

Design under Uncertainty using a Combination of Evidence Theory and a Bayesian Approach

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ABSTRACT

Early in the engineering design cycle, it is difficult to quantify product reliability due to insufficient data or information to model uncertainties. Probability theory can not be therefore, used. Design decisions are usually based on fuzzy information which is imprecise and incomplete. Various design methods such as Possibility-Based Design Optimization (PBDO) and Evidence-Based Design Optimization (EBDO) have been developed to systematically treat design with non-probabilistic uncertainties. In practical engineering applications, information regarding the uncertain variables and parameters may exist in the form of sample points, and uncertainties with sufficient and insufficient information may exist simultaneously. Most of the existing optimal design methods under uncertainty can not handle this form of incomplete information. They have to either discard some valuable information or postulate the existence of additional information. In this paper, a design optimization method is proposed based on evidence theory, which can handle a mixture of epistemic and random uncertainties. Instead of using "expert" opinions to form the basic probability assignment, a Bayesian approach is used with a limited number of sample points. A pressure vessel example demonstrates the merit of the proposed design optimization method. The results are compared with those from existing design methodologies under uncertainty.

1. INTRODUCTION

Engineering design under uncertainty has recently gained a lot of attention. Uncertainties are usually modeled using probability theory. In Reliability-Based Design Optimization (RBDO), variations are represented by standard deviations which are typically assumed constant, and a mean performance is optimized subject to probabilistic constraints [1-5]. In general, probability theory is very effective when sufficient data is available to quantify uncertainty using probability distributions.

However, when sufficient data is not available or there is lack of information due to ignorance, the classical probability methodology may not be appropriate. For example, during the early stages of product development, quantification of the product's reliability or compliance to performance targets is practically very difficult due to insufficient data for modeling the uncertainties. A similar problem exists when the reliability of a complex system is assessed in the presence of incomplete information on the variability of certain design variables, parameters, operating conditions, boundary conditions etc.

Uncertainties can be classified in two general types; aleatory (stochastic or random) and epistemic (subjective) [6-10]. Aleatory or irreducible uncertainty is related to inherent variability and is efficiently modeled using probability theory. However, when data is scarce or there is lack of information, the probability theory is not useful because the needed probability distributions cannot be accurately constructed. In this case, epistemic uncertainty, which describes subjectivity, ignorance or lack of information, can be used. Epistemic uncertainty is also called reducible because it can be reduced with increased state of knowledge or collection of more data.

Formal theories to handle uncertainty have been proposed in the literature including evidence theory (or Dempster – Shafer theory) [9, 10], possibility theory [11] and interval analysis [12]. Two large classes of fuzzy measures, called belief and plausibility measures, respectively, characterize the mathematical theory of evidence. They are mutually dual in the sense that one of them can be uniquely determined from the other. Evidence theory uses plausibility and belief (upper and lower bounds of probability) to measure the likelihood of events. When the plausibility and belief measures are equal, the general evidence theory reduces to the classical probability theory. Therefore, the classical probability theory is a special case of evidence theory. Possibility theory handles epistemic uncertainty if there is no conflicting evidence among experts. It is a special subclass of dual possibility and belief measures.

called possibility and necessity measures, respectively. In possibility theory, a fuzzy set approach is common, where membership functions characterize the input uncertainty [13]. Even if a probability distribution is not available due to limited information, lower and upper bounds (intervals) on uncertain design variables are usually known. In this case, interval analysis [12, 14, 15] and fuzzy set theory [13] have been extensively used to characterize and propagate input uncertainty in order to calculate the interval of the uncertain output. An efficient method for reliability estimation with a combination of random and interval variables is presented in [16]. However, it is not implemented in a design optimization framework. A few design optimization studies have been also reported, where some or all of the uncertain design variables are in interval form [17-19].

Optimization with input ranges has also been studied under the term anti-optimization [20, 21]. Anti-optimization is used to describe the task of finding the “worst-case” scenario for a given problem. It solves a two-level (usually nested) optimization problem. The outer level performs the design optimization while the inner level performs the anti-optimization. The latter seeks the worst condition under the interval uncertainty [21]. A decoupled approach is suggested in [21] where the design optimization alternates with the anti-optimization rather than nesting the two. It was mentioned that this method takes longer to converge and may not even converge at all if there is strong coupling between the interval design variables and the rest of the design variables. A “worst-case” scenario approach using interval variables has also been considered in multidisciplinary systems design [19, 22].

Very recently, possibility-based design algorithms have been proposed [23-25] where a mean performance is optimized subject to possibilistic constraints. It was shown that more conservative results are obtained compared with the probability-based RBDO. A comprehensive comparison of probability and possibility theories is given in [26] for design under uncertainty.

