

Constructing Equations of Motion for a Vehicle Rigid Body Model

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ABSTRACT

This paper focuses on generating equations of motion for a Vehicle model. The generated equations can be used for rigid body analysis such as vibration analysis, frequency domain analysis, and time domain analysis. The paper starts with the equations of motion of a 6-degree-of-freedom (DOF) system and demonstrates detailed approach in building the equations. Next, the equations of motion are built for a 12-DOF vehicle model, which can be used as an approximated Vehicle model in most cases. Finally, the rigid body of an accurate Vehicle model is created and the equations of motion are constructed following the presented method.

Nomenclature

Symbols

θ	Rotation along Y-axis(Pitch)
ϕ	Rotation along X-axis(Roll)
ψ	Rotation along Z-axis(Yaw)
Z	Displacement along Z-direction
K	Stiffness of R
C	Damping coefficient
I	Moment inertia of Vehicle component

Subscripts

w	Wheel
c	Cab
f	Frame
x	X-direction
y	Y-direction
z	Z-direction

INTRODUCTION

In vehicle design and analysis, NVH analysis (noise, vibration, and harshness) and rigid body analysis (RBA) have become important considerations for designers. In order to perform these analyses, a qualified rigid body model (RBM) for the vehicle and a set of equations of motion need to be generated.

Various rigid body models have been developed for different applications. Song et. al. [1] created simple and

efficient rigid body models and used them for perturbation analysis and simulation of systems with frictional contacts. Hegazy et. al. [2] presented a complicated vehicle model for nonlinear rigid body analysis. This model had 94 degrees of freedom and comprised of front/rear suspensions, steering system, wheels/tires, and vehicle inertia. The model was successfully used for the vehicle handling analysis. Zhang et. al. [3] used flexible and rigid beams to create a vehicle model which was used for road surface analyses in order to find dynamic properties of the vehicle chassis/suspension system. Pereira et. al.[4] presented a multibody dynamic model for train vehicles and constructed formulations based on it. The model and the formulations were used for crashworthiness analysis and design of the train vehicles. Mousseau et. al. [5] successfully developed a comprehensive vehicle dynamics model for simulating the dynamics response of ground vehicles on different road surfaces. In their research, the multibody system simulations (MSS) program was used to simulate the vehicle and a nonlinear finite element model was used for the tires. Lee et. al. [6] created a computer model for a vehicle system and used it for frequency response analyses. The computer model consisted of vehicle body, suspension systems, and tires. A coupled formulation is also obtained for the vehicle model by using the modal synthesis technique. Hunt [7] devised a method of creating a multi-axle vehicle model and used this model to represent all vehicles on the road for vibration analysis. The effects of vehicle mass, speed, and wheelbase on its ground vibration were discussed. Ibrahim et. al. [8] applied finite element method to create a truck frame model in order to analyze the vehicle dynamic responses. The effects of the frame flexibility on the ride vibration behavior of trucks were investigated through the FE model.

The purpose of this paper is to build a concept vehicle model with necessary components and develop the equations of motion by applying such model. The developed model and equations can be used for rigid body analysis and correctly reflect the vehicle's frequency and time responses.

6-DOF SYSTEM

EOF FOR EXAMPLE SYSTEM

This project starts with a simpler case: modeling and formulating a 6-DOF system, which is the basic component in a vehicle assembly. Figure1 shows a cab model, which is simplified to be a 6-DOF rigid body. From this figure, the cab body is connected to ground through 4 mounts, and each mount has 3 springs and dampers along X-, Y-, and Z-direction. Conventionally, the three pairs of spring and damper elements are defined as a 3-DOF viscoelastic element. In other words, in our model, each mount has a 3-DOF viscoelastic element. For simplification, it is assumed that the stiffness/damping along the same direction are equals at each mount.

