

How to Play a Disc Brake: A Dissipation-Induced Squeal

Oleg N. Kirillov

Technische Universität Darmstadt
Institute of Mechanics, Moscow State Lomonosov University

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ABSTRACT

The eigenvalues of an elastic body of revolution, rotating about its axis of symmetry, form a 'spectral mesh'. The nodes of the mesh in the plane 'frequency' versus 'gyroscopic parameter' correspond to the double eigenfrequencies. With the use of the perturbation theory of multiple eigenvalues, deformation of the spectral mesh caused by dissipative and non-conservative perturbations, originating from the frictional contact, is analytically described. The key role of indefinite damping and non-conservative positional forces in the development of the subcritical flutter instability is clarified. A clear mathematical description is given for the mechanism of excitation of particular modes of rotating structures in frictional contact, such as squealing disc brakes and singing wine glasses.

INTRODUCTION

An axially symmetric shell, like a wine glass, can easily produce sound when a wet finger is rubbed around its rim or wall, as it was observed already in 1638 by Galileo Galilei in his Dialogues Concerning Two New Sciences [6, 29]. This principle is used in playing the glass harmonica invented by Benjamin Franklin in 1761, which he called 'armonica', where, in order to produce sound, one should touch by a moist finger an edge of a glass bowl rotating about its axis of symmetry [12, 19, 29, 47]. This remarkable phenomenon has been studied experimentally; see e.g. [35]. However, an adequate theory for its description seems to be still missing. Another closely related example of acoustics of friction is the squealing disc brake [19, 23, 34]. This mechanical system produces sound due to transverse vibrations of a rotating annular plate caused by its interaction with the brake pads. Despite intensive experimental and theoretical study, the problem of predicting and controlling the squeal remains an important issue [10, 16, 19, 21, 23, 34, 42, 43, 48, 49]. Significant but still poorly understood phenomena are squealing and barring of calender rolls in paper mills causing a noise and reducing the quality of the paper [30].

The presence of multiple eigenvalues in the spectra of free vibrations of axially symmetric shells and plates is

well known. Already Rayleigh, studying the acoustics of bells, recognized that, if the symmetry of a bell were complete, the nodal meridians of a transverse vibration mode would have no fixed position but would travel freely around the bell, as do those in a wine glass driven by the moistened finger [19]. This is a reflection of the fact that spectra of free vibrations of a bell, a wine glass, an annular plate, and other bodies of revolution contain double purely imaginary semi-simple eigenvalues with two linearly independent eigenvectors.

Rotation causes the double eigenvalues of an axially symmetric structure to split [1]. The newborn pair of simple eigenvalues corresponds to the forward and backward traveling waves, which propagate along the circumferential direction [1, 2, 3, 7, 13, 16, 17]. Viewed from the rotational frame, the frequency of the forward traveling wave appears to decrease and that of the backward traveling wave appears to increase, as the spin increases. Due to this fact, double eigenvalues originate again at non-zero angular velocities, forming the nodes of the spectral mesh in the plane 'eigenfrequency' versus 'angular velocity'. The spectral meshes are characteristic for the rotating circular strings, rings, discs, and cylindrical and hemispherical shells. The phenomenon is apparent also in hydrodynamics, in the problem of stability of a vortex tube [24] and in magnetohydrodynamics in the problem of instability of the spherically symmetric MHD α^2 -dynamo [38].

It is known that striking the wine glass excites a number of modes, but rubbing the rim with a finger generally excites a single mode [12]. Similar phenomenon is observed for the squealing disc brake [19, 23, 34, 39, 40, 41]. For this reason, we formulate the main problem of acoustics of friction of rotating elastic bodies of revolution as the description of the mechanism of activating a particular mode of the continuum by its contact with an external body.

