

Open Chain Systems Based on Oriented Graph-Matroid Theory

Ilie Talpasanu

Wentworth Institute of Technology

Pilaka Murty

West Texas A&M University

Copyright © 2008 SAE International

ABSTRACT

This paper presents a novel technique for the kinematic analysis of planar and spatial open chain systems, called the *Incidence and Transfer Method* (IT), which is based on the incidence matrices associated with the edge-oriented graph attached to the mechanism and the transfer joints. Relative to such joints, a set of independent equations can be automatically generated for the efficient computation of manipulator's joint positions and velocities. Kinematics of open chain manipulators with common joints such as revolute (R), prismatic (P), cylindrical (C), and helical (H) can be solved by using the sparse matrices derived from the manipulator's graph model. Complete kinematic equations are obtained in matrix form using a base of circuits from a cycle matroid. The proposed method has general applicability and can be employed for systems with any number of links and degrees of freedom, as illustrated by the numerical example presented. This method has applicability in kinematic optimization for mechanisms usage in automobiles.

Key Words: *graph, transfer matrix, relative velocity, relative acceleration, absolute velocity, absolute acceleration*

1 INTRODUCTION

The kinematics of mechanical systems consists of two different analytical methods: the direct kinematics and the inverse kinematics. In the direct kinematics problem, the geometry of the mechanical system is provided, and kinematic equations for the positions, velocities and accelerations relative to the reference frame are determined.

The inverse kinematics problem consists in the computation of joint kinematic variables for desired positions, velocities or accelerations. These two

methods take into account the relative motions between links and are disregarding the forces that produce such relative motions.

LITERATURE REVIEW OF PREVIOUS WORK FOR MATRIX METHODS

Dimentberg [6] used the matrix method for kinematic analysis. Independent sets of kinematic equations were presented in vector form. Uicker [23] formulated a 4x4 matrix approach, and Nikravesh [14], [15] used Euler parameters for the computational kinematics. Asada [1] Corke [5] by considering the Euler's parameters as a (4 x 1) vector, known as a unit quaternion, accomplished the analysis of robotic systems.

Considerable research material is available for open chain mechanical systems especially for robot manipulators. Roth [18], defined the performance evolution of open chain systems by the terms of "workspace", Lee and Yang [11], [12] found that for a given manipulator structure the ratio of the volume of the workspace to the cube of its total link length is a constant, Tsai and Soni [21] developed an algorithm for the workspace determination of an n-R robot. Snyman, J.A, Plessis, L.J, and Duffy, J. [20], presented an optimization approach to the determination of the boundaries of manipulator workspace. Algorithms for determining extreme distances between the reference point and the end effector of a robot arm were developed by Kumar and Waldron [9], and by Duffy and Sugimoto [7]. Yih T.C. [27], developed a model for kinematic analysis of spatial systems based on 5 Parameters (C/B Notation) to describe a spatial binary linkage. Cheng, H.H and Thompson, S. [4], developed a method of analysis for singularities based on complex dual numbers. A new polynomial solution to the geometric design of spatial robot manipulators was proposed by Mavroidis, C., Lee, E, and Alam, M, [10], [13].

The application of *matroid theory* which combines the properties of graphs and matrices, to mechanisms community is less known in literature. Whitney [26] introduced the concept of matroid as a *common generalization of graphs and matrices*. Recski, A [17], Shai O [19] described the interconnection of elements in an electric network or links-joints in a static frame by a graph, while the properties of different electric devices or the rigidity of a static frame in terms of vectors and matrices. Talpasanu I., [22] described the interconnection links-joints in kinematics dynamics of mechanical systems by the oriented edges graph. Kaveh. A [8], used the matroid approach in structural mechanics.

2 ORIENTED GRAPH ATTACHED TO THE OPEN CHAIN MECHANICAL SYSTEM

2.1 LINK AND JOINT NUMBERING

For the case of an open chain with n mobile links and k joints, link numbering begins with n_0 assigned to the ground element continuing up to n_n for the other mobile links. The k joints are assigned numbers between 1 and k , with $k = n$. Additionally, the fictitious (transfer) joints assigned to $^*k$ end effectors are labeled with numbers between $n+1$ and $n+^*k$.

Figure 1 shows the numbering in a planar manipulator with $n = 6$ $\{n_1, n_2, n_3, n_4, n_5, n_6\}$ mobile links and $k = 6$ joints $\{1, 2, 3, 4, 5, 6\}$. The $^*k$ end effectors are labeled $\{^*7, ^*8\}$.

