use in RBDO. In section 5, the efficiency and accuracy of the proposed methodology is demonstrated on probabilistic analysis and RBDO of a complex vehicle model. Conclusions are drawn in section 6.

2. REVIEW OF EXISTING METHODS FOR DETERMINISTIC STRUCTURAL ANALYSIS

2.1 REDUCED ORDER MODELING FOR DYNAMIC RESPONSE

For an undamped structure with stiffness and mass matrices $\mathbf{K}$, and $\mathbf{M}$ respectively, under the excitation force vector $\mathbf{F}$, the equations of motion (EOM) for frequency response are

$$[\mathbf{K} - \omega^2 \mathbf{M}] \mathbf{d} = \mathbf{F}, \quad (1)$$

where the displacement $\mathbf{d}$ is calculated at the forcing frequency ω . If the response is required at multiple frequencies, the repeated direct solution of Eq. (1) is computationally very expensive and therefore, impractical for large scale finite-element models.

The reduced order model (ROM) can be constructed using a subspace projection method. Instead of solving the original response equations, it is assumed that the solution can be approximated in a subspace spanned by the dominant mode shapes. A modal response approach can be used to calculate the response more efficiently. A set of eigen-frequencies ω_i and corresponding eigenvectors (mode shapes) ϕ_i are first obtained. Then, the displacement $\mathbf{d}$ is approximated in the reduced space formed by the first n modes as

$$\mathbf{d} = \Phi \mathbf{U}, \quad (2)$$

where $\Phi = [\phi_1 \ \phi_2 \ \dots \ \phi_n]$ is the modal basis and $\mathbf{U}$ is the vector of principal coordinates or modal degrees of freedom (DOF). Then the EOM of Eq. (1) can be transformed from the original physical to the modal degrees of freedom as

$$[\Phi^T \mathbf{K} \Phi - \omega^2 \Phi^T \mathbf{M} \Phi] \mathbf{U} = \Phi^T \mathbf{F}. \quad (3)$$

Solving for the modal response $\mathbf{U}$ and projecting it back to the physical coordinates, the response $\mathbf{d}$ can be recovered. If the maximum excitation frequency is $\omega_{\max}$, it is common practice to retain the mode shapes in the frequency range of $0 - 2\omega_{\max}$. The system modes in the high frequency range can be safely truncated with minimal loss of accuracy.

Due to the modal truncation, the size of the ROM is drastically reduced, compared to the original model. However, the size increases as the maximum excitation frequency increases. An added benefit of the ROM is that Eq. (3) is decoupled because of the orthogonality properties of mode shapes. Therefore they can be solved separately, reducing further the overall computational effort.

2.2 PARAMETRIC REDUCED-ORDER MODELING METHOD

The parametric reduced-order modeling (PROM) method is a subspace approximation method developed to improve the efficiency of re-analysis. It assumes that the dynamic response of a new design, which is obtained by relatively large parameter changes from the baseline design, can be approximated in the subspace spanned by the mode shapes of some representative designs.

The quality of the approximation depends on the selection of the representative designs. Balmes et al. [30, 31] suggested a procedure for obtaining such designs. Consider a design defined in terms of m design parameters (for example plate thicknesses, spot weld diameters, spot weld pitch of a car body, etc.). The ranges in which these parameters and their baseline values can vary are estimated by the designer. In Balmes et al. [30, 31], the first representative design point $\mathbf{p}_0$ is the baseline design at the center of the design space, with all design parameters at the middle of their allowed ranges. Additional m representative design points $\mathbf{p}_i, i=1, \dots, m$ are chosen as the center points of each face of the design space, with one parameter at its upper bound and all other parameters at their middle values. Therefore, the total number of representative design points is $(m+1)$. In Zhang et al. [32, 33], a different set of representative design points is chosen as follows. The first design $\mathbf{p}_0$ is at the point with all design parameters at their lower limits. The remaining m designs are obtained by perturbing the design parameters from their lower limits to their upper limits, one at a time. The points representing these designs in the space of the design parameters will be called *corner points*. The total number of representative design points is again $(m+1)$, but the collection of $(m+1)$ sets of modes from those design points are less likely to be linearly dependent and they produce more accurate results when used as a subspace to approximate the dynamic response of any other design point [33].