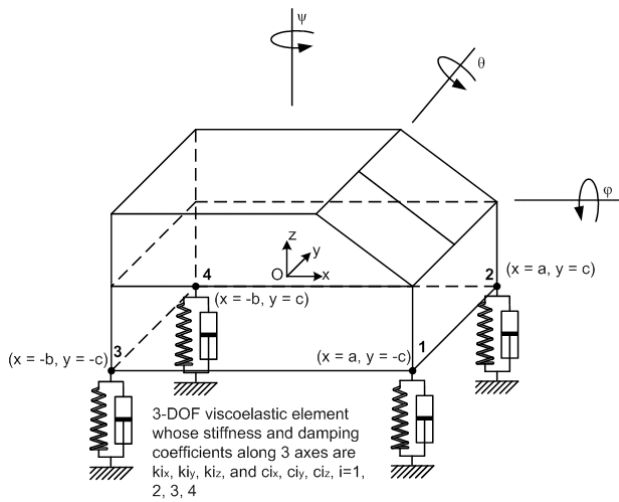


Fig. 1. 6-DOF cab model

In order to derive the EOM for such system, relationship among the degrees of freedom should be investigated at first. On analyzing figure 1, it is found that the cab's vertical displacement (Z), rolling (ϕ) and pitching (θ) are related to each other. In another words, the model's Z-displacement will change if it rotates about X- or Y-direction while stay unchanged if the cab only rotates about Z-direction. Similarly, the cab's lateral displacements (X and Y) and its yawing (ψ) are related and coupled together. Thus, the EOM of such 6-DOF rigid body can be represented as two uncoupled 3-DOF system. The EOM systems can be derived following Newton's law as

Forces along X-direction

$$m\ddot{x} = -k_{1x}(x - c\psi) - k_{2x}(x + c\psi) - k_{3x}(x - c\psi) - k_{4x}(x + c\psi) - c_{1x}(\dot{x} - c\dot{\psi}) - c_{2x}(\dot{x} + c\dot{\psi}) - c_{3x}(\dot{x} - c\dot{\psi}) - c_{4x}(\dot{x} + c\dot{\psi}) \quad (1-a)$$

Forces along Y-direction

$$m\ddot{y} = -k_{1y}(y + a\psi) - k_{2y}(y + a\psi) - k_{3y}(y - b\psi) - k_{4y}(y - b\psi) - c_{1y}(\dot{y} + a\dot{\psi}) - c_{2y}(\dot{y} + a\dot{\psi}) - c_{3y}(\dot{y} - b\dot{\psi}) - c_{4y}(\dot{y} - b\dot{\psi}) \quad (1-b)$$

Forces along Z-direction

$$m\ddot{z} = -k_{1z}(z - a\theta + c\phi) - k_{2z}(z - a\theta - c\phi) - k_{3z}(z + b\theta + c\phi) - k_{4z}(z + b\theta - c\phi) - c_{1z}(\dot{z} - a\dot{\theta} + c\dot{\phi}) - c_{2z}(\dot{z} - a\dot{\theta} - c\dot{\phi}) - c_{3z}(\dot{z} + b\dot{\theta} + c\dot{\phi}) - c_{4z}(\dot{z} + b\dot{\theta} - c\dot{\phi}) \quad (1-c)$$

Moments about X-axis

$$I_{xx}\ddot{\phi} = -k_{1z}(z - a\theta + c\phi)c + k_{2z}(z - a\theta - c\phi)c - k_{3z}c(z + b\theta + c\phi) + k_{4z}c(z + b\theta - c\phi) - c_{1z}(\dot{z} - a\dot{\theta} + c\dot{\phi})c + c_{2z}(\dot{z} - a\dot{\theta} - c\dot{\phi})c - c_{3z}c(\dot{z} + b\dot{\theta} + c\dot{\phi}) + c_{4z}c(\dot{z} + b\dot{\theta} - c\dot{\phi}) \quad (1-d)$$

Moments about Y-axis

$$I_{yy}\ddot{\theta} = k_{1z}a(z - a\theta + c\phi) + k_{2z}a(z - a\theta - c\phi) - k_{3z}b(z + b\theta + c\phi) - k_{4z}b(z + b\theta - c\phi) + c_{1z}a(\dot{z} - a\dot{\theta} + c\dot{\phi}) + c_{2z}a(\dot{z} - a\dot{\theta} - c\dot{\phi}) - c_{3z}b(\dot{z} + b\dot{\theta} + c\dot{\phi}) - c_{4z}b(\dot{z} + b\dot{\theta} - c\dot{\phi}) \quad (1-e)$$

Moments about Z-axis

$$I_{zz}\ddot{\psi} = -k_{1y}a(y + a\psi) - k_{2y}a(y + a\psi) + k_{3y}b(y - b\psi) + k_{4y}b(y - b\psi) - c_{1y}a(\dot{y} + a\dot{\psi}) - c_{2y}a(\dot{y} + a\dot{\psi}) + c_{3y}b(\dot{y} - b\dot{\psi}) + c_{4y}b(\dot{y} - b\dot{\psi}) + k_{1x}c(x - c\psi) - k_{2x}c(x + d\psi) + k_{3x}c(x - c\psi) - k_{4x}c(x + d\psi) + c_{1x}c(\dot{x} - c\dot{\psi}) - c_{2x}c(\dot{x} + d\dot{\psi}) + c_{3x}c(\dot{x} - c\dot{\psi}) - c_{4x}c(\dot{x} + d\dot{\psi}) \quad (1-f)$$

where a, b are distances from the gravity center of the cab model to its front and rear mounts; c, d are distances from the center to the left and right mounts. Here it is assumed that for a concept cab model, the distances to the two front mounts are equals, the same with the other mounts.

