

# Equivalent Accelerated Life Testing Plans and Application to Reliability Prediction

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## ABSTRACT

Accelerated life testing (ALT) is widely used to determine the failure time distribution of a product and the associated life-stress relationship in order to predict the product's reliability under normal operating conditions. Many types of stress loadings such as constant-stress, step-stress and cyclic-stress can be utilized when conducting ALT. Extensive research has been conducted on the analysis of ALT data obtained under a specified stress loading. However, the equivalency of ALT experiments involving different stress loadings has not been investigated. In this paper, a definition is provided for the equivalency of various ALT plans involving different stress loadings. Based on this definition, general equivalent ALT plans and some special types of equivalent ALT plans are explored. For demonstration, a constant-stress ALT and a ramp-stress ALT for miniature lamps are presented and their equivalency is investigated.

## INTRODUCTION

Accelerated life testing (ALT) is widely used to estimate the reliability of a product in a shorter time by applying harsher-than-normal operating conditions during tests (Nelson, 1990; Elsayed, 1996; Meeker and Escobar, 1998). The accuracy of the reliability estimate depends on both ALT model and ALT plan. An appropriate ALT model provides a statistically accurate fit to ALT data for extrapolating the product's reliability. Likewise, an optimal ALT plan, which determines relevant experimental variables, such as the stress loadings (constant-stress, ramp-stress, cyclic-stress, etc.) and allocation of the test units at each stress level, results in the improvement of estimation accuracy.

In the literature, extensive research has been conducted on the optimal design of ALT plans (Nelson, 2005) following the work done by Chernoff (1962). Nelson and Kielpinski (1976) obtained the optimal plans and the best traditional plans for estimating the medians of the normal and lognormal distributions respectively under constant-stress ALT. Maxim *et al.* (1977) determined the optimal ALT plans using the D-optimality criterion and assuming a bivariate exponential or Weibull model. Meeker and Hahn (1977) considered the optimal allocation of test units to accelerated stress conditions with the objective of minimizing the estimate of the product reliability under the design stress condition. Nelson and Meeker (1978) provided the optimal test plans to estimate the percentiles of the Weibull and extreme-value distributions at a specified design stress. Meeker (1984) compared the statistically optimal test plans to a more practical test plan involving three stress levels. Nelson (1990) reviewed statistically optimal and compromise plans for the single stress ALT planning problem. Meeter and Meeker (1994) extended the maximum likelihood theory for planning ALT to the nonconstant scale parameter model and presented test plans for a large range of practical tests. Yang (1994) proposed an optimal design of 4-level constant-stress ALT plans considering different censoring times. Regarding the step-stress ALT, Miller and Nelson (1983) presented the cumulative exposure model and obtained the optimal test plans. The optimal plans minimize the asymptotic variance of the maximum likelihood estimate (MLE) of the mean life at the design stress. Bai *et al.* (1989) extended the results to the case where a prescribed censoring time is involved. Bai and Chun (1991) obtained the optimal simple step-stress ALT with competing causes of failure. The optimal test plans minimize the sum, over all failure causes, of asymptotic variances of the MLEs of the log-mean lives at the use

condition. Chung and Bai (1998) considered the optimal design of step-stress ALT in which a cumulative exposure model for the effect of changing stress is assumed. Khamis and Higgins (1996) presented 3-step step-stress plans assuming a linear or quadratic relationship between the life and the stress. Xiong (1998) addressed the statistical inferences on the parameters of a simple step-stress ALT model with Type II censoring. Xiong and Milliken (1999) studied the statistical models in step-stress ALT when the stress change times are random. Xiong and Ji (2004) studied the optimal design of a simple step-stress test plan involving grouped and censored data.

As stated above, most of the research has been focused on the optimal design of ALT plans for specific stress loadings. Considering broader applications of ALT, it is possible to have multiple choices in stress loadings (e.g., cyclic-stress and ramp-stress) other than the constant-stress and step-stress when planning ALT which can be conducted in a much shorter time. This has raised some practical questions such as: Can those ALT plans involving different stress loadings be equivalent? How do we design the optimal ALT plan when the stress loading is not specified beforehand? The solutions to these questions have not been well investigated. This paper defines the equivalency of ALT plans and investigates special types of equivalent plans as well as the computational algorithms and statistical inference methods.