In the PROM approach [32, 33], the mode shapes of a new design, with stiffness and mass matrices $\mathbf{K}_p$ and $\mathbf{M}_p$, are approximated in the subspace $\mathbf{P}$ spanned by the modes of the above $(m+1)$ designs,

$$\mathbf{P} = [\Phi_0 \quad \Phi_1 \quad \dots \quad \Phi_m], \quad (4)$$

where Φ_0 is the modal matrix composed of dominant mode shapes for the baseline design, and Φ_i is the modal matrix for the i^{th} design point, which is the same as the baseline design except that its i^{th} parameter is at its upper limit.

For m design parameters, $(m+1)$ eigenvalue problems must be solved in order to form matrix $\mathbf{P}$ of Eq. (4). Therefore, both the cost of obtaining modal matrices Φ_i and the size of matrix $\mathbf{P}$ increase linearly with the number of parameters. Basis $\mathbf{P}$ captures the dynamic characteristics in each dimension of the parameter space because it consists of mode shapes corresponding to the centers of the faces of a box representing the ranges of the parameters. Therefore, the response at a new design point of the parameter space can be approximated accurately using basis $\mathbf{P}$.

The following algorithm is used by the PROM approach to compute the mode shapes of a new design:

1) Find the modes of the baseline design and the designs corresponding to the m points in the design space, and form subspace basis $\mathbf{P}$.

2) Condense matrices $\mathbf{K}_p$ and $\mathbf{M}_p$ as

$$\mathbf{K}_R = \mathbf{P}^T \mathbf{K}_p \mathbf{P} \quad \mathbf{M}_R = \mathbf{P}^T \mathbf{M}_p \mathbf{P}, \quad (5)$$

3) Perform an eigen-analysis on the reduced matrices $\mathbf{K}_R$ and $\mathbf{M}_R$ to calculate eigenvector Θ .

4) Reconstruct the approximate eigenvectors $\tilde{\Phi}_p$ using

$$\tilde{\Phi}_p = \mathbf{P} \Theta. \quad (6)$$

In the above procedure, step 1 is performed only once. For any repeated re-analysis, only steps 2 to 4 need to be repeated. For a small number of modes and a few number of design parameters, the cost of steps 2 to 4 should be much smaller than the cost of a full analysis. Therefore, the PROM method could be a very efficient re-analysis method in such a case.

However the computational cost of steps 2 to 4 could become very high when the number of modes and/or the number of parameters m are large. The reason is that the cost of forming the condensed matrices (by computing the triple matrix products in Eq. (5)) could be high because subspace basis $\mathbf{P}$ is a large matrix with high density.

Balmes et al. [31] have proposed a 'parametric' approach to reduce the condensation cost, for the

special case where matrices $\mathbf{K}_p$ and $\mathbf{M}_p$ can be approximated by a n^{th} order polynomial functions of the m design parameters p_i with matrices $\mathbf{K}_{ij}$ and $\mathbf{M}_{ij}$ as coefficients. For example, for most parameters of material properties used in finite element modeling, a third order polynomial is sufficient. The condensed stiffness and mass matrices can be calculated as follows

$$\begin{aligned}\mathbf{K}_R &= \sum_{i=1}^m \sum_{j=0}^n (\mathbf{P}^T \mathbf{K}_{ij} \mathbf{P}) p_i^j = \sum_{i=1}^m \sum_{j=0}^n \bar{\mathbf{K}}_{ij} p_i^j \\ \mathbf{M}_R &= \sum_{i=1}^m \sum_{j=0}^n (\mathbf{P}^T \mathbf{M}_{ij} \mathbf{P}) p_i^j = \sum_{i=1}^m \sum_{j=0}^n \bar{\mathbf{M}}_{ij} p_i^j\end{aligned}\quad (7)$$

Because $\mathbf{K}_{ij}$ and $\mathbf{M}_{ij}$ are constant matrices, the condensations $\mathbf{P}^T \mathbf{K}_{ij} \mathbf{P}$ and $\mathbf{P}^T \mathbf{M}_{ij} \mathbf{P}$ are performed only once and then the condensed matrices $\bar{\mathbf{K}}_{ij}$ and $\bar{\mathbf{M}}_{ij}$ are used to form matrices $\mathbf{K}_R$ and $\mathbf{M}_R$ in Eq. (7) for each analysis of a modified design. The cost of matrix condensation is largely reduced because it now involves only a sum of matrices of reduced size for any re-analysis.