2.2 ASSOCIATED GRAPH

A graph $G^\circ = (V, E)$ is defined as a collection of $m = n + 1$ vertices $\{V = n_0, n_1, n_2, \dots, n_n\}$ connected by a set of k edges $\{E = 1, 2, 3, \dots, k\}$, [2], [3]. In a connected graph G , a spanning tree (further noted T_k) contains all the m vertices, and has $m - 1 = n = k$ edges. The set of $^*k$ chords, represented as dashed curved lines, is called a co-tree set and is noted T_{*k} . Any branch in the tree is called an edge and a branch in the co-tree is further defined as a chord. A path is defined as a sequence $\{n_0, 1, n_1, 2, n_2, 3, \dots, n_n\}$ with all vertices n_0 to n_n being different. A loop is defined as a path with $n_0 = n_n$ which includes one chord. If $^*k$ is one of the chords from the co tree set, then $T_k \cup \{^*k\}$ will contain one loop, noted C_{*k} . By adding to the tree the chords

from co-tree, one at a time, one would get a set of $^*k$ fundamental loops. Let us consider a system of closed surfaces (cuts), each intersecting once, only one edge of the tree. The cut intersecting the edge k from the tree is

noted Q_k . The set of k cuts of edges in the tree is a fundamental cutset. The graph attached to the open chain manipulators has the topology of a spanning tree, in which the nodes are assigned to the links and the edges being represented by the joints.

Figure 1a shows an example of an anthropomorphic system for which the tree and the co-tree are represented in Figure 1c and Figure 1d, respectively. The tree has $m = 4$ vertices $\{V = n_0, n_1, n_2, n_4\}$ and $k = 3$ $E = \{1, 2, 3\}$. The co-tree set contains one chord $T_{*k} = \{^*4\}$ as shown in curved dashed line which is assigned to the fictitious joint located in the end effector ^{*4} . The fundamental loop set $\{C_{*k}\}$ has one loop, which is obtained adding to the tree the chord from the co-tree, thus: $C_{*k} = \{1, 2, 3, ^*4\}$. The fundamental cutset $\{Q_1, Q_2, Q_3\}$ is obtained by cutting one at a time the edges from the tree, therefore :

$Q_1 = \{1, ^*4\}$, $Q_2 = \{2, ^*4\}$, $Q_3 = \{3, ^*4\}$, as shown in Figure 2.

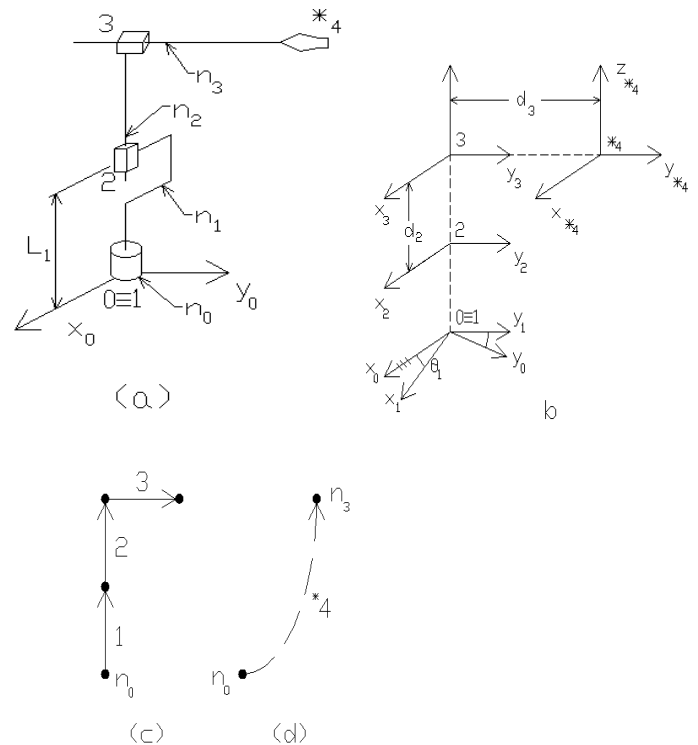


Figure1 a) Manipulator b) Frames c) Tree, and d) Co-tree.