## ASSUMPTIONS AND ALT MODEL

Assumptions:

1. The stress level  $S_0$  under the product's use condition is constant.
2. The failure time distribution of the product can be described by a log-location-scale distribution as:

$$F(t; \mu, \sigma) = \Phi\left(\frac{\log(t) - \mu}{\sigma}\right), \quad t > 0, \quad (1)$$

where  $\sigma$  is the scale parameter,  $\mu$  is the location parameter, and  $\Phi(x)$  is the cumulative distribution function (Cdf) of a standardized location-scale distribution (with the probability density function (pdf) given by  $\phi(x) = d\Phi(x)/dx$ , for which  $\mu = 0$  and  $\sigma = 1$ ).

3. The scale parameter  $\sigma$  is constant within the range of the stress considered; the location parameter  $\mu$  depends on the stress level, which can be described by a life-stress relationship:  $\mu = g(S; \beta)$  for a stress level  $S$ , where  $\beta$  is the vector of parameters in the relationship.
4. Test units in ALT are independent and are subject to Type I censoring with the censoring time of  $t_c$ .

Without loss of generality, to facilitate mathematical derivations in planning ALT, the levels of stress  $S$  are

normalized first. Let  $S_H$  be the highest stress level that can be used in the ALT experiment. The normalized stress level  $j$  is given by:

$$Z_j = (S_j - S_0)/(S_H - S_0), j=0, 1, \dots, k. \quad (2)$$

As a result, the use condition  $S_0$  becomes  $Z_0=0$ ,  $S_H$  becomes  $Z_H=1$ , and  $0 \leq Z_j \leq 1$  for  $j=1, \dots, k$ . Let  $\Theta$  be the vector of model parameters after the standardization procedure.

Let  $N$  be the total number of units used in the ALT experiment,  $t_{ji}$  be the failure time of unit  $i$  under  $Z_j$ . In this paper, it should be regarded as the total test time accumulated on this unit which may have experienced a series of stress levels ending with  $Z_j$ . Under Type I censoring, the log-likelihood function of ALT data obtained under a specific set of stress loadings can be generally expressed as:

$$L(\Theta) = \sum_{i=1}^N \sum_{j=1}^k \sum_{m=1}^j \delta_{mji} [\delta_{ji} \log(f_{\{m \rightarrow j\}}(t_{ji}; \Theta))] + (1 - \delta_{ji}) \log(1 - F_{\{m \rightarrow j\}}(t_c; \Theta)), \quad (3)$$

where  $F_{\{m \rightarrow j\}}(\cdot, \Theta)$  and  $f_{\{m \rightarrow j\}}(\cdot, \Theta)$  are the equivalent Cdf and pdf of a unit's failure time if the unit experiences a series of stress levels  $\{Z_m, Z_{m+1}, \dots, Z_j\}$ , respectively;

$\delta_{mji} = \begin{cases} 1 & \text{if unit } i \text{ experiences } Z_m \text{ and ending at } Z_j, \\ 0 & \text{otherwise.} \end{cases}$

and  $\delta_{ji} = \begin{cases} 1 & \text{if unit } i \text{ fails at } Z_j, \\ 0 & \text{otherwise.} \end{cases}$

Under the regularity conditions, the information matrix associated with an ALT plan can be expressed as:

$$I_{\Theta} = -E \left[ \frac{\partial^2 L}{\partial \Theta \partial \Theta^T} \right]. \quad (4)$$

From the ML theory, the asymptotic distribution of the model parameter estimates follows the multivariate normal distribution with the pdf of  $MVN(\hat{\Theta}, \Sigma_{\hat{\Theta}})$  where the covariance matrix  $\Sigma_{\hat{\Theta}}$  can be estimated by  $I_{\hat{\Theta}}^{-1}$  evaluated at the MLE  $\hat{\Theta}$ . For a function of the parameter estimates, such as the estimate of reliability function at a specified life and the estimates of failure time quantiles. Let  $G(\hat{\Theta})$  be such a function. Using the delta method (Meeker and Escobar,

1998), the asymptotic variance  $\text{Asvar}(G(\hat{\underline{\Theta}}))$  can be approximated by:

$$\text{Asvar}(G(\hat{\underline{\Theta}})) = \left[ \frac{\partial G(\hat{\underline{\Theta}})}{\partial \hat{\underline{\Theta}}} \right]^T \hat{\Sigma}_{\hat{\underline{\Theta}}} \left[ \frac{\partial G(\hat{\underline{\Theta}})}{\partial \hat{\underline{\Theta}}} \right]. \quad (5)$$

Note that, in general the covariance matrix  $\Sigma_{\hat{\underline{\Theta}}}$  depends on those decision variables in planning ALT. Therefore, an ALT planning process can be regarded as a matrix design problem where the elements in  $\Sigma_{\hat{\underline{\Theta}}}$  are functions of the decision variables.