In case of the disc brake, the frictional contact of the brake pads with the rotating disc introduces dissipative and non-conservative positional forces into the system [34, 43, 46, 48]. Since the nodes of the spectral mesh correspond to the double eigenvalues, they are most sensitive to perturbations, especially to those breaking

the symmetries of the system. Consequently, the instability will most likely occur at the angular velocities close to that of the nodes of the spectral mesh. The unstable modes of the perturbed system will have the frequencies close to that of the double eigenvalues at the nodes. This picture qualitatively agrees with the existing experimental data [12, 19, 23, 34, 39, 40, 41].

In the following using the perturbation theory of multiple eigenvalues [28, 32, 33, 38] we will show that independently on the definiteness of the damping matrix, there exist combinations of dissipative and non-conservative positional forces, causing the flutter instability in the vicinity of the nodes of the spectral mesh for the angular velocities from the subcritical range. Zero and negative eigenvalues in the spectrum of the damping matrix encourage the development of the localized subcritical flutter instability while zero eigenvalues in the matrix of non-conservative positional forces suppress it. Explicit expressions describing the movements of eigenvalues due to change of the system parameters will be obtained. Conditions will be derived for the eigenvalues to move to the right part of the complex plane. Approximations of the domain of asymptotic stability will be found and singularities of the stability boundary responsible for the development of instability will be described and classified. The methodology developed for the study of the two-dimensional system will finally be employed to the detailed investigation of the stability of a rotating circular string constrained by a stationary load system.

THE SPECTRAL MESH OF A GYROSCOPIC SYSTEM WITH TWO DEGREES OF FREEDOM

Consider an autonomous non-conservative system

$$(1) \ddot{\mathbf{x}} + (2\Omega\mathbf{G} + \delta\mathbf{D})\dot{\mathbf{x}} + ((\beta^2 - \Omega^2)\mathbf{I} + \kappa\mathbf{K} + \nu\mathbf{N})\mathbf{x} = 0,$$

where a dot over a symbol denotes time differentiation, $\mathbf{x} \in R^2$, and $\mathbf{I}$ is the identity matrix. The real matrices $\mathbf{D} = \mathbf{D}^T$, $\mathbf{G} = -\mathbf{G}^T$, $\mathbf{K} = \mathbf{K}^T$, and $\mathbf{N} = -\mathbf{N}^T$ are related to dissipative (damping), gyroscopic, potential, and non-conservative positional (circulatory) forces with magnitudes controlled by scaling factors δ , Ω , κ , and ν respectively; $\beta > 0$ is the frequency of free vibrations of the potential system, corresponding to $\delta = \Omega = \kappa = \nu = 0$. The parameters and variables of the system are assumed to be non-dimensional quantities. Without loss in generality it is accepted that $\det \mathbf{G} = 1$ and $\det \mathbf{N} = 1$. Equation (1) frequently appears as a two-mode approximation of the models of rotating elastic bodies of revolution in frictional contact after their linearization and discretization [16, 23, 46].

Separating time by setting $\mathbf{x}(t) = \mathbf{u} \exp(\lambda t)$ we arrive at the eigenvalue problem $\mathbf{L}\mathbf{u} = 0$, where

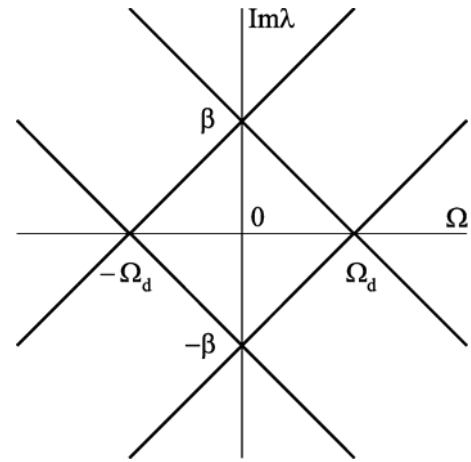


Figure 1.

