

An RBDO Method for Multiple Failure Region Problems using Probabilistic Reanalysis and Approximate Metamodels

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ABSTRACT

A Reliability-Based Design Optimization (RBDO) method for multiple failure regions is presented. The method uses a Probabilistic Re-Analysis (PRRA) approach in conjunction with an approximate global metamodel with local refinements. The latter serves as an indicator to determine the failure and safe regions. PRRA calculates very efficiently the system reliability of a design by performing a single Monte Carlo (MC) simulation. Although PRRA is based on MC simulation, it calculates “smooth” sensitivity derivatives, allowing therefore, the use of a gradient-based optimizer. An “accurate-on-demand” metamodel is used in the PRRA that allows us to handle problems with multiple disjoint failure regions and potentially multiple most-probable points (MPP). The multiple failure regions are identified by using a clustering technique. A maximin “space-filling” sampling technique is used to construct the metamodel. A vibration absorber example highlights the potential of the proposed method.

INTRODUCTION

Due to competitive pressures, engineers and scientists are facing an ever-shortening design and validation cycle. The desire for fast development often comes at the cost of reliability and robustness. Early in the design phase it is necessary to make decisions based on

incomplete information and with a minimum of physical and virtual tests (FEA). In order to facilitate this decision-making, improved methods are required to assess system reliability and use it in design.

In order to quantify the performance of a system under uncertainty, a method is needed to determine the reliability of a design. By estimating the system reliability at a given design point, an optimization can be performed to determine the design that meets all specifications in terms of probability of failure.

In optimization, an objective is optimized while certain constraints are satisfied. The deterministic optimum though, *does not* necessarily have high reliability. To ensure that the optimum design is also reliable, the optimization formulation must include reliability constraints. Such a formulation is commonly referred as Reliability-Based Design Optimization (RBDO). RBDO approaches focus on maintaining design feasibility (for design constraints) at expected probabilistic levels. They account for inherent input and parameter variations in manufacturing processes, materials, loading, etc.

In RBDO, probability distributions describe the stochastic nature of the design variables and model parameters. Variations are represented by standard deviations (typically assumed to be constant) and a *mean* performance measure is optimized subject to *probabilistic* constraints. RBDO can be a powerful

design tool, since it provides optimum designs in the presence of uncertainty.

Efficiency and accuracy are the two main challenges in RBDO. The reliability, or equivalently the probability of failure, of each constraint must be calculated accurately and efficiently. The commonly used analytical methods (e.g. FORM) are usually efficient but inaccurate, especially for problems with “noisy” limit states, which may exhibit multiple failure regions and potentially multiple most-probable points. Such problems are common in structural dynamics. To alleviate this problem, simulation-based reliability methods may be used. However, they are computationally very expensive and therefore impractical for many engineering problems.

This paper presents a simulation-based RBDO method which is both accurate and efficient, using a PRRA method and an “accurate-on-demand” metamodel which allows us to handle problems with multiple disjoint failure regions, and potentially multiple MPPs. After an introduction to the analytical and simulation-based RBDO methods in section 1.1, we present an overview of the proposed RBDO method in section 2 providing details on the PRRA method in section 2.1, and details on the metamodel construction in section 2.2. Section 3 demonstrates the proposed methodology using a vibration absorber example. Finally, summary and conclusions are presented in section 4.

1. ANALYTICAL AND SIMULATION-BASED METHODS IN RBDO

For large-scale systems, the reliability prediction is usually based on efficient computational methods. Both analytical and simulation-based methods are available. The analytical methods are based on the MPP concept. They include the well-known first-order reliability method (FORM) that has been widely used [1], second-order reliability methods (SORM) [2], and multi-point approximation methods [3]. Among the simulation-based methods, the Monte Carlo Simulation (MCS) method is very simple and accurate. However, its computational cost can be prohibitively high. For this reason, more efficient simulation-based techniques have been proposed [4, 5]. Among them, the adaptive importance sampling (AIS) techniques are popular [6,7]. A combination of analytical and simulation-based methods has also been used [8]. The analytical methods are generally simple and efficient, but for complex problems, their accuracy cannot be guaranteed. In simulation-based methods, the accuracy can be controlled but the efficiency is generally not satisfactory.

For system reliability analysis involving multiple failure modes (limit states), the joint failure probability must be taken into account. Due to the difficulty in determining the joint failure probability of more than two failure modes except through MCS, approximations using first-order and second-order bounds have been developed [9, 10].