Rearranging the variables in the above formula and we have

$$m\ddot{x} + (k_{1x} + k_{2x} + k_{3x} + k_{4x})x + c(-k_{1x} - k_{3x} + k_{2x} + k_{4x})\psi + (c_{1x} + c_{2x} + c_{3x} + c_{4x})\dot{x} + c(-c_{1x} - c_{3x} + c_{2x} + c_{4x})\dot{\psi} = 0 \quad (2-a)$$

$$m\ddot{y} - y(k_{1y} + k_{2y} + k_{3y} + k_{4y}) - \psi c(k_{1y} + k_{2y} - k_{3y} - k_{4y}) + \dot{y}(c_{1y} + c_{2y} + c_{3y} + c_{4y}) + \dot{\psi} c(c_{1y} + c_{2y} - c_{3y} - c_{4y}) = 0 \quad (2-b)$$

$$m\ddot{z} + (k_{1z} + k_{2z} + k_{3z} + k_{4z})z + (-ak_{1z} - ak_{2z} + bk_{3z} + bk_{4z})\theta + c(k_{1z} + k_{2z} - k_{3z} - k_{4z})\phi + (c_{1z} + c_{2z} + c_{3z} + c_{4z})\dot{z} + (-ac_{1z} - ac_{2z} + bc_{3z} + bc_{4z})\dot{\theta} + c(c_{1z} + c_{3z} - c_{2z} - c_{4z})\dot{\phi} = 0 \quad (2-c)$$

$$\begin{aligned}
& I_{xx}\ddot{\theta} + (ck_{1z} + ck_{3z} - dk_{2z} - dk_{4z})z + c(-ak_{1z} + ak_{2z} + bk_{3z} - bk_{4z})\theta \\
& + c^2(k_{1z} + k_{2z} + k_{3z} + k_{4z})\dot{\theta} + c(c_{1z} + c_{2z} - c_{3z} - c_{4z})\dot{z} \\
& + c(-ac_{1z} + ac_{2z} + bc_{3z} - bc_{4z})\dot{\theta} + c^2(c_{1z} + c_{2z} + c_{3z} + c_{4z})\dot{\theta} = 0
\end{aligned}
\quad (2-d)$$

$$\begin{aligned}
& I_{yy}\ddot{\theta} + (-ak_{1z} - ak_{2z} + bk_{3z} + bk_{4z})z + (a^2k_{1z} + a^2k_{2z} + b^2k_{3z} + b^2k_{4z})\theta \\
& + c(-ak_{1z} + ak_{2z} + bk_{3z} - bk_{4z})\dot{\theta} + (-ac_{1z} - ac_{2z} + bc_{3z} + bc_{4z})\dot{z} \\
& + (a^2c_{1z} + a^2c_{2z} + b^2c_{3z} + b^2c_{4z})\dot{\theta} + c(-ac_{1z} + ac_{2z} + bc_{3z} - bc_{4z})\dot{\theta} = 0
\end{aligned}
\quad (2-e)$$

$$\begin{aligned}
& I_{zz}\ddot{\psi} + (ak_{1y} + ak_{2y} - bk_{3y} - bk_{4y})y + (ac_{1y} + ac_{2y} - bc_{3y} - bc_{4y})\dot{y} \\
& + (a^2k_{1y} + a^2k_{2y} + b^2k_{3y} + b^2k_{4y} + c^2k_{1x} + c^2k_{2x} + c^2k_{3x} + c^2k_{4x})\psi \\
& + (a^2c_{1y} + a^2c_{2y} + b^2c_{3y} + b^2c_{4y} + c^2c_{1x} + c^2c_{2x} + c^2c_{3x} + c^2c_{4x})\dot{\psi} \\
& + c(-k_{1x} - k_{3x} + k_{2x} + k_{4x})x + c(-c_{1x} - c_{3x} + c_{2x} + c_{4x})\dot{x} = 0
\end{aligned}
\quad (2-f)$$

Eq. (1) and (2) explain how to create EOM for a 6-DOF model with 4 mounts connected to the ground. However, a more general EOM has to be developed for any 6-DOF model which connects to the ground through arbitrary number of mounts.

