

A Re-Analysis Methodology for System RBDO Using a Trust Region Approach with Local Metamodels

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ABSTRACT

A simulation-based, system reliability-based design optimization (RBDO) method is presented that can handle problems with multiple failure regions and correlated random variables. Copulas are used to represent the correlation. The method uses a Probabilistic Re-Analysis (PRRA) approach in conjunction with a trust-region optimization approach and local metamodels covering each trust region. PRRA calculates very efficiently the system reliability of a design by performing a single Monte Carlo (MC) simulation per trust region. Although PRRA is based on MC simulation, it calculates “smooth” sensitivity derivatives, allowing therefore, the use of a gradient-based optimizer. The PRRA method is based on importance sampling. It provides accurate results, if the support of the sampling PDF contains the support of the joint PDF of the input random variables. The sequential, trust-region optimization approach satisfies this requirement. Local metamodels are constructed sequentially for each trust region taking advantage of the potential overlap of the trust regions. The metamodels are used to determine the value of the indicator function in MC simulation. An example with correlated input random variables demonstrates the accuracy and efficiency of the proposed RBDO method.

1. INTRODUCTION

Due to competitive pressures, engineers and scientists are facing an ever-shortening design and validation cycle. The desire for fast development often comes at the cost of reliability and robustness. Early in the design phase it is necessary to make decisions based on incomplete information and with a minimum of physical and virtual tests. In order to

facilitate this decision-making, improved methods are required to assess system reliability and its use in design.

In order to quantify the performance of a system under uncertainty, a method is needed to determine the reliability of a design. By estimating the system reliability at a given design, an optimization can be performed to determine the design that meets all specifications in terms of probability of failure.

In optimization, an objective is optimized while certain constraints are satisfied. The deterministic optimum though, is often overdesigned because of the use of overly conservative safety factors and design loads. The deterministic optimum could be unsafe in applications involving novel materials and geometries. To ensure that the optimum design is both efficient and reliable, the optimization formulation must include reliability constraints. Such a formulation is commonly referred as Reliability-Based Design Optimization (RBDO). RBDO approaches focus on maintaining design feasibility (for design constraints) at expected probabilistic levels. They account for inherent input and parameter variations in manufacturing processes, materials, and loading.

In RBDO, probability distributions describe the stochastic nature of the design variables and model parameters. Variations are represented by standard deviations (typically assumed constant) and a *mean* performance measure is optimized subject to *probabilistic* constraints. RBDO can be a powerful design tool, since it provides optimum designs in the presence of uncertainty.

Efficiency and accuracy are the two main challenges in RBDO. The reliability, or equivalently the probability of failure, of each constraint or of the entire system, must be calculated accurately and efficiently. The commonly used analytical methods (e.g. FORM) are usually efficient but inaccurate, especially for problems with “noisy” limit states, which may exhibit multiple failure regions and potentially multiple most-probable points (MPPs). Such problems are common in structural dynamics. To alleviate this problem, simulation-based reliability methods may be used. However, they are computationally very expensive and therefore, impractical for many engineering problems.

This paper presents an algorithm for simulation-based RBDO using a trust-region approach in conjunction with Probabilistic Re-Analysis (PRRA) and approximate (“accurate-on-demand”) local metamodels (surrogate models). The algorithm performs well for problems with disjoint failure regions with highly non-linear limit states and multiple MPPs. Also, it can handle problems with uncorrelated or correlated input random variables and parameters.

It is well known that the analytical reliability methods are computationally efficient, and relatively accurate depending on the application. However, their accuracy deteriorates for problems with highly non-linear and/or “noisy” limit states. Also, they are not appropriate for problems with disjoint failure domains, highly correlated variables and multiple MPPs. Simulation-based reliability methods can address the shortcomings of the analytical methods. However, they are usually expensive for many real-world engineering systems.

Although the proposed methodology is a simulation-based RBDO method, it is computationally efficient because 1) the PRRA method needs only one Monte Carlo simulation per trust region, and 2) the method uses “accurate-on-demand” local metamodels to assess the system reliability within each trust region. The high computational effort of most RBDO methods is due to the repeated reliability assessments. The PRRA method drastically reduces the effort to estimate the system reliability. Considering that even one MC simulation within each trust region can be very expensive, we use local approximate metamodels with local refinements to perform the single MC simulation efficiently.