Simulation-based methods are also used for reliability analysis involving single or multiple limit states. Among MCS-based methods seeking to improve computational efficiency, adaptive importance sampling (AIS) techniques use an importance sampling density function, which is gradually refined to reflect the increasing state of knowledge of the failure domain. The importance sampling methods are divided into direct methods [5, 11], updating methods [12], spherical schemes [11], directional sampling [13], and adaptive schemes [6]. All methods, except the adaptive schemes, require prior knowledge of the failure domain.

2. OVERVIEW OF PROPOSED RBDO METHOD

A simulation-based Probabilistic Re-Analysis (PRRA) method is presented in this paper for the design of real-life engineering systems under uncertainty. The method is efficient because it uses a combination of an “accurate-on-demand” global metamodel (surrogate model) with local refinements to assess the reliability of constraints in the RBDO problem. The PRRA method allows the design to change during the optimization process without requiring additional computational effort because it uses a single Monte Carlo simulation.

It is well known that the analytical reliability methods of section 1.1 are computationally efficient, and relatively accurate depending on the application. However, their accuracy deteriorates for problems with highly non-linear and/or “noisy” limit states. Also, they are not appropriate for problems with disjoint failure domains and multiple MPPs. Although simulation-based reliability methods can address the shortcomings of the analytical methods, they are prohibitively expensive for many real-world engineering systems.

The proposed methodology is a simulation-based RBDO method which is computationally efficient and accurate. The high computational effort of most RBDO methods is due to the repeated evaluations of the reliability of each constraint. The PRRA method in this work drastically reduces the effort to estimate the reliability of a constraint. Quite often even a single MC simulation can be very expensive. For this reason, we use an “accurate-on-demand” global metamodel with local refinements to perform the single MC simulation very efficiently. We describe the basics of the PRRA method in section 2.1, and the “accurate-on-demand” metamodeling technique in section 2.2.

2.1. ESTIMATION OF FAILURE PROBABILITY IN PRRA

The probability of failure of a system is the integral of the joint PDF $f_{\mathbf{x}}(\mathbf{x})$ times the failure indicator function, over the range of the random variables,

$$p_f = \int_{\mathbf{x}} I(\mathbf{x}) f_{\mathbf{x}}(\mathbf{x}) d\mathbf{x} \quad (1)$$

In Importance Sampling (IS), sample values of the random variables are generated using a sampling PDF

$f_X(\mathbf{x})$ that yields many failures with high probability of occurrence, instead of the true PDF $f_X(\mathbf{x})$. An estimation of the probability of failure in this case is

$$p_f = \int_{\mathbf{x}} I(\mathbf{x}) \frac{f_X(\mathbf{x})}{f_X^S(\mathbf{x})} f_X^S(\mathbf{x}) d\mathbf{x} \quad (2)$$

where $I(\mathbf{x})$ is the failure indicator function. An unbiased estimator of the probability of failure is,

$$\hat{p}_f = \frac{1}{N} \sum_{i=1}^N I(\mathbf{x}_i) \frac{f_X(\mathbf{x}_i)}{f_X^S(\mathbf{x}_i)} = \frac{1}{N} \sum_{i=1}^N \frac{f_X(\mathbf{x}_i)}{f_X^S(\mathbf{x}_i)} \quad (3)$$

where $\mathbf{x}_i, i=1, \dots, N$ are the sample values of the random design variables generated from the sampling PDF $f_X^S(\mathbf{x})$, and N_f is the number of failures. The sum of the right hand side is only for those replications in which the system fails.

Assume that random variables $\mathbf{X}$ and $\mathbf{Y}$ are statistically independent, $f_{\mathbf{X},\mathbf{Y}}(\mathbf{x},\mathbf{y}) = f_X(\mathbf{x})f_Y(\mathbf{y})$, where $f_{\mathbf{X},\mathbf{Y}}(\mathbf{x},\mathbf{y})$ is the joint PDF of $\mathbf{X}$ and $\mathbf{Y}$, and $f_X(\mathbf{x})$ and $f_Y(\mathbf{y})$ are the marginal PDFs of these variables. A designer can control the mean value of $\mathbf{X}$ but not $\mathbf{Y}$. The key idea of PRRA is that it is sufficient to calculate the failure indicator function of a system only for a single sample of values, $\{\mathbf{x}_i, \mathbf{y}_i, i = 1, \dots, N_f\}$ of random variables $\mathbf{X}$ and $\mathbf{Y}$ in order to estimate the system failure probability for other PDFs. Indeed, if one generates a sample from $f_{\mathbf{X},\mathbf{Y}}^S(\mathbf{x},\mathbf{y})$ and calculates and saves the sample values that caused system failure $\{\mathbf{x}_i, \mathbf{y}_i, i = 1, \dots, N_f\}$, then one can reuse these sample values to estimate the probability of failure p_f for another PDF, $f_{\mathbf{X},\mathbf{Y}}(\mathbf{x},\mathbf{y})$,

$$\hat{p}_f = \frac{1}{N} \sum_{i=1}^N \frac{f_X(\mathbf{x}_i) f_Y(\mathbf{y}_i)}{f_{\mathbf{X},\mathbf{Y}}^S(\mathbf{x}_i, \mathbf{y}_i)} \quad (4)$$