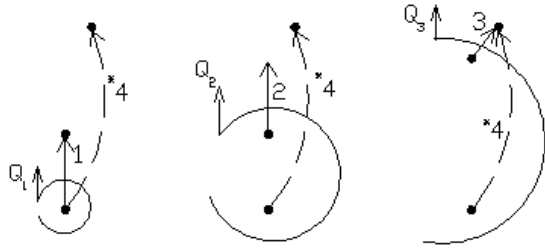


Figure 2. Cut-set

2.3 GRAPH EDGES ORIENTATION

An oriented edge graph is called a digraph. For each edge k , a pair of numbers (n^-, n^+) is assigned, and the edge is represented as an arrow oriented from vertex n^- to vertex n^+ . For example, if joint 3 (edge tree) connects link n_3 to link n_2 , then the pair $(n^-, n^+) = (n_3, n_2)$ will be assigned to it. The fictitious joint *4 (chord from the co-tree) connects link n_3 to the fixed link n_0 , then the pair $(n^-, n^+) = (n_0, n_3)$ will be assigned to it.

3 MATRIX DESCRIPTION OF THE ORIENTED GRAPH

3.1 INCIDENCE LINK-JOINT MATRIX : ${}^\circ G$

One of the most common matrix representations of a graph is the incidence vertex-edge tree matrix. The $(m \times k)$ matrix ${}^\circ \Gamma$ entries $g_{m,k}$ are equal to -1 if $n^- = n$. The entries are equal to $+1$ if $n^+ = n$, and for all the others vertices are equal 0 . Consequently, each column k will have a -1 and a $+1$ entry corresponding to the two vertices connected by the respective edge. The incidence vertex-edge co-tree matrix ${}^\circ \Gamma_{*k}$ has the form $(m \times k)$. The system's incidence link (vertex in graph)-joint (edge or chord in graph) is

$${}^\circ G = [{}^\circ \Gamma_k \mid {}^\circ \Gamma_{*k}] \quad (1)$$

For the graph in Figure 1, the incidence matrices for the tree and co-tree will therefore be

$${}^\circ G = [{}^\circ \Gamma_k \mid {}^\circ \Gamma_{*k}] = \begin{matrix} & \begin{matrix} 1 & 2 & 3 & *4 \end{matrix} \\ \begin{matrix} n_0 \\ n_1 \\ n_2 \\ n_3 \end{matrix} & \begin{bmatrix} -1 & 0 & 0 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \end{matrix} \quad (2)$$

3.2 REDUCED INCIDENCE LINK-JOINT MATRIX: G

This $n \times (k + k)$ matrix G is obtained by deleting the first row corresponding to the fixed link from the matrix

shown in Equation 1.

For the oriented graph in Figure 1, the reduced incidence matrix will therefore be

$$G = \begin{matrix} & \begin{matrix} 1 & 2 & 3 & *4 \end{matrix} \\ \begin{matrix} n_1 \\ n_2 \\ n_3 \end{matrix} & \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \end{matrix} \quad (3)$$

3.3 THE ROW-REDUCED CUTSET MATRIX: Q

This $k \times (k + k)$ matrix Q is obtained by row-reducing the entries in incidence matrix G , and has the form:

$$Q = [U_k \mid Q_{*k}] \quad (4)$$

The matrix Q has $n = k$ rows, the same as the number of cuts. Since the total number of columns is ${}^\circ k + k$, matrix Q is partitioned in two submatrices: a $({}^\circ k \times k)$ unit matrix U_k with the edges in the tree T_k as column labels and a $(k \times k)$ submatrix Q_{*k} , with the chords in the co-tree T_{*k} as column labels.

For the graph in Figure 1, the reduced cutset matrix Q , and the Q_{*k} submatrices are:

$$Q = \begin{matrix} & \begin{matrix} 1 & 2 & 3 & *4 \end{matrix} \\ \begin{matrix} Q_1 \\ Q_2 \\ Q_3 \end{matrix} & \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \end{matrix} \Rightarrow Q_{*k} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad (5)$$

Rule1: Matrix Q is the row-reduced form of matrix G . A fast computation of matrix Q is obtained by row-reducing the entries in matrix G using *Mathematica*. The row-reduced form of G matrix for the graph in Figure 1 is the matrix Q :

$$Q = \text{RowReduce}[G] = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \quad (6)$$

One could notice that the reference direction of a cut is chosen to be the direction of the edge, as shown in Figure 2. The entries $c_{k,j}$ in matrix Q are defined as follows:

$c_{k,j} = +1$ if the edge (chord) belongs to the cut and oriented in the direction of the cut.