Evidence theory is more general than probability and possibility theories, even though the methodologies of uncertainty propagation are completely different [27, 28]. It can be used in design under uncertainty if limited, and even conflicting, information is provided from experts. Furthermore, the basic axioms of evidence theory allow to combine aleatory (random) and epistemic uncertainty in a straightforward way without any assumptions [28]. Evidence theory however, has been barely explored in engineering design. One of the reasons may be its high computational cost due mainly to the discontinuous nature of uncertainty quantification. Evidence-based methods have been only recently used to propagate epistemic uncertainty [28, 29] in large-scale engineering systems. Although a computationally efficient method is proposed in [28, 29], the design issue is not addressed. We are aware of only one study which propagates epistemic uncertainty using evidence theory and also performs a design optimization [30]. The optimum

design is calculated for multidisciplinary systems under uncertainty using a trust region sequential approximate optimization method with surrogate models representing the uncertain measures as continuous functions.

In engineering design, information regarding the uncertain quantities is usually available in the form of a set of finite samples, either from historical data or from actual measurements. These samples are not enough to infer a probability distribution. However, if we collapse them into intervals, we discard valuable information. Collecting more samples is often not possible due to the cost or time limitations. So RBDO, PBDO (Possibility-Based Design Optimization) [23, 24], and EBDO (Evidence-Based Design Optimization) [31] may not satisfactorily address the presence of incomplete information. We must utilize Bayesian inference to estimate design reliability with incomplete information.

Bayesian inference is an approach to statistics in which all forms of uncertainty are expressed in terms of probability. A Bayesian approach starts with the formulation of a model to describe the situation of interest. A prior distribution is formulated over the unknown parameters of the model, which is meant to capture the belief about the situation before seeing the data. Using available data, we apply Bayesian's rule to obtain a posterior distribution for these unknowns, which accounts for both the prior and the new data.

In this paper, a Bayesian approach is used to account for uncertainty in the design when limited information is provided by a limited number of sample points. A Bayesian approach is proposed using the extreme value distribution of the smallest value. The approach can handle both a mixture of epistemic and random uncertainties or pure epistemic uncertainties. The accuracy of predictions improves with the use of more sample points. Previous research, such as in [32-34] illustrate how to use a Bayesian approach in design utilizing the confidence percentile concept. In this paper, the available methodologies are improved by using the extreme value distribution of the smallest value instead of the conventional beta distribution. The extreme value distribution approach is necessary because we have only a small set of sample points which are different at each experiment. A Bayesian approach to design optimization (BADO) using the extreme value distribution is proposed. We show that the optimal design is conservative.

The proposed BADO approach can handle epistemic uncertainties or a mixture of aleatory and epistemic uncertainties. Also if only the number of sample points within a certain range is known instead of the exact distribution of the sample points, we propose a design methodology which combines the evidence theory and the Bayesian approach.

The difference between Possibility-Based Design Optimization (PBDO) and Bayesian-Based Design Optimization (BADO) is in the format of uncertain

variables. Possibilistic variables are in the form of intervals and Bayesian uncertain variables are in the form of sample points. The latter provide more information compared with the possibilistic variables. Both of them are based on the confidence percentile concept. In PBDO, a membership function is constructed for each possibilistic variables. The PBDO approach provides a worst-case design because there is a minimal amount of information in the form of intervals. However, more information is available for the Bayesian uncertain variables in the form of sample points. For this reason, we will show that the BADO design is less conservative than the PBDO design.

The paper is organized as follows. Section 2 gives an introduction to the fundamentals of evidence theory. Section 3 presents an overview of an Evidence-Based Design Optimization (EBDO) algorithm. Section 4 presents the proposed Bayesian-Based Design Optimization (BADO) procedure and a methodology to estimate the BPA structure from limited available data using Bayesian statistics. The concepts in section 4 are demonstrated with a pressure vessel example. Comparisons among RBDO, EBDO, PBDO and BADO are also provided in order to demonstrate the value of added information in design. Finally, a summary and conclusions are given in section 5.

2. FUNDAMENTALS OF EVIDENCE THEORY

This section gives the fundamentals of evidence theory, how it can be used in design optimization and an introduction to fuzzy measures. Detailed information is provided in [8, 9, 11, 31, 35]. The role of fuzzy measures and the axiomatic definition of evidence theory are explained.

Evidence theory is based on the belief (Bel) and Plausibility (Pl) fuzzy measures. Fuzzy measures provide the foundation of fuzzy set theory. Before we introduce the basics of fuzzy measures, it is helpful to review the used notation on set representation. A universe X represents the entire collection of elements having the same characteristics. The individual elements in the universe X are denoted by x , and are usually called singletons. A set A is a collection of some elements of X . All possible sets of X constitute a special set called the power set $\wp(X)$.

