



Fuzzy Boundary Element Method for Geometric Uncertainty in Elasticity Problem

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ABSTRACT

Solutions to partial differential equations describing behavior of physical systems are often imprecise. This uncertainty is due to numerical approximations and uncertainty in physical parameters. In elastostatics, these parameters include uncertain material behavior, uncertain boundary conditions, and uncertain geometry of the system. This paper addresses the treatment of geometrical uncertainty for elasticity problems. The new method predicts fuzzy responses for the given membership functions, describing the range of tolerances for the system's geometry. To obtain exact bounds on the solution to the resulting fuzzy linear system of equations, fuzzy matrix parameterization is developed. Numerical example is shown to illustrate the behavior of the method.

INTRODUCTION

An unavoidable problem in engineering practice results from the incapability to manufacture system elements to the prescribed dimensions. Although the majority of parts in the automobile industry are manufactured to be within prescribed tolerances, the true dimension of each component is unknown. Even though these geometry alterations are usually small, they may affect the reliability of vehicle performance, especially in a longer time span. Depending on the material used, small changes in the geometry of the system have a potential to cause fairly large and unforeseen stresses. Since usually changing the geometry of the entire system to

adjust to geometrical alteration of individual components is either impossible, due to design constraints, or unfeasible, a common procedure is to force the misfit members into their designed geometry. Doing so may introduce additional parameters that must be considered to fully assess the capability of those components. For geometrically defined problems, fabrication errors introduce additional displacements while for geometrically over-defined problems an addition of fabrication errors introduces internal stresses that may add to or counteract the stresses induced by the applied external loads. Since, in general, the magnitude of the fabrication errors is unknown; the magnitude of the stresses these misfits produce cannot be correctly quantified. Furthermore, the system is usually too complicated to be analyzed exactly and numerical techniques are used to predict the behavior of the system. In this work boundary element method is used to obtain approximate solutions to the system's behavior.

The solutions to the elasticity problem obtained through boundary element method are affected by numerous errors and uncertainties such as uncertainty in boundary conditions and system parameters, and errors due to numerical integration, rounding, and discretization. The uncertainty in the geometry of the system also affects the reliability of the numerical solutions. In previous works, uncertainty in boundary conditions, geometry and material has been considered as fuzzy numbers (Zadeh 1978, Ross 1995) for finite element method (FEM) resulting in fuzzy finite element method (FFEM) (Muhanna and Mullen 1999). Interval bounds for the

solutions of finite element buckling analysis considering uncertainty in the input material properties were also computed (Modares et al. 2006). Interval boundary element method (IBEM) was developed to obtain worst case bounds on the solutions considering discretization, truncation, and rounding errors, and uncertainty in boundary conditions (Zalewski and Mullen 2008, Zalewski and Mullen 2008, Zalewski et al. 2008, Zalewski and Mullen 2008).

An alternative approach of characterizing geometric uncertainty is by the use of random variables (Papoulis 1965), interval numbers (Moore 1966), and clouds (Neumaier 2004). Probabilistic approach assumes a random geometrical misfit, characterized by a cumulative distribution function (CDF), which in practice may not be obtainable. An interval characterization of the geometric uncertainty provides a similar approach as fuzzy numbers. However, it doesn't provide such flexibility as the fuzzy approach, especially when considering bounds which cannot be rigorously proven to be guaranteed, such as those obtained from expert opinions, and therefore vary depending on the confidence of the source. Clouds consider a certain probability of occurrence of a variable in a fuzzy set. Therefore, same limitations concerning the CDF as in probabilistic approach hold. A fuzzy approach circumvents most of these shortcomings as it allows for a computation of a reliable and guaranteed solution while considering flexibility in the formulation of the input bounds.