The spectral mesh of system (1) when $\delta = \kappa = \nu = 0$.

$$(2) \mathbf{L} = \mathbf{I}\lambda^2 + (2\Omega\mathbf{G} + \delta\mathbf{D})\lambda + (\beta^2 - \Omega^2)\mathbf{I} + \kappa\mathbf{K} + \nu\mathbf{N}$$

Applying the Leverrier-Barnett algorithm [9] to the operator $\mathbf{L}$ we find the characteristic polynomial

$$\begin{aligned} P(\lambda) = & \lambda^4 + \delta \text{Tr} \mathbf{D} \lambda^3 \\ (3) & + (2(\beta^2 + \Omega^2) + \delta^2 \det \mathbf{D} + \kappa \text{Tr} \mathbf{K}) \lambda^2 \\ & + (4\Omega\nu + \delta(\beta^2 - \Omega^2) \text{Tr} \mathbf{D} + \delta\kappa(\text{Tr} \mathbf{K} \text{Tr} \mathbf{D} - \text{Tr} \mathbf{K} \mathbf{D})) \lambda \\ & + \kappa^2 \det \mathbf{K} + \kappa \text{Tr} \mathbf{K} (\beta^2 - \Omega^2) + (\beta^2 - \Omega^2)^2 + \nu^2, \end{aligned}$$

where Tr stands for the trace of a matrix. In the absence of dissipative, external potential, and non-conservative positional forces the polynomial (3), corresponding to the operator $\mathbf{L}_0(\Omega) = \mathbf{I}\lambda^2 + 2\lambda\Omega\mathbf{G} + (\beta^2 - \Omega^2)\mathbf{I}$, has four purely imaginary roots [18]

$$(4) \lambda_p^\pm = i\beta \pm i\Omega, \quad \lambda_n^\pm = -i\beta \pm i\Omega$$

In the plane $(\Omega, \text{Im} \lambda)$ equations (4) describe straight lines intersecting with each other (Fig. 1) - the *spectral mesh* [38]. Two nodes of the mesh at $\Omega = 0$ correspond to the double semi-simple eigenvalues $\lambda = \pm i\beta$. At the other two nodes at $\Omega = \pm\Omega_d$ there exist double semi-simple eigenvalues $\lambda = 0$. The range $|\Omega| < \Omega_d = \beta$ is called *subcritical* for the gyroscopic parameter Ω [18]. The double semi-simple eigenvalue $i\beta$ at $\Omega = \Omega_0 = 0$ has two linearly-independent eigenvectors $\mathbf{u}_1$ and $\mathbf{u}_2$

$$(5) \mathbf{u}_1 = \frac{1}{\sqrt{2\beta}} \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \mathbf{u}_2 = \frac{1}{\sqrt{2\beta}} \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

The eigenvectors are orthogonal $\mathbf{u}_i^T \mathbf{u}_j = 0$, $i \neq j$, and satisfy the normalization condition $\mathbf{u}_i^T \mathbf{u}_i = (2\beta)^{-1}$.

Under variation of the gyroscopic parameter $\Omega = \Omega_0 + \Delta\Omega$, the double eigenvalue $i\beta$ into two simple ones bifurcates. The asymptotic formula for the perturbed eigenvalues is [32]

$$(6) \lambda_p^\pm = i\beta + i\Delta\Omega \frac{f_{11} + f_{22}}{2} \pm i\Delta\Omega \sqrt{\frac{(f_{11} - f_{22})^2}{4} + f_{12}f_{21}}$$

where the quantities f_{ij} are

$$(7) f_{ij} = \mathbf{u}_j^T \frac{\partial \mathbf{L}_0(\Omega)}{\partial \Omega} \mathbf{u}_i \Big|_{\Omega=0, \lambda=i\beta} = 2i\beta \mathbf{u}_j^T \mathbf{G} \mathbf{u}_i.$$

The skew symmetry of $\mathbf{G}$ yields $f_{11} = f_{22} = 0$, $f_{12} = -f_{21} = i$, so that (6) gives the exact result (4).