A fuzzy measure is defined by a function $g: \wp(X) \rightarrow [0,1]$ which assigns to each crisp subset of X a number in the unit interval $[0,1]$. The assigned number in the unit interval for a subset $A \in \wp(X)$, denoted by $g(A)$, represents the degree of available evidence or belief that a given element of X belongs to the subset A .

In order to qualify as a fuzzy measure, the function g must have certain properties. These properties are defined by axioms that are *weaker* than the probability theory axioms [8, 9]. Every fuzzy measure obeys the following three axioms:

Axiom 1 (boundary conditions): $g(\emptyset)=0$ and $g(X)=1$.

Axiom 2 (monotonicity): For every $A, B \in \wp(X)$, if $A \subseteq B$, then $g(A) \leq g(B)$.

Axiom 3 (continuity): For every sequence $(A_i \in \wp(X), i=1,2,\dots)$ of subsets of $\wp(X)$, if either $A_1 \subseteq A_2 \subseteq \dots$ or $A_1 \supseteq A_2 \supseteq \dots$ (i.e., the sequence is monotonic), then $\lim_{i \rightarrow \infty} g(A_i) = g(\lim_{i \rightarrow \infty} A_i)$.

A belief measure is a function $Bel: \wp(X) \rightarrow [0,1]$ which satisfies the three axioms of fuzzy measures and the following additional axiom [9]

$$Bel(A_1 \cup A_2) \geq Bel(A_1) + Bel(A_2) - Bel(A_1 \cap A_2) \quad (1)$$

The axiom (1) can be expanded for more than two sets. For $A \in \wp(X)$, $Bel(A)$ is interpreted as the degree of belief, based on available evidence, that a given element of X belongs to the set A .

A plausibility measure is a function

$$Pl: \wp(X) \Rightarrow [0,1] \quad (2)$$

which satisfies the three axioms of fuzzy measures and the following additional axiom [9]

$$Pl(A_1 \cap A_2) \leq Pl(A_1) + Pl(A_2) - Pl(A_1 \cup A_2) \quad (3)$$

Every belief measure and its dual plausibility measure can be expressed with respect to the non-negative function

$$m: \wp(X) \Rightarrow [0,1] \quad (4)$$

such that $m(\emptyset) = 0$ and

$$\sum_{A \in \wp(X)} m(A) = 1. \quad (5)$$

The function m is called Basic Probability Assignment (BPA) due to the resemblance of Eq. (5) with a similar equation for probability distributions. The basic probability assignment $m(A)$ is interpreted either as the degree of evidence supporting the claim that a specific element of X belongs to the set A or as the degree to which we believe that such a claim is warranted. Every set $A \in \wp(X)$ for which $m(A) > 0$ is called a focal element of m . Focal elements are subsets of X on which the available evidence focuses; i.e. available evidence exists.

Given a BPA m , a belief measure and a plausibility measure are uniquely determined by

$$Bel(A) = \sum_{B \subseteq A} m(B) \quad (6)$$

and

$$Pl(A) = \sum_{B \cap A \neq \emptyset} m(B). \quad (7)$$

which are applicable for all $A \in \wp(X)$.

In Eq. (6), $Bel(A)$ represents the total evidence or belief that the element belongs to A as well as to various subsets of A . The $Pl(A)$ in Eq. (7) represents not only the total evidence or belief that the element in question belongs to set A or to any of its subsets but also the additional evidence or belief associated with sets that overlap with A . Therefore,

$$Pl(A) \geq Bel(A). \quad (8)$$

It should be noted that belief and plausibility are complementary in the sense that one of them can be uniquely derived from the other.

Probability theory is a subcase of evidence theory. When the additional axiom of belief measures (see Eq. (1)) is replaced with the stronger axiom

$$Bel(A \cup B) = Bel(A) + Bel(B) \text{ where } A \cap B = \emptyset \quad (9)$$

we obtain a special type of belief measures which are the classical probability measures. In this case, the right hand sides of Eq. (6) and (7) become equal and therefore,

$$Bel(A) = Pl(A) = \sum_{x \in A} m(x) = \sum_{x \in A} p(x) \quad (10)$$

for all $A \in \wp(X)$, where $p(x)$ is the probability distribution function (PDF). Note that the BPA $m(x)$ is equal to $p(x)$. Therefore with evidence theory, we can simultaneously handle a mixture of input parameters. Some of the inputs can be described probabilistically (random uncertainty) and some can be described through expert opinions (epistemic uncertainty with incomplete data). In the first case, the range of each input parameter will be discretized using a finite number of intervals. The BPA value for each interval must be equal to the PDF area within the interval.