The computational cost of PROM consists of the costs involved in the following calculations: a) $(m+1)$ full eigen-analyses to form subspace basis $\mathbf{P}$ in Eq. (4), b) identification of matrices $\mathbf{K}_{ij}$ and $\mathbf{M}_{ij}$, and establishment of the relations in Eq. (7) and c) the cost of re-analysis of each new design. The cost involved in steps a) and b) is called *overhead cost* of PROM because it cannot be attributed to the calculation of the response of a particular design; it is required in order to obtain the information needed to apply PROM to a given problem. The cost of re-analysis of a new design (cost of c) is trivial compared to the overhead cost.

The following two observations can be made on the PROM approach. First, the overhead cost is proportional to the number of design parameters. Therefore the efficiency of the PROM method is best demonstrated when the number of re-analysis is significantly larger than the number of design parameters. This is the case in simulation-based probabilistic analysis. Second, further improvement of the efficiency of the PROM approach requires reduction of the overhead cost.

In section 3 of this paper, a modified combined approximations (MCA) method and a Kriging interpolation method will be introduced and integrated with the PROM method to improve the efficiency of the PROM method by largely reducing its overhead cost.

2.3 THE COMBINED APPROXIMATIONS (CA) METHOD

The PROM method requires calculation of the eigenvectors of multiple designs to form a basis for approximating the eigenvectors at other designs. In this

study, the CA method is used to circumvent the need to solve the eigenvalue problems of some representative designs in order to form a basis. This section briefly describes the fundamentals of the CA method [37-41].

In the CA method a subspace basis is formed through a recursive process for solving an eigenvalue problem to find the natural frequencies and modes of a system. Assuming that the stiffness and mass matrices of the original (baseline) design are $\mathbf{K}_0$ and $\mathbf{M}_0$, the exact mode shapes Φ_0 are obtained by solving the following eigen-problem

$$\mathbf{K}_0 \Phi_0 = \lambda_0 \mathbf{M}_0 \Phi_0. \quad (8)$$

We want to approximate the mode shapes of a modified design (subscript p) with stiffness and mass matrices

$$\mathbf{K}_p = \mathbf{K}_0 + \Delta \mathbf{K} \quad \mathbf{M}_p = \mathbf{M}_0 + \Delta \mathbf{M} \quad (9)$$

where $\Delta \mathbf{K}$ and $\Delta \mathbf{M}$ represent large perturbations. The objective of the CA method is to estimate the new eigenvalues λ_p and eigenvectors Φ_p accurately and efficiently, without solving the exact eigen-problem for the modified design.

The eigenvalue problem for the modified design can be expressed in the following form:

$$\Phi_p = \lambda_p \mathbf{K}_0^{-1} \mathbf{M}_p \Phi_p - \mathbf{K}_0^{-1} \Delta \mathbf{K} \Phi_p. \quad (10)$$

The following recursive equation can be used to solve this equation

$$\Phi_{p,j} = \lambda_p \mathbf{K}_0^{-1} \mathbf{M}_p \Phi_{p,j-1} - \mathbf{K}_0^{-1} \Delta \mathbf{K} \Phi_{p,j-1}. \quad (11)$$

This represents a sequence of approximations of the mode shapes $\Phi_{p,j}$, $j=1,2,\dots,s$ which converge to the true mode shape Φ_p of the new design. The CA method uses the changes in subsequent estimates of the mode shapes, $\mathbf{R}_j = \Phi_{p,j} - \Phi_{p,j-1}$, to form a subspace basis for approximating the modes of the new design. In order to simplify the calculations of the basis vectors, $\lambda_p \mathbf{K}_0^{-1} \mathbf{M}_p \Phi_{p,j-1}$ in Eq. (10) is replaced with $\lambda_p \mathbf{K}_0^{-1} \mathbf{M}_p \Phi_0$. Then the recursive Eq. (11) becomes

$$\Phi_{p,j} = \lambda_p \mathbf{K}_0^{-1} \mathbf{M}_p \Phi_0 - \mathbf{K}_0^{-1} \Delta \mathbf{K} \Phi_{p,j-1}. \quad (12)$$

From Eq. (12) we find that the basis vectors satisfy the following recursive equation

$$\mathbf{R}_j = -\mathbf{K}_0^{-1} \Delta \mathbf{K} \mathbf{R}_{j-1} \quad j = 2, \dots, s, \quad (13)$$

where the first basis vector is assumed to be $\mathbf{R}_1 = \mathbf{K}_0^{-1} \mathbf{M}_p \Phi_0$.