DEFORMATION OF THE SPECTRAL MESH

Consider a perturbation of the gyroscopic system $\mathbf{L}_0(\Omega) + \Delta\mathbf{L}(\Omega)$, assuming that the size of the perturbation $\Delta\mathbf{L}(\Omega) = \delta\lambda \mathbf{D} + \kappa \mathbf{K} + \nu \mathbf{N} \sim \varepsilon$ is small, where $\varepsilon = \|\Delta\mathbf{L}(0)\|$. The behavior of the perturbed double eigenvalue $i\beta$ for small Ω and ε is described by the asymptotic formula [32]

$$(8) \lambda = i\beta + i\Omega \frac{f_{11} + f_{22}}{2} + i \frac{\xi_{11} + \xi_{22}}{2} \pm i \sqrt{\frac{(\Omega(f_{11} - f_{22}) + \xi_{11} - \xi_{22})^2}{4} + (\Omega f_{12} + \xi_{12})(\Omega f_{21} + \xi_{21})}$$

where f_{ij} are given by (7) and

$$(9) \xi_{ij} = \mathbf{u}_j^T \Delta\mathbf{L}(0) \mathbf{u}_i = i\beta \delta \mathbf{u}_j^T \mathbf{D} \mathbf{u}_i + \kappa \mathbf{u}_j^T \mathbf{K} \mathbf{u}_i + \nu \mathbf{u}_j^T \mathbf{N} \mathbf{u}_i$$

are small complex numbers of order ε . With the vectors (5) we obtain

$$(10) \lambda = i\beta + i \frac{\rho_1 + \rho_2}{4\beta} \kappa - \frac{\mu_1 + \mu_2}{4} \delta \pm \sqrt{c},$$

$$(11) c = \left(\frac{\mu_1 - \mu_2}{4} \right)^2 \delta^2 - \left(\frac{\rho_1 - \rho_2}{4\beta} \right)^2 \kappa^2 + \left(i\Omega + \frac{\nu}{2\beta} \right)^2 - i\delta\kappa \frac{2\text{Tr}\mathbf{K}\mathbf{D} - \text{Tr}\mathbf{K}\text{Tr}\mathbf{D}}{8\beta},$$

where μ_1, μ_2 and ρ_1, ρ_2 are eigenvalues of $\mathbf{D}$ and $\mathbf{K}$, respectively, and thus satisfy the equations

$$(12) \mu^2 - \mu \text{Tr}\mathbf{D} + \det \mathbf{D} = 0, \quad \rho^2 - \rho \text{Tr}\mathbf{K} + \det \mathbf{K} = 0.$$

Separation of real and imaginary parts in (10) yields

$$(13) \text{Re } \lambda = -\frac{\mu_1 + \mu_2}{4} \delta \pm \sqrt{\frac{\text{Re } c + \sqrt{(\text{Re } c)^2 + (\text{Im } c)^2}}{2}},$$

$$\text{Im } \lambda = \beta + \frac{\rho_1 + \rho_2}{4\beta} \kappa \pm \sqrt{\frac{-\text{Re } c + \sqrt{(\text{Re } c)^2 + (\text{Im } c)^2}}{2}},$$

where

$$(14) \text{Im } c = \frac{\Omega\nu}{\beta} - \delta\kappa \frac{2\text{Tr}\mathbf{K}\mathbf{D} - \text{Tr}\mathbf{K}\text{Tr}\mathbf{D}}{8\beta}$$

$$\text{Re } c = \left(\frac{\mu_1 - \mu_2}{4} \right)^2 \delta^2 - \left(\frac{\rho_1 - \rho_2}{4\beta} \right)^2 \kappa^2 - \Omega^2 + \frac{\nu^2}{4\beta^2}.$$

The formulas (10) – (14) describe splitting of the double eigenvalues at the nodes of the spectral mesh due to variation of parameters, including those corresponding to dissipative and non-conservative positional forces. This leads to the deformation of the mesh, in particular, to the *veering* and *merging* of eigenvalue branches.