Evidence obtained from independent sources or experts must be combined. If the BPA's m_1 and m_2 express evidence from two experts, the combined evidence m can be calculated by the following Dempster's rule of combining [36]

$$m(A) = \frac{\sum_{B \cap C = A} m_1(B)m_2(C)}{1 - K} \text{ for } A \neq \emptyset \quad (11)$$

where

$$K = \sum_{B \cap C = \emptyset} m_1(B)m_2(C) \quad (12)$$

represents the *conflict* between the two independent experts. Dempster's rule filters out any conflict, or contradiction among the provided evidence, by normalizing with the complementary degree of conflict. It is usually appropriate for relatively small amounts of conflict where there is some consistency or sufficient agreement among the opinions of the experts. Yager [10] has proposed an alternative rule of combination where all degrees of contradiction are attributed to total ignorance. Other rules of combining can be found in [36].

2.1. ASSESSING BELIEF AND PLAUSIBILITY WITH DEMPSTER-SHAFER THEORY

The previous section described a methodology to quantify epistemic uncertainty, even when the experts provide conflicting evidence. This section shows how to propagate epistemic uncertainty through a given model (transfer function). We will illustrate that, using the following simple transfer function

$$y = f(a, b) \quad (13)$$

where $a \in A, b \in B$ are two independent input parameters and y is the output. The combined BPA's for both a and b are obtained from Dempster's rule of combining of Eq. (11) if multiple experts have provided evidence for either a or b . With combined information for each input parameter, we define a vector $c = [a_{ci}, b_{cj}]$, needed to calculate the output y as

$$C = A \times B = \{c = [a_{ci}, b_{cj}] | a_{ci} \in A, b_{cj} \in B\} \quad (14)$$

where subscript c stands for "combined" and i, j indicate focal elements.

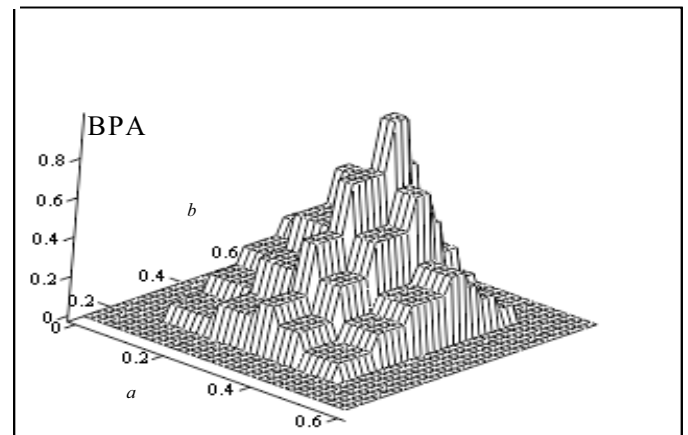


Figure 1. Representative BPA structure for two parameters a and b .

Taking advantage of assumed parameter independency, the BPA for c is

$$m_c(h_{ij}) = m(a_{ci})m(b_{cj}) \quad (15)$$

where $h_{ij} = [a_{ci}, b_{cj}]$ and a_{ci}, b_{cj} denote intervals such that $a \in a_{ci}$ and $b \in b_{cj}$. Eq. (15) can be used to calculate

the combined BPA structure for the entire domain C . For every $(a,b) \in c \mid c \in C$, needed to evaluate the output y , the combined BPA m_c is used. A representative combined BPA structure is shown in Figure 1.

The Cartesian product C of Eq. (14) is also called frame of discernment (FD) in the literature. It consists of all focal elements (rectangles in Figure 1 with nonzero combined BPA) and can be viewed as the finite sample space in probability theory.

If a domain F is defined as

$$F = \{g : g = f(a,b) - y_0 > 0, (a,b) \in c, c = [a_c, b_c] \subset C\} \quad (16)$$

where y_0 is a specified value, $Bel(F)$ and $Pl(F)$ can be calculated from Eqs (6) and (7) where set F replaces set A . According to evidence theory, $Bel(F)$ and $Pl(F)$ bracket the true probability $p_f = P(g > 0)$ [9,27]; i.e.