Within each trust region, we use approximate local metamodels of the limit states that serve as indicators to determine the “failure” and “safe” regions. High accuracy is only needed close to the limit states where sample points have a “low” absolute value. In this work, points with “low” limit-state absolute value define a transition region between the safe and failure regions. All sample points in the transition region are identified, and some of them are used to refine the local metamodels.

If a limit state has multiple failure regions (that may be even disjoint), or if multiple limit states exist due to a system reliability assessment, the transition region points may be grouped in clusters. A clustering technique can be used to identify these groupings so that local metamodels can be constructed for each group. This is essential for handling multiple MPP problems that often exist in a variety of engineering applications, such as problems involving vibratory systems. Recently, some work has been reported for multiple MPP problems [1, 2].

Although various metamodeling techniques can be successfully used, we have chosen the Cross-Validated Moving Least Squares (CVMLS) [3] method. To construct a metamodel, a “space-filling” sampling algorithm is needed. Commonly used algorithms such as Latin Hypercube (LH) and Optimal Symmetric LH (OSLH) sampling [4] among others, place most of the samples in the interior of the domain. In this work, an efficient “space-filling” sampling is used based on a maximin distance algorithm [5], which efficiently selects a few sample points with “space-filling” properties (see [6] for details). The maximin sampling method selects points from the existing MC sample points. The selected points are used to construct the local approximate metamodels. It is important to model the dependence of random variables because dependence can affect significantly structural reliability. The proposed method uses copulas to represent dependence.

The paper is organized as follows. After an introduction to the analytical and simulation-based RBDO methods in section 2, we present an overview of correlated random variables using copulas in section 3. Section 4 describes the proposed simulation-based RBDO method using a trust-region optimization approach with PRRA and approximate local metamodels. A mathematical example is presented in section 5 to illustrate all steps of the proposed method using correlated random variables. Finally, summary and conclusions are presented in section 6.

2. ANALYTICAL AND SIMULATION-BASED METHODS IN RBDO

For large-scale systems, the reliability prediction is usually based on efficient computational methods. Both analytical and simulation-based methods are available. The analytical methods are based on the MPP concept. They include the well-known first-order reliability method (FORM) that has been widely used [7], second-order reliability methods (SORM) [8], and multi-point approximation methods [9]. Among the simulation-based methods, the Monte Carlo Simulation (MCS) method is very simple and accurate. However, its computational cost can be prohibitively high. For this reason, more efficient simulation-based techniques have been proposed [10, 11]. Among them, the adaptive importance sampling (AIS) techniques are popular [12, 13].

A combination of analytical and simulation-based methods has also been used [14]. The analytical methods are generally simple and efficient, but for complex problems, their accuracy cannot be guaranteed. In simulation-based methods, the accuracy can be controlled but the efficiency is generally not satisfactory.

For system reliability analysis involving multiple failure modes (limit states), the joint failure probability must be taken into account. Due to the difficulty in determining the joint failure probability of more than two failure modes except through MCS, approximations using first-order and second-order bounds have been developed [15, 16].

Simulation-based methods are also used for reliability analysis involving single or multiple limit states. Among MCS-based methods seeking to improve computational efficiency, adaptive importance sampling (AIS) techniques use an importance sampling density function, which is gradually refined to reflect the increasing state of knowledge of the failure domain. The importance sampling methods are divided into direct methods [11, 17], updating methods [18], spherical schemes [17], directional sampling [19], and adaptive schemes [12, 20]. All methods, except the adaptive schemes, require prior knowledge of the failure domain.

3. CORRELATED RANDOM VARIABLES USING COPULAS

The proposed RBDO method can easily handle correlated random variables because it is based on simulation. This section gives an overview of how to express correlation using copulas and how to generate correlated sample points if the copula is known.

In many engineering problems the value that a variable assumes may be statistically related to the value of another variable [21]. When this occurs the variables are statistically correlated, or more simply correlated. For example in many engineering applications involving the fatigue of metals, many of the material properties are correlated [22, 23]. Although statistical correlation adds a layer of complexity, it is still relatively simple to consider if the joint PDF (or joint CDF) is known. However, obtaining the joint PDF of two or more random variables is very challenging. For this reason, most research on reliability assessment or RBDO has been performed under the arbitrary assumption of independent random variables. This assumption may lead however, to conservative or even unreliable designs. Because of this, a lot of research has been devoted recently on the topic of identifying the type of the joint distribution using copulas [24, 25].