Although the veering phenomenon in the systems with gyroscopic coupling was studied both numerically and analytically, e.g. in [3, 4, 7, 8, 13, 17, 31], the explicit expressions (10) – (14) for the splitting of the double eigenvalues due to action of forces of all types were not previously derived. The approach used in our paper is distinct of that of the cited works. It is based on the perturbation theory of multiple eigenvalues [28, 32, 33]. The spectrum of the perturbed system is described by means of only the derivatives of the operator with respect to parameters and the eigenvectors of the multiple eigenvalue calculated directly at a node of the spectral mesh.

THE EFFECT OF POTENTIAL FORCES

We first study the influence of a conservative perturbation with the matrix $\mathbf{K}$ on the spectral mesh of the gyroscopic system. Substituting $\delta = \nu = 0$ into the formulas (13) and (14) and transforming them yields

$$(15) \left(\text{Im } \lambda - \beta - \frac{\rho_1 + \rho_2}{4\beta} \right)^2 - \Omega^2 = \left(\frac{\rho_1 - \rho_2}{4\beta} \right)^2 \kappa^2,$$

$$\text{Re } \lambda = 0.$$

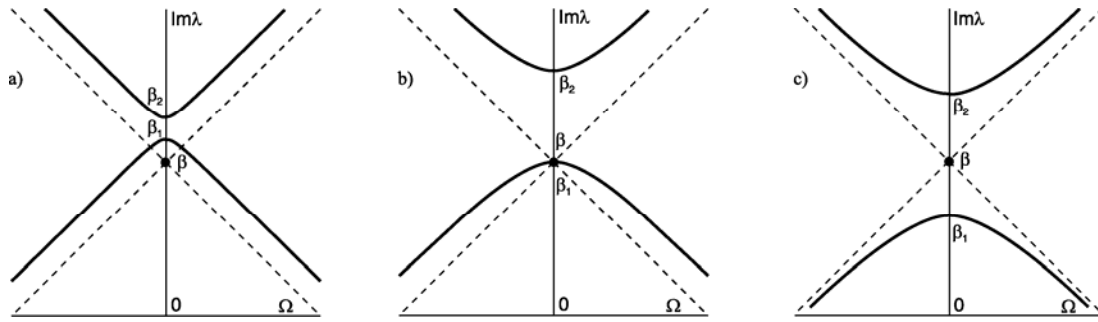


Figure 2.

Conservative deformation of the spectral mesh ($\kappa > 0$): $\mathbf{K} > 0$ (a); $\mathbf{K}$ is positive semi-definite (b); $\mathbf{K}$ is indefinite (c).

When $\kappa \neq 0$, equation (15) describes a hyperbola in the plane $\text{Im } \lambda$ versus Ω with the asymptotes

$$(16) \quad \text{Im } \lambda = \beta + \frac{\rho_1 + \rho_2}{4\beta} \kappa \pm \Omega.$$

The asymptotes cross each other above the node $(0, \beta)$ of the non-deformed spectral mesh for $\text{Tr} \mathbf{K} > 0$, exactly at the node for $\rho_1 = -\rho_2$, and below the node for $\text{Tr} \mathbf{K} < 0$. The branches of the hyperbola intersect the axis $\Omega = 0$ at the points

$$(17) \quad \beta_1 = \beta + \frac{\rho_1}{2\beta} \kappa, \quad \beta_2 = \beta + \frac{\rho_2}{2\beta} \kappa.$$

If the eigenvalues $\rho_{1,2}$ have the same sign, the intersection points are always located above or below the node of the spectral mesh for $\mathbf{K} > 0$ or $\mathbf{K} < 0$, respectively, see Fig. 2(a). In case when one of the eigenvalues $\rho_{1,2}$ is zero, implying semi-definiteness of the matrix $\mathbf{K}$, one of the branches of the hyperbola goes through the node of the spectral mesh and the other crosses the axis $\Omega = 0$ above the node, if $\mathbf{K}$ is positive semi-definite, or below it, if $\mathbf{K}$ is negative semi-definite, Fig. 2(b). If $\mathbf{K}$ is indefinite, one of the intersection points $\beta_{1,2}$ is located above the node and another one below the node, as indicated in Fig. 2(c).