## EQUIVALENT ALT PLANS

In the literature, a variety of optimization criteria have been used in planning ALT. Most ALT plans minimize the asymptotic variance of the MLE of a model parameter, or a function of parameter estimates such as a specified failure time quantile or reliability function at the use condition. These objective functions can be generally expressed by Eq. (5). To study the equivalency of various ALT plans involving different stress loadings we provide a measure of equivalency as given below.

**Definition 1 (Equivalency of ALT Plans).** *Two ALT plans are equivalent if they result in the same statistical estimation accuracy of a reliability estimate as the primary objective while satisfying other secondary criteria and/or constraints, if any.*

Most practical work in ALT planning uses the variance of an estimator of a reliability measure at normal operating conditions to quantify the statistical estimation precision of an ALT plan. This is known as c-optimality as expressed by Eq. (5). In this paper, a c-optimality criterion is used as the primary criterion, and others such as D-optimality, the shortest test time, and the least number of test units are treated as secondary criteria in planning equivalent ALT. Let  $X$  and  $Y$  be the asymptotic variance-covariance matrices of two ALT plans (note:  $X^{-1}$  and  $Y^{-1}$  are the corresponding

information matrices), respectively, and  $a$  be  $\left[ \frac{\partial G(\hat{\underline{\Theta}})}{\partial \hat{\underline{\Theta}}} \right]$  in Eq. (5). Because both  $X$  and  $Y$  are positive definite symmetric matrices, without loss of generality, let

$$X = \begin{bmatrix} x_{1,1} & x_{1,2} & \cdots & x_{1,n} \\ x_{1,2} & x_{2,2} & \ddots & x_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{1,n} & x_{2,n} & \cdots & x_{n,n} \end{bmatrix} \text{ and } Y = \begin{bmatrix} y_{1,1} & y_{1,2} & \cdots & y_{1,n} \\ y_{1,2} & y_{2,2} & \ddots & y_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ y_{1,n} & y_{2,n} & \cdots & y_{n,n} \end{bmatrix}$$

be two  $n$  by  $n$  symmetric and positive definite matrices, and  $\underline{a} = [a_1, a_2, \dots, a_n]^T \neq 0$ . In mathematical terms, the two ALT plans are equivalent in terms of Definition 1 if and only if the following equation holds:

$$\underline{a}^T X \underline{a} = \underline{a}^T Y \underline{a}. \quad (6)$$

Eq. (6) provides another definition for equivalent ALT plans in a mathematical form.

**Definition 2 (General Equivalent ALT Plans).** *For a given ALT plan with the covariance matrix  $X$ , another ALT plan is called a statistically equivalent plan in a general sense if its associated covariance matrix  $Y$  satisfies Eq. (6).*

It can be seen that multiple equivalent plans may be obtained for a given ALT plan. In particular, the following special case, called Strict Statistically Equivalent ALT Plan is of a special interest.

**Definition 3 (Strict Statistically Equivalent ALT Plan).** *For a given ALT plan with the covariance matrix  $X$ , an equivalent ALT plan is called the "Strict Statistically Equivalent ALT Plan" if its associated covariance matrix  $Y = X$ , i.e.,  $y_{i,j} = x_{i,j}$  for all  $i, j$ .*

In other words, under the strictest constraints, a Strict Statistically Equivalent ALT plan generates the same covariance matrix as the given plan. Let  $\underline{D}$  be an  $m$ -dimensional vector of decision variables, and  $\Omega$  be the design space for planning ALT. A Strict Statistically Equivalent ALT plan attempts to solve the following system of nonlinear equations:

$$y_{i,j} |_{\underline{D}} = x_{i,j}, \text{ for all } i \in [1, n], j \in [1, n], \text{ and } \underline{D} \in \Omega. \quad (7)$$

Problems of this type can be solved using Newton's method or Trust-Region method. A difficult issue in solving the system of nonlinear equations is to provide a "good" initial guess so that Newton's method converges. The following modified procedure based on the one-parameter operator imbedding technique (Meyer, 1968) can be utilized to seek a desired strict statistically equivalent plan without relying on a "good" initial guess.

**One-parameter operator imbedding algorithm for obtaining a strict statistically equivalent ALT plan:**