In this paper geometrical uncertainty in the elasticity problem is considered as a fuzzy number. A fuzzy number is a convex set whose bounds vary dependent on the level of the belief about the correctness of these bounds. The bounds on the fuzzy number are assumed to come from a variety of sources. These include but are not limited to: precision of machine used to manufacture mechanical components, experimental results for the deformation of material over time, and expert opinions about the possible deformation of the system. It can be easily concluded that all these sources of lower and upper bounds on the geometrical uncertainty vary dependent on how certain the sources themselves are. It is assumed that experts are knowledgeable and they behave rationally. It is further assumed that all the sources give a smaller value of a lower bound if their certainty is low and a larger value of a lower bound if their certainty is high. It is also assumed that all the sources give a larger value of an upper bound if their certainty is low and a smaller value of an upper bound if their certainty is high. Therefore, the difference between the upper and lower bounds decreases with increased certainty of the sources. The validity of the given bounds and their possible uncertainty is outside the scope of this paper and it is assumed that all bounds guarantee the enclosure of the true uncertainty.

In this work, geometrical uncertainty problem is substituted with the problem of fabrication error and dealt with in a usual approach as in strength of materials

(Timoshenko and Gear 1984, Shames and Pitaressi 2000). The fabrication error results in a fuzzy vector and fuzzy matrix parameterization is developed to consider the existence of the same fabrication error on multiple elements. This is a more realistic approach as the fabrication error is independent of the mesh. The parameterization allows computing exact bounds for the considered mesh. A numerical example illustrates the behavior of the method and shows that the bounds are exact independent of the mesh size.

FUZZY BOUNDARY ELEMENT METHOD (FBEM)

BOUNDARY ELEMENT FORMULATION FOR THE ELASTICITY PROBLEM

Boundary Element Formulation of the Navier's Equation

A detailed boundary element formulation for the behavior of an isotropic and homogeneous can be found in (Brebbia and Dominguez, 1992). The following section reviews the plain strain boundary element formulation for the elasticity problem. The elasticity problem is:

$$\left. \begin{aligned} \sigma_{ij,j} + b_i &= 0 \text{ in } \Omega \\ u_i &= \hat{u}_i \text{ on } \Gamma_1, \quad t_i = \hat{t}_i \text{ on } \Gamma_2 \\ \bigcup_{i=1}^2 \Gamma_i &= \Gamma \text{ and } \bigcap_{i=1}^2 \Gamma_i = \emptyset \end{aligned} \right\} \quad (1)$$

where Ω is the domain of the system, Γ is the boundary of the system, σ_{ij} is the stress tensor, b_i is the vector of body force, u_i is the displacement vector with a forced boundary condition $\hat{u}_i$ on Γ_1 , and t_i is the traction vector with a natural boundary condition $\hat{t}_i$ on Γ_2 . Since the exact solution to Eq. (1) cannot be generally obtained, approximate methods are used to obtain approximate solutions. For Navier's equation the starting point of the boundary element formulation is Betti's reciprocal theorem:

$$\int_{\Gamma} b_i^* u_i d\Gamma + \int_{\Gamma} t_i^* u_i d\Gamma = \int_{\Gamma} u_i^* b_i d\Gamma + \int_{\Gamma} u_i^* t_i d\Gamma \quad (2)$$

Equilibrium equation $\sigma_{ij,j}^* = -b_i^*$ is substituted into Eq. (2) yielding:

$$-\int_{\Gamma} \sigma_{ij,j}^* u_i d\Gamma + \int_{\Gamma} t_i^* u_i d\Gamma = \int_{\Gamma} u_i^* b_i d\Gamma + \int_{\Gamma} u_i^* t_i d\Gamma \quad (3)$$

In order to decrease the dimension of the integral equation, Eq. (3), the weighted residual function is set to

be the Green's function which is obtained by applying a concentrated force in direction a_i . This can be expressed as:

$$\sigma_{ij,j}^* = -\delta(x - \xi)a_i \quad (4)$$

where ξ is a source point at which a concentrated force in direction a_i is applied and x is a field point at which the response of the concentrated force is measured. The resulting fundamental solution is:

$$u_i^* = u_{ji}^* a_j, \quad t_i^* = t_{ji}^* a_j \quad (5)$$

where u_{ji}^* and t_{ji}^* are i components of the displacements and tractions, respectively, due to a concentrated force in the j direction, and a_j is a unit vector in the direction of the applied concentrated force. The kernel functions u_{ji}^* and t_{ji}^* are given as:

$$u_{ij}^* = \frac{1}{8\pi(1-\nu)G} \left[(4\nu-3)\ln(r)\delta_{ij} + \frac{(\bar{x}-\bar{\xi}) \cdot \bar{i}}{r} \cdot \frac{(\bar{x}-\bar{\xi}) \cdot \bar{j}}{r} \right],$$

$$t_{ij}^* = \frac{-1}{4\pi(1-\nu)r} \left\{ \left[(1-2\nu)\delta_{ij} + 2 \frac{(\bar{x}-\bar{\xi}) \cdot \bar{i}}{r} \cdot \frac{(\bar{x}-\bar{\xi}) \cdot \bar{j}}{r} \right] \cdot \frac{(\bar{x}-\bar{\xi}) \cdot \bar{n}}{r} - (1-2\nu) \left[\frac{(\bar{x}-\bar{\xi}) \cdot \bar{i}}{r} n_y - \frac{(\bar{x}-\bar{\xi}) \cdot \bar{j}}{r} n_x \right] \right\} \quad (6)$$

Where G is the shear modulus and ν is Poisson's ratio. Substituting Eq. (4-5) into Eq. (3) and interchanging indices in all integral terms yields:

$$u_i(\xi)a_i + \int_{\Gamma} t_{ij}^* a_i u_j d\Gamma = \int_{\Gamma} u_{ij}^* a_i b_j d\Gamma + \int_{\Gamma} u_{ij}^* a_i t_j d\Gamma, \quad \xi \in \Omega \quad (7)$$

Canceling out a_i coefficients, which are constant, and for convenience assuming that the body force is zero in Eq. (7) results in:

$$u_i(\xi) + \int_{\Gamma} t_{ij}^* u_j d\Gamma = \int_{\Gamma} u_{ij}^* t_j d\Gamma, \quad \xi \in \Omega \quad (8)$$

Eq. (8) is integrated such that the source point, ξ , is enclosed by a half-circular boundary of radius ε , as $\varepsilon \rightarrow 0$. This results in the right-hand side integral vanishing. For constant elements the left-hand side integral results in $-u(\xi)/2$. Eq. (8) can be thus rewritten on the boundary as:

$$\frac{1}{2} u_i(\xi) + \int_{\Gamma} t_{ij}^*(x, \xi) u_j(x) dx = \int_{\Gamma} u_{ij}^*(x, \xi) t_j(x) dx, \quad \xi \in \Gamma \quad (9)$$

Constant Element Boundary Discretization

The boundary integral equation, Eq. (9), can be discretized into boundary elements consisting of nodes and polynomial shape functions between nodes. At each node either the boundary value of the displacement or traction must be known. Also at least one value of the displacement in each dimension must be known for the problem to have a unique solution. In this paper, constant boundary elements are used for illustrative purpose; the extension to higher order elements is straightforward. Constant element discretization leads to the function to be approximated as:

$$u_i(x) = \Phi_J u_{ji} \quad (10)$$

$$q_i(x) = \Phi_J t_{ji} \quad (11)$$

where u_i and t_i are the components of the vectors of nodal values of u and t , respectively, at node J , and Φ_J is the vector of constant shape functions. The discretized form of Eq. (9) is written as:

$$\frac{1}{2} u_{ji} + \sum_{\text{Elements } \Gamma} \int_{\Gamma} t_{ij}^*(x, \xi) \Phi_J dx u_{jj} = \sum_{\text{Elements } \Gamma} \int_{\Gamma} u_{ij}^*(x, \xi) \Phi_J dx t_{jj} \quad (12)$$

Eq. (12) can be written in a matrix form:

$$Hu = Gt \quad (13)$$

where matrix H is singular. Eq. (13) is rearranged based on boundary conditions and solved as a linear algebra problem:

$$Ax = f \quad (14)$$

FUZZY SETS

In this paper uncertainty in system's geometry is treated using fuzzy numbers (Zadeh 1978, Ross 1995). A fuzzy set is a convex set consisting of interval sets which vary monotonically in width dependent on the level of presumption which is bounded between 0 and 1 (Ross 1995). The highest level of belief is level 1 and the lowest level is 0. The lower bounds of interval sets increase monotonically with increasing level of presumption while the higher bounds decrease monotonically (Figure 1).