Therefore, the conservative deformation of the spectral mesh does not shift the eigenvalues from the imaginary axis, preserving the marginal stability. However, the deformation pattern depends on the definiteness of the perturbation matrix $\mathbf{K}$. In particular, in the degenerate case when $\det \mathbf{K} = 0$, one of the eigenvalue branches, originated after the splitting of the double eigenvalue, always passes through the point corresponding to the node of the unperturbed spectral mesh, Fig. 2(b).

CREATION AND ACTIVATION OF THE LATENT SOURCES OF INSTABILITY BY DISSIPATION

Consider the effect of dissipative forces on the stability of the gyroscopic system and study its dependence on the properties of the matrix $\mathbf{D}$. Assuming $\nu = \kappa = 0$ in expression (11) we rewrite formula (10) in the form

$$(18) \quad \lambda = i\beta - \frac{\mu_1 + \mu_2}{4} \delta \pm \sqrt{\left(\frac{\mu_1 - \mu_2}{4}\right)^2 \delta^2 - \Omega^2}.$$

Since $\text{Im } c = 0$, equations (13) are transformed into

$$(19) \quad \left(\text{Re } \lambda + \frac{\mu_1 + \mu_2}{4} \delta\right)^2 + \Omega^2 = \frac{(\mu_1 - \mu_2)^2}{16} \delta^2,$$

$$\text{Im } \lambda = \beta$$

when

$$(20) \quad \Omega^2 - \frac{(\mu_1 - \mu_2)^2}{16} \delta^2 < 0,$$

and into

$$(21) \quad \Omega^2 - (\text{Im } \lambda - \beta)^2 = \frac{(\mu_1 - \mu_2)^2}{16} \delta^2,$$

$$\text{Re } \lambda = -\frac{\mu_1 + \mu_2}{4} \delta,$$

for the opposite sign in inequality (20). For a given δ equation (21) defines a hyperbola in the plane $(\Omega, \text{Im } \lambda)$, while (19) is the equation of a circle in the plane $(\Omega, \text{Re } \lambda)$, as shown in Fig. 3(a,c). For tracking the complex eigenvalues due to change of Ω , it is convenient to consider the eigenvalue branches in the three-dimensional space $(\Omega, \text{Im } \lambda, \text{Re } \lambda)$.

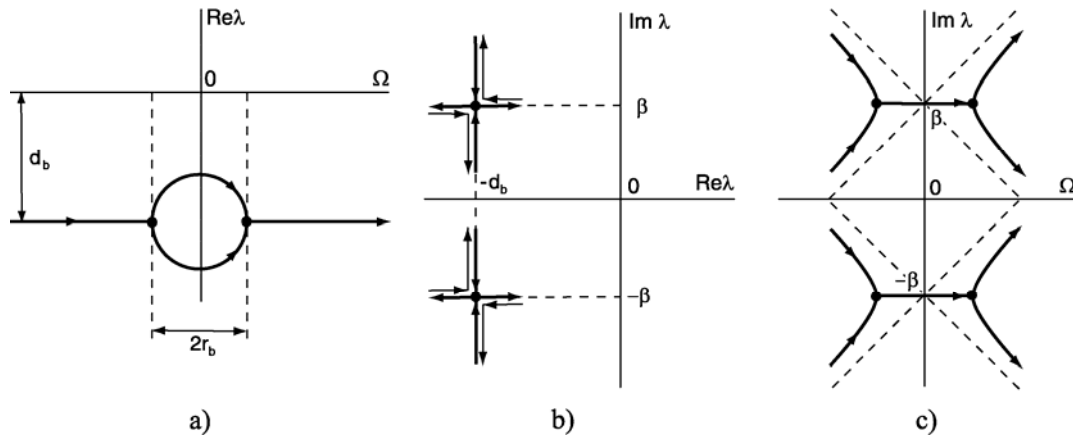


Figure 3. Origination of a latent source of the subcritical flutter instability in presence of full dissipation.