In statistics, Nelsen [26] defined copulas as “functions that join or couple multivariate CDF functions to their one-dimensional marginal CDF functions. Alternatively, copulas

are multivariate CDF functions whose one-dimensional marginal distributions are uniform on the interval [0, 1].”

According to Sklar's theorem [26], if random variables $X_1, X_2, \dots, X_n$ have marginal CDFs, then there exists a n -dimensional copula C such that

$$F_{X_1, \dots, X_n}(x_1, \dots, x_n) = C(F_{X_1}(x_1), \dots, F_{X_n}(x_n) | \theta) \quad (1)$$

where θ is the correlation matrix among $X_1, \dots, X_n$. If the marginal CDFs are all continuous, C is unique. Conversely, if C is a n -dimensional copula and $F_{X_1}(x_1), \dots, F_{X_n}(x_n)$ are marginal CDFs, the joint CDF is a n -dimensional function of marginal CDFs. By differentiating Equation (1) with respect to X , the joint PDF is obtained as

$$f(x_1, \dots, x_n) = f(F_{X_1}(x_1), \dots, F_{X_n}(x_n) | \theta) \prod_{i=1}^n f_{X_i}(x_i) \quad (2)$$

where

$$c(u_1, \dots, u_n) = \frac{\partial^n C(u_1, \dots, u_n)}{\partial u_1 \dots \partial u_n} \quad (3)$$

In Equation (3), $u_i = F(x_i)$ and $f_{X_i}(x_i)$ is the marginal PDF of X_i , $i = 1, \dots, n$. To model the joint CDF using a copula, the correlation matrix θ must be obtained from available data. Because various types of copulas have their own correlation parameters, it is desirable to have a common correlation measure to obtain the correlation parameters from the experimental data.

3.1. CORRELATION MEASURE AND COMMONLY USED COPULAS

To measure the correlation between two random variables, Pearson's ρ and Kendall's τ are commonly used. Pearson's ρ , which is also called a product moment correlation coefficient, was first introduced by Bravais (1846) and was defined by Pearson (1896). It indicates the degree of linear relationship between two random variables X and Y as

$$\rho_{XY} = \frac{\text{Cov}(X, Y)}{\sigma_X \sigma_Y} \quad (4)$$

where σ_X and σ_Y are the standard deviations of X and Y , respectively, and $\text{Cov}(X, Y)$ is the covariance between X and

Y. Since Pearson's ρ indicates only the linear relationship between two random variables, it is valid only when the joint CDF is Gaussian. Therefore, it is not a good measure for a nonlinear relationship between two random variables, which is often the case in many practical engineering applications.

Unlike Pearson's ρ , Kendall's τ does not assume that the relationship between two random variables is linear. Because it measures the correspondence of rankings between correlated random variables, it is called a rank correlation coefficient. It was first introduced by Kendall (1938) and is defined as

$$\tau = 4 \iint_2 C(u, v | \theta) dC(u, v) - 1 \quad (5)$$

where $I^2 = I \times I (I = [0, 1])$, and $u = F_{X_1}(x_1)$ and $v = F_{X_2}(x_2)$ are the marginal CDFs of X_1 and X_2 . For two variables correlation matrix θ reduces to Pearson's correlation coefficient ρ . Figures 1 and 2 illustrate the relationship between the correlation matrix θ for two variables and Kendall's τ for the two commonly used Frank and Clayton copulas.