$$Bel(F) \leq p_f \leq Pl(F). \quad (17)$$

The $Bel(F)$ and $Pl(F)$ are calculated using Eqs (6) and (7) where set A is equal to set F of Eq. (16) and B is a rectangular domain (focal element) such that $B \subseteq A$ for Eq. (6) and $B \cap A \neq \emptyset$ for Eq. (7). In other words, $B \subseteq A$ means that the focal element must be entirely within the domain $g > 0$ and $B \cap A \neq \emptyset$ means that the focal element must be entirely or partially within the domain $g > 0$ (see Fig. 2). In general, in order to identify if a focal element B satisfies $B \subseteq A$ or $B \cap A \neq \emptyset$, the following minimum and maximum values of g must be calculated

$$[g_{\min}, g_{\max}] = [\min_{\mathbf{X}} g(\mathbf{X}), \max_{\mathbf{X}} g(\mathbf{X})] \quad (18)$$

for $\mathbf{X}^L \leq \mathbf{X} \leq \mathbf{X}^U$ where $(\mathbf{X}^L, \mathbf{X}^U)$ defines the focal element domain. For monotonic functions, the vertex method [37] can be used to calculate the minimum and maximum values in Eq. (18) by simply identifying the minimum and maximum values among all vertices of the focal element domain. If for a focal element, $g_{\min}$ and $g_{\max}$ are both positive, the focal element will contribute to the calculation of belief and plausibility according to Eqs (6) and (7). On the other hand, if $g_{\min}$ and $g_{\max}$ are both negative, the focal element will not contribute to the calculation of belief or plausibility. If however, $g_{\min}$ is negative and $g_{\max}$ is positive, the focal element will not contribute to the belief but it will contribute to the plausibility calculation. This is shown schematically in Figure 2.

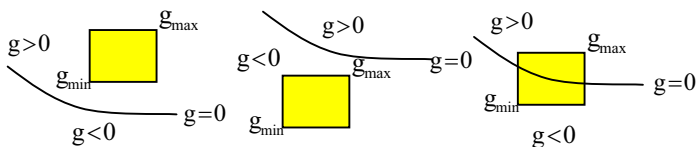


Figure 2. Schematic illustration of focal element contribution to belief and plausibility measures.

In summary the following tasks are performed in order to calculate the belief and plausibility of the failure region:

- 1) For each input parameter, combine the evidence from the experts by combining the individual BPA's from each expert using Dempster's rule of combining (Eq. (11)).
- 2) Construct the BPA structure for the m -dimensional frame of discernment, where m is the number of input parameters. Assuming independent input parameters, Eq. (15) is used.
- 3) Identify the failure region space (set F of Eq. (16)).
- 4) Use Eqs (6) and (7) to calculate the belief and plausibility measures of the failure region. The failure region must be identified only within the frame of discernment. The true probability of failure is bracketed according to Eq. (17).

3. EVIDENCE-BASED DESIGN OPTIMIZATION (EBDO)

In deterministic design optimization, an objective function is minimized subject to satisfying each constraint. In Reliability-Based Design Optimization (RBDO), where all design variables are characterized probabilistically, an objective function is usually minimized subject to the probability of satisfying each constraint, being greater than a specified high reliability level. In this section, a methodology is presented on how to use evidence theory in design. We will show that the evidence theory-based design is conservative compared with all RBDO designs obtained with different probability distributions.

If feasibility of a constraint g is expressed with the non-negative null form $g \geq 0$, we have shown in the previous section that $P(g \geq 0)$ is bracketed by the belief $Bel(g \geq 0)$ and plausibility $Pl(g \geq 0)$; i.e. $Bel(g \geq 0) \leq P(g \geq 0) \leq Pl(g \geq 0)$. Therefore,

$$P(g < 0) \leq p_f \text{ is satisfied if } Pl(g < 0) \leq p_f \quad (19)$$

where p_f is the probability of failure which is usually a small prescribed value. The above statement is equivalent to

$$P(g \geq 0) \geq R \text{ is satisfied if } Bel(g \geq 0) \geq R \quad (20)$$

where $R = 1 - p_f$ is the corresponding reliability level.

An evidence theory-based design optimization (EBDO) problem can be formulated as

$$\begin{aligned} & \min_{\mathbf{d}, \mathbf{X}} f(\mathbf{d}, \mathbf{X}, \mathbf{P}^N) \\ & \text{s.t. } Pl(g_i(\mathbf{d}, \mathbf{X}, \mathbf{P}) < 0) \leq p_{f_i}, \quad i = 1, \dots, n \\ & \mathbf{d}_L \leq \mathbf{d} \leq \mathbf{d}_U \\ & \mathbf{X}_L^N \leq \mathbf{X}^N \leq \mathbf{X}_U^N \end{aligned} \quad (21)$$

where $\mathbf{d} \in R^k$ is the vector of deterministic design variables, $\mathbf{X} \in R^m$ is the vector of uncertain design