

A Generalized Anisotropic Hardening Rule Based on the Mroz Multi-Yield-Surface Model and Various Classical Yield Functions

K. S. Choi* and J. Pan

The University of Michigan, Ann Arbor

Copyright © 2009 SAE International

ABSTRACT

In this paper, a generalized anisotropic hardening rule based on the Mroz multi-yield-surface model is derived. The evolution equation for the active yield surface is obtained by considering the continuous expansion of the active yield surface during the unloading/reloading process. The incremental constitutive relation based on the associated flow rule is then derived for a general yield function. Detailed incremental constitutive relations for materials based on the Mises yield function, the Hill quadratic anisotropic yield function and the Drucker-Prager yield function are derived as the special cases. The closed-form solutions for one-dimensional stress-plastic strain curves are plotted for materials under cyclic loading conditions based on the three yield functions. For materials based on the Mises and the Hill anisotropic yield functions, the stress-plastic strain curves show closed hysteresis loops under uniaxial cyclic loading conditions and the Masing hypothesis is applicable. For materials based on the Drucker-Prager yield function, the stress-plastic strain curves do not close and show the ratcheting effect under uniaxial cyclic loading conditions. The ratcheting effect is due to different strain ranges for a given stress range for the unloading and reloading processes. A user material subroutine based on the Mises yield function, the anisotropic hardening rule and the constitutive relation was written and implemented into ABAQUS. Computations were conducted for a simple plane strain finite element model under uniaxial monotonic and cyclic loading conditions based on the anisotropic hardening rule, the isotropic and nonlinear kinematic hardening rules of ABAQUS. The results indicate that the plastic

response of the material follows the intended input stress-strain data for the anisotropic hardening rule whereas the plastic response depends upon the input strain ranges of the stress-strain data for the nonlinear kinematic hardening rule.

INTRODUCTION

Different hardening rules were proposed in the past to specify the evolution of the yield surface during plastic deformation. Under monotonically increasing nearly proportional loading conditions, the isotropic hardening rule appears to give a good approximation of the material plastic hardening behavior. On the other hand, the kinematic hardening rule [1,2] was proposed to model the Bauschinger effect, which cannot be modeled by the isotropic hardening rule. However, under complicated plastic deformation histories, substantial deviations of the actual plastic behaviors from those predicted by both isotropic and kinematic hardening rules are often observed. Different plasticity theories were therefore proposed to model the cyclic hardening behavior [3-7].

Based on some general observations of the uniaxial cyclic plastic behavior of many metals, Mroz [4,8] introduced the concept of a field of work-hardening moduli corresponding to an infinite number of initially concentric yield surfaces in the deviatoric stress space. In his model, the translation of each surface does not intersect each other but these surfaces are consecutively in contact with each other. Although the Mroz hardening rule gives a good prediction of the material behavior [9,10], its numerical implementation

* Current address: Pacific Northwest National Laboratory, P.O. Box 999, Richland, WA 99352, USA

has been considered to be inefficient and complex because the positions of all the yield surfaces need to be continuously tracked in the stress space and require a large memory storage. Due to this computational complexity, the Mroz model was modified by Krieg [11] and Dafalias and Popov [12] where all the yield surfaces in the Mroz model were replaced by an inner yield surface and an outer yield surface with a continuum of intermediate surfaces that are analytically determined.

In simulations of sheet metal forming processes from binder wrap to springback after flanging operations, sheet metals can experience a few cycles of loading and unloading. For example, a sheet metal element can have a few cycles of bending and unbending when passing through a draw bead during a forming operation. Therefore, a mathematically consistent and computationally convenient cyclic plasticity theory is needed to examine the cyclic plastic response of the sheet metal element under forming operations.

In this paper, an evolution equation for the active yield surface is directly derived by considering the continuous expansion of the active yield surface during the unloading/reloading process. The incremental constitutive relation based on the associated flow rule is then derived for a general yield function. Detailed incremental constitutive relations for materials based on the Mises yield function, the Hill quadratic anisotropic yield function and the Drucker-Prager yield function are derived as the special cases. The selection of the Drucker-Prager yield function for this investigation is due to recent interest in the pressure-sensitive yielding and the strength differential effects for some steels, aluminum alloys and hcp metals, and to examine the consequence of the general formulation of the Mroz hardening rule for the pressure-sensitive yield function where the yield surface is asymmetric with respect to its center. The closed-form solutions for one-dimensional stress-plastic strain curves are plotted for materials under cyclic loading conditions based on the three yield functions. Finally, the plastic responses of the Mises materials based on the anisotropic hardening rule, the isotropic and nonlinear kinematic hardening rules of ABAQUS are examined in order to investigate the effects of the hardening rules.