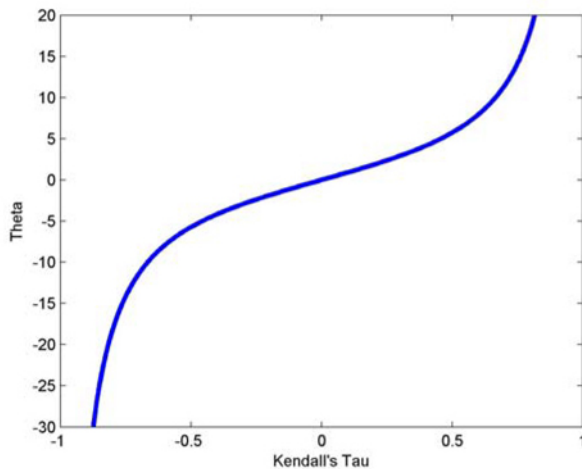


Figure 1. Relationship between Kendall's τ and θ for Frank copula

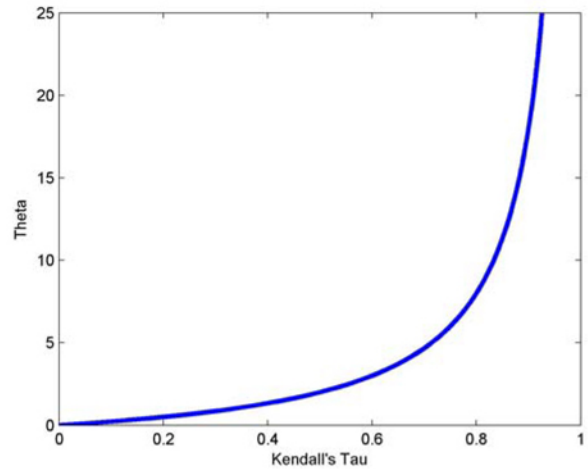


Figure 2. Relationship between Kendall's τ and θ for Clayton copula

There are various types of copulas, such as the elliptical copula and the Archimedean copula. In this section, the Gaussian copula, which belongs to an elliptical copula family, and several Archimedean copulas, such as Clayton, Ali- Mikhail-Haq (AMH), Gumbel, Frank, A12, and A14, are introduced (see Table 1). Table 2 shows the relationship between θ and τ and the range of τ for the same copulas. If the marginal CDFs are Gaussian, the joint CDF of the Gaussian copula is the joint Gaussian CDF. Sample points of two correlated random variables X_1 and X_2 according to the Frank and Clayton copulas, are shown in Figures 3 and 4 for Kendall's $\tau = -30$ (Frank copula) and $\tau = 0.5$ (Clayton copula).

Table 1. Common Copula functions and their parameter domains

Copula	$C(u, v \theta)$	$\tau \in \Omega^\tau$
A12	$\left\{1 + \left[u^{-1} - 1\right]^\theta + \left[v^{-1} - 1\right]^\theta\right\}^{-1/\theta}$	$[1, \infty)$
A14	$\left\{1 + \left[u^{-1/\theta} - 1\right]^\theta + \left[v^{-1/\theta} - 1\right]^\theta\right\}^{-\theta}$	$[1, \infty]$
AMH	$\frac{uv}{[1 - \theta(1-u)(1-v)]}$	$[-1, 1]$
Clayton	$(u^{-\theta} + v^{-\theta} - 1)^{-1/\theta}$	$(0, \infty]$
FGM	$uv + \theta u(1-u)(1-v)$	$[-1, 1]$
Frank	$-\frac{1}{\theta} \ln \left[1 + \frac{(e^{-\theta u} - 1)(e^{-\theta v} - 1)}{(e^{-\theta} - 1)} \right]$	$(-\infty, \infty)$
Gaussian	$\int_{-\infty}^{\Phi^{-1}(u)} \int_{-\infty}^{\Phi^{-1}(v)} \frac{\exp\left(\frac{2\theta w - s^2 - w^2}{2(1-\theta^2)}\right)}{2\pi\sqrt{1-\theta^2}} ds dw$	$[-1, 1]$
Gumbel	$\exp\left\{-\left[(-\ln(u))^\theta + (-\ln(v))^\theta\right]^{1/\theta}\right\}$	$[1, \infty]$

Table 2. Kendall's τ and its domain for common copulas

Copula	$\tau = g(\theta)$	$\tau \in \Omega^\tau$
A12	$1 - 2/3\theta$	$[1/3, 1]$
A14	$1 - 2/1 + 2\theta$	$[1/3, 1]$
AMH	$1 - \frac{2}{3} \cdot \frac{(\theta-1)^2 \cdot \ln(1-\theta) + \theta}{\theta^2}$	$[-0.171726, 1/3]$
Clayton	$1 - 2/(2 + \theta)$	$(0, 1]$
FGM	$2\theta/9$	$[-2/9, 2/9]$
Frank	$1 - \frac{4}{\theta} \left(1 - \frac{1}{\theta} \int_0^\theta \frac{t}{e^t - 1} dt \right)$	$[-1, 1] \setminus \{0\}$
Gaussian	$(2/\pi) \arcsin(\theta)$	$[-1, 1]$
Gumbel	$1 - \theta^{-1}$	$[0, 1]$