The CA method forms a subspace basis

$$\mathbf{R} = [\mathbf{R}_1 \quad \mathbf{R}_2 \quad \dots \quad \mathbf{R}_s] \quad (14)$$

where s is usually chosen between 3 and 6 for a balance between accuracy and efficiency [35-39]. It is then assumed that the mode shapes of the new $(\mathbf{K}_p, \mathbf{M}_p)$ design, can be approximated in the subspace spanned by $\mathbf{R}_p$, by the following algorithm:

1) Condense the stiffness and mass matrices

$$\mathbf{K}_R = \mathbf{R}^T \mathbf{K}_p \mathbf{R} \quad \mathbf{M}_R = \mathbf{R}^T \mathbf{M}_p \mathbf{R}, \quad (15)$$

2) Solve the reduced eigenvalue problem to calculate eigenvector Θ .

3) Reconstruct the approximate eigenvectors of the new design $\tilde{\Phi}_p$ as

$$\tilde{\Phi}_p = \mathbf{R} \Theta. \quad (16)$$

The eigenvectors for the new design are approximated by the eigenvectors $\tilde{\lambda}_p$ of the reduced eigenvalue problem.

Note that steps (15) and (16) are similar to steps (5) and (6) of the PROM method, except that subspace basis $\mathbf{R}$ is used instead of $\mathbf{P}$. The cost of matrix condensation of Eq. (15) is much lower than that of Eq. (5), because the size (number of columns) of basis $\mathbf{R}$ is not proportional to the number of parameters m as is the case with Φ . Instead, it is equal to s (Eq. 11), which is usually chosen between 3 and 6, independently of the number of parameters.

The CA method does not require exact calculation of the eigenvalues and eigenvectors $\Phi_i, i = 1, \dots, m$, of the m designs obtained by perturbing the design parameters. Thus, the CA method reduces the overhead cost of the PROM method required for constructing basis $\mathbf{P}$. Moreover, the CA approximation is accurate because it updates the basis $\mathbf{R}$ for each new design instead of using the same basis for all designs. Updating basis $\mathbf{R}$ could increase the cost of the CA method for a large number of re-analyses, but this cost is low because the CA method decomposes stiffness matrix $\mathbf{K}_0$ only once instead the decomposing the stiffness matrix $\mathbf{K}_p$ for each new design as shown in Eq. (13). The accuracy

and efficiency of the CA method has been mostly tested on problems involving structures with up to few thousands of DOFs, such as frames or trusses [34-41]. We have tested the CA method for the structural dynamics response of a medium size (65,000 DOF), finite-element model. Due to its high modal density, more than two hundred dominant modes were retained in the frequency range of zero to 100 Hz. It was observed that the computational savings of the CA method, using the recursive process of Eq. (13), which avoids repeated matrix decompositions, was not substantial. Therefore, we introduce a modified combined approximations method (MCA) by modifying the recursive process of Eq. (13). The new method, which is described next in section 3.1, is much more efficient than the original CA method for re-analysis of dynamic problems with a large number of dominant modes.

3. METHOD FOR DETERMINISTIC STRUCTURAL RE-ANALYSIS

Three major developments are introduced in this section. First, a Modified Combined Approximations (MCA) method is presented that uses an improved set of high-quality basis vectors. The new basis vectors are more expensive to calculate than the basis of the original CA method, but only few of these new basis vectors (i.e. one to three) are required to approximate accurately the response of a complex real world design. Then the efficiency and generality of the PROM method are improved by using the MCA method. Finally, Kriging interpolation is employed to calculate efficiently the reduced mass and stiffness matrices of the reduced eigen-problem.

3.1 MODIFIED COMBINED APPROXIMATIONS (MCA) METHOD

The cost of calculating the subspace basis in equations in Eq. (13) consists of one matrix decomposition (DCMP) and one forward-backward substitution (FBS) of matrix $\mathbf{K}_0$. The DCMP cost is only related to the size and density of the symmetric stiffness matrix, while the FBS cost depends on both the size and density of the stiffness matrix and the number of columns of Φ_0 .

As we mentioned in section 2.1, the higher the frequency range of interest the more modes are needed for predicting the structural response accurately. In such a case, although a single DCMP is needed in Eq. (13), the number of columns in Φ_0 may increase drastically, thereby increasing the cost of the repeated FBS. When the number of dominant modes becomes very large (e.g. more than 200), the cost of performing the calculations in Eq. (13) can be dominated by the FBS cost. For example, the vehicle mode used in section 4 (see

Figure 1) has about 65,000 degrees of freedom. If Φ_0 consists of only one mode, then the cost of one DCMP is 4 seconds and the cost of one FBS is 0.1 seconds, using a SUN ULTRA workstation and the NASTRAN v2001 finite-element code. In this case, the total solution cost is dominated by the DCMP, and the use of the CA method reduces considerably the cost of one re-analysis. However, if Φ_0 consists of 220 modes then the cost of FBS increases to 22 seconds (0.1 seconds per mode times 220 modes), dominating the cost of the DCMP.