$c_{k,j} = -1$ if the edge (chord) belongs to the cut and oriented in the opposite direction of the cut, and

$c_{k,j} = 0$ if the edge (chord) does not belong to the cut.

3.4 THE TRANSPOSED ROW-REDUCED CUTSET

MATRIX: Q^T

The transposed Q^T of the matrix Q is a matrix with the rank equal with the number of joints, k .

$$Q^T = \begin{bmatrix} U_k \\ Q_k^T \end{bmatrix} \quad (7)$$

The rows represent the edges/chords and the columns represent the fundamental cut-set $\{Q_1, Q_2, Q_3, \dots, Q_k\}$. This set of independent columns is a base of cut-vectors in the $(k+^*k)$ dimensional vector space. For the graph in Figure 1 the Q^T incidence cutset matrix is

$$Q^T = \begin{matrix} & Q_1 & Q_2 & Q_3 \\ \begin{matrix} 1 \\ 2 \\ 3 \\ *4 \end{matrix} & \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \end{matrix} \quad (8)$$

3.5 THE REDUCED LOOP-EDGE MATRIX: C

This $^*k \times (k+^*k)$ matrix C has $^*k$ rows equal to the number of fundamental loops and is partitioned into two submatrices: a $(^*k \times ^*k)$ matrix B_k , with edges from the tree T_k as column labels, and a $(^*k \times ^*k)$ unit matrix $U_{^*k}$, with the chords from the co-tree $T_{^*k}$ as column labels. The late matrix is a unit matrix, since one chord at a time was added to generate the k fundamental loops.

$$C = [B_k \mid U_{^*k}] \quad (9)$$

For oriented graphs, the entries in the $(^*k \times ^*k)$ submatrix B_k are determined based on the $Q_{^*k}$ submatrix, as follows:

3.6 ORTHOGONALITY EQUATION

The row vectors (loop base) from matrix C are orthogonal to the column vectors (cut base) from Q^T

$$C * Q^T = 0 \quad (10)$$

Using Equation 7 and Equation 9, Equation 10 has the form: (11)

$$B = -Q_{^*k}^T \quad (11)$$

and therefore Equation 9 becomes

$$C = [-Q_{^*k}^T \mid U_{^*k}] \quad (12)$$

From Equation 12, for the graph in Figure 1 the reduced (fundamental) loop matrix C has the form as shown in Equation 13.

$$C = C_{^*4} = \begin{bmatrix} 1 & 2 & 3 & 4^* \\ -1 & -1 & -1 & 1 \end{bmatrix} \quad (13)$$

Rule 2: Matrix C is the null set of matrix G .

From orthogonality Equation (10) one could conclude that matrix C is a null set of matrix Q . Since matrix Q is the row-reduced form of matrix G , one could determine matrix C as the null set of matrix G . Therefore, a fast computation of reduced loop-edges matrix C from the incidence vertex-edges matrix G is obtained.

Using *Mathematica*, for the graphs from Figure 1 the following null sets are determined

$$C = NullSpace[G] = [-1 \quad -1 \quad -1 \mid 1] \quad (14)$$

It is noticed that the reference direction of the walk along the edges in a loop is considered the direction of the chord $^*k$. An $i_{^*k,k}$ entry could be equal to $-1, +1, \text{ or } 0$. A -1 entry denotes the edge belonging to the loop and is oriented in the opposite direction of the path. A $+1$ entry denotes the edge belonging to the loop and having the same direction as the path, while a 0 entry indicates that the respective edge does not belong to the loop.

The graph in Figure 1 has one fundamental loop $C_{^*4}$.

The reference counterclockwise direction of chords ^{*4} is the direction of the walk along the edges of the loop.

3.7 THE TRANSPOSED LOOP-EDGE MATRIX: C^T

The transposed of the matrix C is the matrix C^T with the rank equal with the number of end effectors, $^*k$. The rows represent the edges/chords and the columns represent the fundamental loop-set $\{C_{k+1}, \dots, C_{k+^*k}\}$. It is readily seen that this set of columns is a base of loop-vectors in the $(k+^*k)$ dimensional vector space.

Our goal is to write independent kinematic equations of motion, therefore the independency of the columns in matrix C^T will be further investigated, thus conducting to the concept of matroid.

3.8 MATROID FUNDAMENTALS

Further, the two reduced matrices C and Q , with their columns numbered in the same order and also related to Equation 10, are considered for their columns independency investigation.