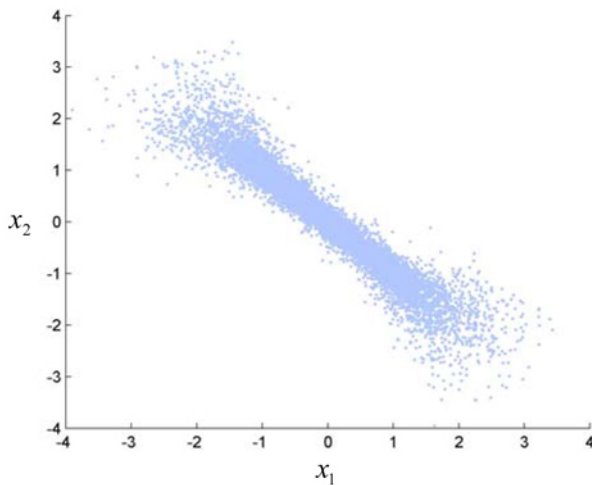


Figure 3. Frank copula with $\tau = -30$

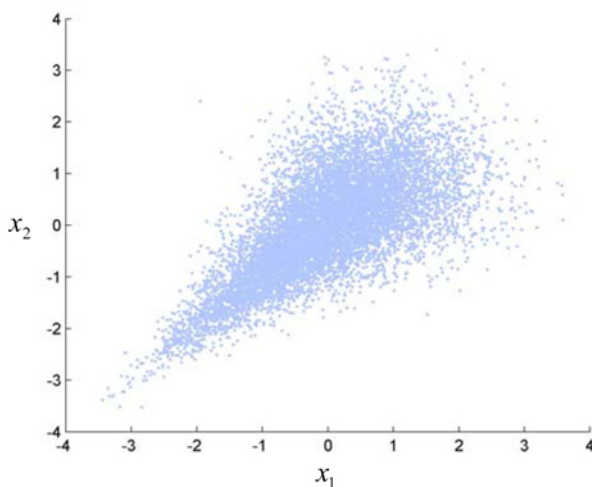


Figure 4. Clayton copula with $\tau = 0.5$

An Archimedean copula is constructed in a completely different way from the Gaussian copula. An important component of constructing Archimedean copulas is a generator function φ_θ with respect to θ . For example, the generator function of the Clayton copula is $\varphi_\theta(t) = t^\theta - 1$. If φ_θ is a continuous and strictly decreasing function from $[0, 1]$ to $[0, \infty)$ such that $\varphi_\theta(0) = \infty$ and $\varphi_\theta(1) = 0$ and its inverse φ_θ^{-1} is monotonic on $[0, \infty)$, then the Archimedean copula is defined as

$$C(u_1, \dots, u_n | \theta) = \varphi^{-1}[\varphi_\theta(u_1) + \dots + \varphi_\theta(u_n)] \quad (6)$$

Each Archimedean copula has a corresponding unique generator function. If the generator function is known, Kendall's τ can be obtained as

$$\tau = 1 + 4 \int_0^1 \frac{\varphi_\theta(t)}{\varphi_\theta(t)} dt \quad (7)$$

The Archimedean copula can have a multivariate CDF, but it is hard to expand to a n -dimensional copula because, as shown in Equation (6), it has one generator function and thus has only one correlation parameter even if the n variables are correlated. Therefore, most copula applications consider bivariate data. For multivariate data, the data is analyzed pair by pair using a bivariate copula. In this paper, we consider only bivariate copulas.

3.2. GENERATING CORRELATED SAMPLE POINTS USING COPULAS

To provide the reader with some guidance on how to generate sample points using copulas, the following procedure is provided using the generator function. First we select the copula which provides the appropriate correlation between two random variables U and V , and select the value of Kendall's τ . There are numerous methods for selecting copulas to generate correlated sample points based on available data. Here for demonstration purposes, we will use the Frank copula which is defined (see Table 1) as

$$C = -\frac{1}{\theta} \ln \left[1 + \frac{(e^{-\theta u} - 1)(e^{-\theta v} - 1)}{(e^{-\theta} - 1)} \right] \quad (8)$$