Eq. (4) holds for any combination of PDFs in the numerator and denominator such that the support (set of all values for which a function is non zero) of $f_{\mathbf{X},\mathbf{Y}}^S(\mathbf{x},\mathbf{y})$ contains the support of $f_{\mathbf{X},\mathbf{Y}}(\mathbf{x},\mathbf{y})$.

Eq. (4) allows us to estimate the failure probabilities of many designs very efficiently because it only requires calculation of ratios of PDFs; it does not require calculation of the failure indicator function. This reduces the cost of system reliability analysis by several orders of magnitude because the calculation of the failure indicator function dominates the cost of simulation. The probability of system failure as a function of the mean values of the design variables μ_X is,

$$\hat{p}_f(\mu_X) = \frac{1}{N} \sum_{i=1}^N \frac{f_X(\mathbf{x}_i, \mu_X) f_Y(\mathbf{y}_i)}{f_{\mathbf{X},\mathbf{Y}}^S(\mathbf{x}_i, \mathbf{y}_i)} \quad (5)$$

where $f_X(\mathbf{x}, \mu_X)$ is the PDF of $\mathbf{X}$ given the mean value of the vector of the random variables $\mathbf{X}$ as a function of the mean values of these variables μ_X .

An unbiased estimator of the standard deviation of the system failure probability is,

$$\sigma_{\hat{p}_f} = \frac{1}{\sqrt{N(N-1)}} \sqrt{\sum_{i=1}^N \left(\frac{f_X(\mathbf{x}_i, \mu_X) f_Y(\mathbf{y}_i)}{f_{\mathbf{X},\mathbf{Y}}^S(\mathbf{x}_i, \mathbf{y}_i)} \right)^2 - N \cdot \hat{p}_f^2} \quad (6)$$

The $1-\alpha$ level confidence interval of the system failure probability is,

$$\hat{p}_f \pm t_{N-1, 1-\frac{\alpha}{2}} \cdot \sigma_{\hat{p}_f} \quad (7)$$

where $t_{N-1, 1-\frac{\alpha}{2}}$ is the point that has below its probability

$1-\alpha/2$ for Student's t -distribution with $(k-1)$ DOF. This interval covers the true system probability of failure with probability $1-\alpha$.

An algorithm is described below for efficient estimation of the failure probability for many different mean values of the random design variables.

1. Select a sampling PDF $f_{\mathbf{X},\mathbf{Y}}^S(\mathbf{x},\mathbf{y})$ for the random variables.
2. Generate a sample of values of variables $\mathbf{X}$ and $\mathbf{Y}$.
3. Calculate the system failure indicator function, $I(\mathbf{x}_i, \mathbf{y}_i)$, for the sample values in step 2. Select the subset of sample values that caused system failure.
4. Estimate the system failure probability using Eq. (5). The standard deviation of this probability and a confidence interval can be also calculated using Eqs (6) and (7).

The inputs to PRRA are the sampling PDF, the sample of values that caused failure and the PDF for which the failure probability is to be estimated. The output consists of the probability of failure and its confidence bounds if needed. PRRA is non-intrusive because it does not require modifications of the computer codes used for deterministic analysis and reliability analysis through MCS.

A two-dimensional example is used here to demonstrate graphically how the PRRA can be used in design. Although it is a low-dimensional problem, it is very challenging because of the highly non-linear limit state around the MPP. As such, it has been used in previous studies [14, 15]. The non-linear limit state function is defined as $g(x_1, x_2) = x_1^3 + x_2^3 - 180$, where x_1

and x_2 are independent, normally distributed random variables with standard deviation 5. A design optimization algorithm shifts the mean values of the

variables in order to minimize an objective while satisfying all constraints with a given probability of failure (see section 2). The objective of this example is to estimate efficiently the probability of failure as a function of the mean values of these variables and to determine those designs whose failure probability is no greater than some upper limit.