EVOLUTION OF YIELD SURFACE AND CONSTITUTIVE RELATION FOR A GENERAL YIELD FUNCTION

In this study, the well-known isotropic hardening rule is adopted under the initial loading condition. Therefore, the constitutive relation for the initial loading condition is briefly introduced here. During the initial loading process, the yield surface Φ keeps its center at the origin in the stress space. The yield surface Φ is assumed to have its own constant work-hardening modulus based on the concept of the Mroz hardening rule. The yield function Φ can be expressed as

$$\Phi(\sigma_{ij}, \bar{\sigma}) = f(\sigma_{ij}) - \bar{\sigma} = 0 \quad (1)$$

where f represents a function of the stress tensor σ_{ij} and $\bar{\sigma}$ is the generalized effective stress representing the current size of the yield surface Φ .

If the associated flow rule is assumed, the plastic strain increment $d\varepsilon_{ij}^p$ during the initial loading process can be written as

$$d\varepsilon_{ij}^p = \frac{1}{\bar{E}_p} \left(\frac{\partial \Phi}{\partial \sigma_{ij}} \right) \left(\frac{\partial \Phi}{\partial \sigma_{kl}} \right) d\sigma_{kl} \quad (2)$$

where $\bar{E}_p$ represents the plastic modulus determined from the generalized effective stress-strain curve and is considered as a function of the generalized effective stress in this investigation. Here, $\bar{E}_p$ is defined as

$$\bar{E}_p = \frac{d\bar{\sigma}}{d\varepsilon^p} \quad (3)$$

Now, the constitutive relation for the total strain increment during the initial loading process can be written as

$$d\varepsilon_{ij} = \left[C_{ijkl}^e \right]^{-1} + \frac{1}{\bar{E}_p} \left(\frac{\partial \Phi}{\partial \sigma_{ij}} \right) \left(\frac{\partial \Phi}{\partial \sigma_{kl}} \right) d\sigma_{kl} \quad (4)$$

where C_{ijkl}^e represents the elastic stiffness tensor.

The plastic modulus $\bar{E}_p$ in Equation (2) can be related to the plastic modulus E_p of a tensile, compressive or cyclic stress-strain curve under uniaxial loading conditions where σ_{11} is the only nonzero stress component. Here, the plastic modulus E_p is considered as a function of σ_{11} and is defined as

$$E_p = \frac{d\sigma_{11}}{d\varepsilon_{11}^p} \quad (5)$$

Therefore, the relation between $\bar{E}_p$ and E_p can be obtained by writing Equation (2) under uniaxial loading conditions as

$$d\varepsilon_{11}^p = \frac{1}{\bar{E}_p} \left(\frac{\partial \Phi}{\partial \sigma_{11}} \right) \left(\frac{\partial \Phi}{\partial \sigma_{11}} \right) d\sigma_{11} \quad (6)$$

Substituting Equation (5) into Equation (6) gives the relation between $\bar{E}_p$ and E_p as

$$\bar{E}_p = E_p \left(\frac{\partial \Phi}{\partial \sigma_{11}} \right) \left(\frac{\partial \Phi}{\partial \sigma_{11}} \right) \quad (7)$$

Now we derive our new formulation of the anisotropic hardening rule for the unloading/reloading process. Figure 1 shows a schematic plot of the expansion of the active yield surface Φ_A and the evolution of its center α_{ij}^A for a given stress increment $d\sigma_{ij}$ during the unloading/reloading process within a memory yield surface Φ_M , based on the observation of the evolution of the active yield surface in the Mroz model. Note that

the superscript or subscript "A" is used to denote the active yield surface and its center during the unloading/reloading process. In the figure, during the size expansion of the active yield surface, the unloading stress point $\sigma_{ij}^{M,t}$ is kept as a tangential contact point between the two yield surfaces, and the center of the active yield surface, α_{ij}^A , continuously moves along the line joining the unloading stress point $\sigma_{ij}^{M,t}$ to the center of the memory yield surface, α_{ij}^M . The active yield surface Φ_A can be expressed as

$$\Phi_A(\sigma_{ij}, \alpha_{ij}^A, \bar{\sigma}_A) = f(\sigma_{ij} - \alpha_{ij}^A) - \bar{\sigma}_A = 0 \quad (8)$$

where f is the yield function of the relative stress $\sigma_{ij} - \alpha_{ij}^A$ during the unloading/reloading process. $\bar{\sigma}_A$ represents the current size of Φ_A and, therefore, corresponds to the generalized effective stress for the initial loading process.