Therefore, the CA method can improve efficiency only when the number of dominant modes is small so that the solution cost is dominated by the DCMP. Otherwise, the computational savings do not compensate for the loss of accuracy due to the CA approximation from inverting K_0 (stiffness matrix of the baseline design) instead of K_p (stiffness matrix of new design). For this reason, a modified combined approximations (MCA) method is proposed in this paper for problems where the FBS cost dominates the DCMP cost.

In the MCA method, a subspace basis T is used whose columns are constructed using the following recursive process instead of that of Eq. (13)

$$\begin{aligned} T_1 &= K_p^{-1}(M_p \Phi_0) \\ T_i &= K_p^{-1}(M_p T_{i-1}) \quad i = 2, 3, \dots, s \end{aligned} \quad (17)$$

The selection of the appropriate s is discussed later in this section. The only difference between Eqs. (13) and (17) is that matrix K_0 is inverted in the former while matrix K_p is inverted in the latter. The DCMP of K_p must be repeated for every new design. However, the cost of the repeated DCMP does not significantly increase the overall solution cost in Eq. (17), because the latter is dominated by the FBS cost. Note that the iterative process of Eq. (17) provides a continuous mode shape updating of the new design. If the process converges in s iterations, the mode shapes T_s will be identical to the exact mode shapes Φ_p . However, Eq. (13) does not have the same property. For this reason, vectors T_i of the MCA method have better "quality" in approximating the exact mode shapes Φ_p than the R_i vectors of the original CA method. This is an important advantage of the MCA method.

In the MCA method, mode shapes $\tilde{\Phi}_p$ can be approximated in a subspace spanned by basis T . Because the mode shapes T_i in Eq. (17) can converge to the exact mode shapes Φ_p , the basis T of the MCA method may consist of only one set of updated mode shapes

$$T = T_s \quad (18)$$

where T_s is the set of mode shapes obtained at the s^{th} application of recursive Eq. (17). Alternatively, if the above basis results in a slow convergence, multiple sets of updated mode shapes can be used, similarly to the CA method; i.e.

$$T = [\Phi_0 \quad T_1 \quad T_2 \quad \dots \quad T_s]. \quad (19)$$

Note that the above basis also includes the mode shapes Φ_0 of the baseline design. As explained earlier, the approximate modes T_i are of better quality than the CA vectors R_i in approximating the exact mode shapes Φ_p , because modes T_i are updated for each new design. This enables the MCA method to achieve similar accuracy with the CA method using fewer modes. Indeed, the CA method provides accurate approximations with only 3 to 6 basis vectors R_i (i.e. $s = 3$ to 6 in Eqs. (13) and (14)) [35-39], whereas the MCA method achieves good accuracy with only 1 or 2 basis vectors as we will demonstrate using a real-world vehicle mode.

In summary, the proposed MCA method involves four steps to find the eigenvalues and the approximate eigenvectors $\tilde{\Phi}_p$:

- 1) Find the basis T using Eq. (18) or Eq. (19).
 - 2) Calculate the condensed stiffness and mass matrices
- $$K_R = T^T K_p T \quad M_R = T^T M_p T. \quad (20)$$
- 3) Solve the following reduced eigenvalue problem to find the eigenvalues and the projections of the modes in the reduced space spanned by T
- $$(K_R - \lambda M_R) \Theta = 0. \quad (21)$$
- 4) Reconstruct the approximate eigenvectors $\tilde{\Phi}_p$ as

$$\tilde{\Phi}_p = T \Theta. \quad (22)$$

Eq. (18) is preferable to Eq. (19) although the former may require a few more iterations to achieve equivalent accuracy with the latter. The slightly increased cost of using Eq. (19) is usually smaller than the realized savings in steps (20) through (22) due to the smaller size of the reduced basis T . Eq. (19) is recommended, only when Eq. (18) results in a slow convergence. Note that the sizes of both bases of Eqs. (18) and (19) are smaller than the CA basis of Eq. (14) for comparable accuracy. Therefore, the MCA method requires less computational effort for steps (20) through (22), and the