First, we generate 100,000 sample points assuming that random variables x_1 and x_2 have mean values 10 and 9.5, respectively. Figure 1 shows the limit state as a green line and the generated MCS sample points as blue dots. Then, sample points are selected among the MCS points by using the maximin algorithm in section 2.2. This algorithm has the advantage that it places points both on the boundary and on the interior of the domain of the random variables. The limit state function is evaluated at each point, and a metamodel is constructed. The metamodel is then refined (see section 2.4) in order to identify all “failure” points.

When we shift the mean values of random variables x_1 and x_2 , we change the probability of failure (constraint violation). PRRA (Eq. 5) is used to estimate the probability of failure for many combinations of the mean values of these random variables by using information from a single MCS. Figure 1 shows combinations of the mean values of the random variables (corresponding to the red dots) for which the system probability of failure is 0.01.

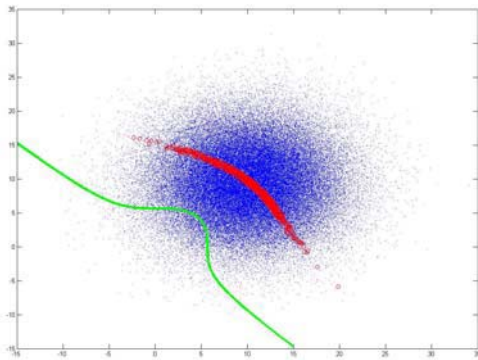


Figure 1. Identification of points with 0.01 probability of failure

2.2 MAXIMIN SAMPLING METHOD

We use an easy to implement maximin method in which the addition of new samples preserves by default, the “space-filling” properties. This is a very important property which allows us to construct “converged” metamodels with a small number of samples. Convergence is practically achieved if the reconstructed metamodel after a few more samples are added, gives similar predictions with the previous version. Similar predictions are judged based only on the sign, and not the estimated actual value, of the limit state.

Consider a design $D_n = \{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n\}$ which is composed of a collection of k -dimensional samples $\mathbf{x}_i \in \mathbb{R}^k, i=1, \dots, n$. A few judiciously selected samples from D_n define the maximin distance design D where, $D \subseteq D_n$. Let $d(\mathbf{u}, \mathbf{v})$ be the Euclidean distance between samples $\mathbf{u}$ and $\mathbf{v}$, where $\mathbf{u}, \mathbf{v} \in D$. Design D simply maximizes the minimum inter-site distance $\min_{\mathbf{u}, \mathbf{v} \in D} d(\mathbf{u}, \mathbf{v})$, i.e.

$$\max_D \min_{\mathbf{u}, \mathbf{v} \in D} d(\mathbf{u}, \mathbf{v}). \quad (8)$$

In this paper, the maximin sampling technique of Eq. (8) is implemented in a straightforward way. Assume that a design D_n of n , k -dimensional samples is available, consisting of n MC samples. Each of the k components of the n MC samples is normalized between zero and one in order to account for potentially different units. If x denotes the k^{th} component, its normalized value is $x^* = (x - x^L) / (x^H - x^L)$, where x^L and x^H are the low and high values respectively, among all n samples.

Point $\mathbf{x}_1^* = \{\bar{x}_1^*, \dots, \bar{x}_i^*, \dots, \bar{x}_k^*\}$, where $\bar{x}_i^*, i=1, \dots, k$ is the mean x_i^* coordinate of all n normalized points, is first selected as a “seed” point. Then the distance of all n normalized points from $\mathbf{x}_1^*$ is calculated and the point $\mathbf{x}_2^*$ with the largest distance is selected. Subsequently, the distances of the remaining $(n-2)$ samples from both $\mathbf{x}_1^*$ and $\mathbf{x}_2^*$ are calculated and the point $\mathbf{x}_3^*$ with the maximum minimum distance is selected. The process is repeated m times in order to create the maximin distance design D with m sample points. It should be noted that the method first places sample points on the boundary of the domain. As more points are created, they are placed in the interior of the domain providing therefore, a “space-filling” design. A very useful property of the method is that the addition of new sample points preserves the “space-filling” properties of the new design. This is essential in order to keep the number of function evaluations low.

2.3 AN “ACCURATE-ON-DEMAND” METAMODEL FOR RELIABILITY ESTIMATION

The method addresses problems with multiple MPPs with disjoint failure domains [14,15]. It can also assess for system reliability assessment in problems with multiple limit states. It can easily handle implicit, nonlinear limit-state functions, with correlated or non-correlated random variables, which are described by any probabilistic distribution. The method is based on an “accurate-on-demand” global metamodel of the limit states with local refinements, which can serve as an