In Figure 1, the similarity condition for the memory yield surface and the active yield surface can be expressed as

$$\frac{\sigma_{ij}^{M,t} - \alpha_{ij}^M}{\bar{\sigma}_M} = \frac{\sigma_{ij}^{M,t} - \alpha_{ij}^A}{\bar{\sigma}_A} \quad (9)$$

where $\bar{\sigma}_M$ and $\bar{\sigma}_A$ represent the sizes of the two yield surfaces, respectively, and α_{ij}^M and α_{ij}^A represent their centers. The evolution equation for α_{ij}^A along the line joining the centers of the two yield surfaces can be obtained from the differentiation of Equation (9) as

$$d\alpha_{ij}^A = - \left(\frac{\sigma_{ij}^{M,t} - \alpha_{ij}^M}{\bar{\sigma}_M} \right) d\bar{\sigma}_A \quad (10)$$

It appears that Equation (10) has not been recognized as an important constraint equation for the evolution of the active yield surface in the past [13,14]. With Equation (10), the evolutions of $\bar{\sigma}_A$ and α_{ij}^A can be uniquely determined with the tangent modulus for the current active yield surface. Substituting Equation (10) into the consistency condition ($d\Phi_A(\sigma_{ij}, \alpha_{ij}^A, \bar{\sigma}_A) = 0$) gives the size increment of the active yield surface, $d\bar{\sigma}_A$, as

$$d\bar{\sigma}_A = \frac{\frac{\partial \Phi_A}{\partial \sigma_{ij}} d\sigma_{ij}}{\frac{\partial \Phi_A}{\partial \alpha_{kl}^A} \left(\frac{\sigma_{kl}^{M,t} - \alpha_{kl}^M}{\bar{\sigma}_M} \right) - \frac{\partial \Phi_A}{\partial \bar{\sigma}_A}} \quad (11)$$

Therefore, the size of the new active yield surface Φ_A^* for a given stress increment $d\sigma_{ij}$ can be determined as $\bar{\sigma}_A + d\bar{\sigma}_A$. Substituting Equation (11) into Equation (10) gives the evolution equation for α_{ij}^A of the active yield surface Φ_A as

$$d\alpha_{ij}^A = - \left[\frac{\frac{\partial \Phi_A}{\partial \sigma_{pq}} d\sigma_{pq}}{\frac{\partial \Phi_A}{\partial \alpha_{kl}^A} \left(\frac{\sigma_{kl}^{M,t} - \alpha_{kl}^M}{\bar{\sigma}_M} \right) - \frac{\partial \Phi_A}{\partial \bar{\sigma}_A}} \right] \left(\frac{\sigma_{ij}^{M,t} - \alpha_{ij}^M}{\bar{\sigma}_M} \right) \quad (12)$$

In Figure 1, when the size of the new active yield surface Φ_A^* for a given stress increment $d\sigma_{ij}$ is determined from Equation (11), its new center α_{ij}^{A*} can be uniquely determined by the similarity condition in Equation (9). Therefore, the center of the new active yield surface does not need to be traced by Equation (12) and stored in each computational step.

When the associated flow rule is assumed, the plastic strain increment $d\varepsilon_{ij}^p$ for the active yield surface Φ_A during the unloading/reloading process can be expressed as

$$d\varepsilon_{ij}^p = \frac{1}{E_p} \left(\frac{\partial \Phi_A}{\partial \sigma_{ij}} \right) \left(\frac{\partial \Phi_A}{\partial \sigma_{kl}} \right) d\sigma_{kl} \quad (13)$$

where E_p is determined based on the size of the active yield surface, $\bar{\sigma}_A$. The constitutive relation for the total strain increment during the unloading/reloading process can be written as

$$d\varepsilon_{ij} = d\varepsilon_{ij}^e + d\varepsilon_{ij}^p = \left[[C_{ijkl}^e]^{-1} + \frac{1}{E_p} \left(\frac{\partial \Phi_A}{\partial \sigma_{ij}} \right) \left(\frac{\partial \Phi_A}{\partial \sigma_{kl}} \right) \right] d\sigma_{kl} \quad (14)$$

CONSTITUTIVE RELATIONS FOR THREE YIELD FUNCTIONS

The constitutive relations derived in the previous section are quite convenient to apply to any given yield function. In this section, the Mises yield function, the Hill quadratic orthotropic yield function and the Drucker-Prager pressure-sensitive yield function are selected as the special cases to demonstrate the convenient use of the constitutive relation derived in the last section. Figure 2 shows a comparison of the normalized yield contours based on the Mises, Hill and Drucker-Prager yield functions. In this figure, the coordinates σ_{11} and σ_{22} are normalized by $\bar{\sigma}_{mises}$, $\bar{\sigma}_{hill}$ and $\bar{\sigma}_{dp}^t$. Here, $\bar{\sigma}_{mises}$ represents the yield stress for the Mises yield function, $\bar{\sigma}_{hill}$ represents the yield stress in the rolling (X_1) direction for the Hill yield function, and $\bar{\sigma}_{dp}^t$ represents the tensile yield stress for the Drucker-Prager yield function. For the brevity of paper, only a few important relations for each yield function are introduced in this section. Other relations, not introduced here, can be easily derived by substituting each yield function into the constitutive relations based on a general yield function, shown in the previous section.

MISES YIELD FUNCTION — The Mises yield function for the initial loading process is given as