MATRICES FOR GENERAL 6-DOF SYSTEM

Assume the coordinate of the i^{th} mount is (x_i, y_i, z_i) , and $k_{ix}, k_{iy}, k_{iz}, c_{ix}, c_{iy}, c_{iz}$ are its spring stiffness and damping coefficients along X-, Y-, and Z- direction. From Eq. (2), the system stiffness and damper matrices caused by the i^{th} mount can be written as

$$K_i = \begin{bmatrix} k_{ix} & 0 & 0 & 0 & 0 & -k_{ix}y_i \\ 0 & k_{iy} & 0 & 0 & 0 & k_{iy}x_i \\ 0 & 0 & k_{iz} & -k_{iz}x_i & k_{iz}y_i & 0 \\ 0 & 0 & -k_{iz}x_i & k_{iz}x_i^2 & -k_{iz}x_iy_i & 0 \\ 0 & 0 & k_{iz}y_i & -k_{iz}x_iy_i & k_{iz}y_i^2 & 0 \\ -k_{ix}y_i & k_{iy}x_i & 0 & 0 & 0 & k_{ix}y_i^2 + k_{iy}x_i^2 \end{bmatrix} \quad (3)$$

$$C_i = \begin{bmatrix} c_{ix} & 0 & 0 & 0 & 0 & -c_{ix}y_i \\ 0 & c_{iy} & 0 & 0 & 0 & c_{iy}x_i \\ 0 & 0 & c_{iz} & -c_{iz}x_i & c_{iz}y_i & 0 \\ 0 & 0 & -c_{iz}x_i & c_{iz}x_i^2 & -c_{iz}x_iy_i & 0 \\ 0 & 0 & c_{iz}y_i & -c_{iz}x_iy_i & c_{iz}y_i^2 & 0 \\ -c_{ix}y_i & c_{iy}x_i & 0 & 0 & 0 & c_{ix}y_i^2 + c_{iy}x_i^2 \end{bmatrix} \quad (3)$$

Eq. (3) and (4) are global stiffness and damping matrices. To simplify those matrices, local moment arm matrix, local stiffness matrix and local damping matrix are introduced.

$$K'_i = \begin{bmatrix} k_{ix} & 0 & 0 \\ 0 & k_{iy} & 0 \\ 0 & 0 & k_{iz} \end{bmatrix} \quad (4)$$

$$C'_i = \begin{bmatrix} c_{ix} & 0 & 0 \\ 0 & c_{iy} & 0 \\ 0 & 0 & c_{iz} \end{bmatrix} \quad (5)$$

$$L_i = \begin{bmatrix} 0 & 0 & -y_i \\ 0 & 0 & x_i \\ -x_i & y_i & 0 \end{bmatrix} \quad (6)$$

where K'_i is the system's local stiffness matrix due to the i^{th} mount, C'_i is the local damping matrix, and L_i is the moment arm matrix related to K'_i and C'_i .

Therefore, the global matrices (eq. (3, 4)) can be simplified by the local matrices and moment arm matrix as

$$K_i = \begin{bmatrix} K'_i & K'_i L_i \\ L_i^T K'_i & K'_i L_i^T L_i \end{bmatrix} \quad (8)$$

$$C_i = \begin{bmatrix} C'_i & C'_i L_i \\ L_i^T C'_i & C'_i L_i^T L_i \end{bmatrix} \quad (9)$$

In eq. (8, 9), the upper left sub-matrices, K'_i and C'_i , represent global translational stiffness and damping due to pure translation motions appears on the i^{th} mount. The upper right sub-matrices, $K'_i L_i$ and $C'_i L_i$ are the global translational stiffness and damping caused by pure rotations. The lower left sub-matrices, $L_i^T K'_i$ and $L_i^T C'_i$ represent the global rotational stiffness and damping due to the pure translational motions, which are transposed from the upper right sub-matrices. The lower right terms, $K'_i L_i^T L_i$ and $C'_i L_i^T L_i$ represent the global rotational stiffness and damping caused by the pure rotations. From eq. (3) and (5), it can also be concluded that the lower right terms $K'_i L_i^T L_i$ and $C'_i L_i^T L_i$ are symmetric matrices and the upper left terms K'_i and C'_i are diagonal matrices.

The whole system's stiffness and damping matrices then can be easily obtained by assembling all the sub stiffness and damping matrices together. These sub stiffness and damping matrices are brought by every single mount in the system (eq. (8, 9)).