In this space the circle belongs to the plane $\text{Im } \lambda = \beta$ and the hyperbola lies in the plane $\text{Re } \lambda = -\delta(\mu_1 + \mu_2)/4$, see Fig. 4(a,c).

Below we show that the circle of the complex eigenvalues - *the bubble of instability* [20] - plays the crucial role in the development and localization of the subcritical flutter instability. Its properties depend on whether the matrix $\mathbf{D}$ is definite or indefinite.

Full dissipation and pervasive damping: A latent state of the bubble of instability

Formulas (19) and (21) show that the radius of the bubble of instability r_b and the distance d_b of its center from the plane $\text{Re } \lambda = 0$ are defined by the eigenvalues μ_1 and μ_2 of the matrix $\mathbf{D}$

$$(22) \quad r_b = \frac{|(\mu_1 - \mu_2)\delta|}{4}, \quad d_b = \frac{|(\mu_1 + \mu_2)\delta|}{4}.$$

Consequently, the bubble of instability is "submerged" under the surface $\text{Re } \lambda = 0$ in the space $(\Omega, \text{Im } \lambda, \text{Re } \lambda)$. It does not intersect the plane $\text{Re } \lambda = 0$ under the conditions

$$(23) \quad \delta(\mu_1 + \mu_2) = \delta \text{tr} \mathbf{D} > 0, \quad \mu_1 \mu_2 = \det \mathbf{D} \geq 0,$$

where the latter is equivalent to the inequality $d_b \geq r_b$. The inequalities (23) imply positive semi-definiteness of the matrix $\delta \mathbf{D}$. Hence, the role of full dissipation or pervasive damping is to deform the spectral mesh in such a way that the double eigenvalue is inflated to the bubble of complex eigenvalues (19) connected with the two branches of the hyperbola (21) at the points

$$(24) \quad \begin{aligned} \text{Im } \lambda &= \beta, \quad \text{Re } \lambda = -\delta(\mu_1 + \mu_2)/4, \\ \Omega &= \pm \delta(\mu_1 - \mu_2)/4, \end{aligned}$$

and to plunge all the eigenvalue curves into the region $\text{Re } \lambda \leq 0$. The eigenvalues at the points (24) are double and have a Jordan chain of generalized eigenvectors of order 2. In the complex plane the eigenvalues of the perturbed system move with the variation of Ω along the lines $\text{Re } \lambda = -d_b$ until they meet at the points (24) and then split in the orthogonal direction; however, they never cross the imaginary axis, see Fig. 3(b).

The bubble of instability has two remarkable properties important for the explanation of the phenomenon of squeal. First, there exist perturbations causing its growth and emersion above the surface $\text{Re } \lambda = 0$ (flutter); second, for small δ the instability is localized in the vicinity of the frequency $\text{Im } \lambda = \beta$ and of the value of the gyroscopic parameter $\Omega = 0$. The bubble is a latent source of subcritical flutter instability localized in a narrow range of change of the gyroscopic parameter with $|\Omega| < \Omega_d$ at the frequency corresponding to the double eigenvalue of the non-rotating system.

Indefinite damping: An active state of the bubble of instability

As it is seen from equations (22), the radius of the bubble of instability is greater than the depth of its submersion under the surface $\text{Re } \lambda = 0$ only if the eigenvalues μ_1 and μ_2 of the damping matrix have different signs, i.e. if the damping is indefinite. The damping with the indefinite matrix is a destabilizing factor [15, 44, 45]. It appears in the systems with frictional contact when the friction coefficient is decreasing with relative sliding velocity [14, 21, 23]. In our case it leads to the emersion of the bubble of instability so that the eigenvalues of the bubble have positive real parts in the range $\Omega^2 < \Omega_{cr}^2$, where the critical value is found from the equation (19)

$$(25) \quad \Omega_{cr} = \frac{\delta}{2} \sqrt{-\det \mathbf{D}}.$$