

Fuzzy Boundary Element Method for Material Uncertainty in Steady State Heat Conduction

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ABSTRACT

The response of the engineering system is often obtained by the use of numerical methods such as finite element method or boundary element method. However, the uncertainty of the acquired solutions cannot be measured using conventional methods. This uncertainty is attributed to two sources: errors in mathematical modeling and uncertainties in the parameter. The following paper addresses the second source of uncertainty for the steady state heat conduction problem where the material conductivity is uncertain. Material uncertainty is implemented into fuzzy boundary element method which obtains the exact worst case bounds on the response given the worst case bounds on the parameter uncertainty. The method assumes that a correct partial membership function is given. Numerical examples are shown to illustrate the behavior of the method.

INTRODUCTION

Most engineering calculations involve at least some components to be obtained from either experimental data or from experience. These components usually involve a relationship between a potential, such as temperature, and a flux variable, such as heat flux. In the case of heat conduction, thermal conductivity of the material provides the necessary relationship between these two. However, since thermal conductivity is measured using experimental techniques, it has an associated uncertainty, which corresponds to the error with which it has been measured. Most of the measurement tools have a measurement tolerance which is given by a manufacturer and potentially can be influenced by field conditions such as temperature. Although a true measurement error of the measured quantity cannot be known, the worst case measurement error, and therefore, the worst case uncertainty, is given directly from the measurement tolerances. It is thus both natural and

convenient to use the given tolerances as the upper and lower bounds on the parameter uncertainty. Uncertainty in system parameters, such as heat conduction, influences the system performance. Stochastic methods or sensitivity analyses can be performed to study the effects of parameter uncertainty on system behavior; however, they are both expensive and do not guarantee that the worst case behavior of the system has been captured. It is beneficial to obtain a computationally inexpensive accurate method, which computes the worst case behavior of the system with uncertain parameters. Since most engineering systems are complex, it is necessary for such method to be numerical. Currently, most engineering systems are solved using finite element method (FEM), finite difference method (FDM), boundary element method (BEM), or a combination of these. In this work boundary element method is used to obtain approximate solutions to partial differential equations.

It should be noted that the solutions obtained via BEM are affected by various errors and uncertainties. These include discretization error, integration error, round-off error, uncertainty in system geometry, uncertainty in boundary conditions, and uncertainty in system parameters. All of these sources of either errors or uncertainties impact the behavior of the system but are not considered in conventional analysis techniques. Uncertainty in boundary conditions, geometry and material has been treated using fuzzy numbers ([Zadeh 1978](#), [Ross 1995](#)) for FEM resulting in fuzzy finite element method (FFEM) ([Muhanna and Mullen 1999](#)). Interval finite element method (IFEM) provides worst case bounds for the solutions of buckling analysis considering unknown but bounded material properties ([Modares et al. 2006](#)). Interval boundary element method (IBEM) and fuzzy boundary element method (FBEM) obtain worst case enclosure of the solution considering discretization, truncation, and round-off errors, and uncertainty in boundary conditions and system geometry ([Zalewski and Mullen 2008](#), [Zalewski and Mullen](#)

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2008, Zalewski and Mullen 2009, Zalewski et al. 2009, Zalewski and Mullen 2009, Zalewski et al. 2009). IBEM was also used to treat uncertainty in system parameters for transient diffusion problems (Piasecka 2007).

There are several approaches for characterizing uncertainty of the heat conduction using random variables (Papoulis 1965), interval numbers (Moore 1966), and clouds (Neumaier 2004); however, all have shortcomings which can be mitigated using a fuzzy approach. The major difficulty in characterizing uncertainty with a probabilistic approach is that random variables are characterized by a cumulative distribution function (CDF), which in practice may not be obtainable. Characterizing uncertainty in conductivity by interval numbers is very similar to characterizing it using fuzzy numbers. However, unlike a fuzzy approach, it does not allow the analyst to consider the degree of uncertainty. Bounds on the conductivity obtained from expert opinions or laboratory measurements vary depending on the confidence of the source and are not set bounds. Cloud numbers consider a certain probability of occurrence of a variable in a fuzzy set. They require the knowledge of the CDF, which may be unknown. Characterizing the conductivity uncertainty using fuzzy numbers allows considering the level of confidence one has in the uncertainty bounds and is computationally inexpensive making its usage attractive for numerical methods.

This paper addresses the treatment of uncertainty in thermal conductivity of the material using fuzzy numbers. A fuzzy number is a convex set whose bounds vary dependent on the level of the confidence in the correctness of these bounds. Bounds of the fuzzy number can be obtained from different sources such as measurement tolerance and expert opinions. It should be noted that the level of uncertainty of these sources varies with the level of confidence of the sources themselves. For example, the price of the thermometer will most likely affect the certainty of its measurement. In this paper, it is assumed that all experts are knowledgeable in the field in question and they behave rationally. It is also assumed that the more resources are used, and the more expensive the measurement tools, the better the quality of the uncertainty. Therefore, all sources give a smaller value of a lower bound if their certainty is low and a larger value of a lower bound if their certainty is high. Also all sources give a larger value of an upper bound if their certainty is low and a smaller value of an upper bound if their certainty is high. Hence, the difference between the upper and lower bounds decreases with increased certainty of the sources. The enclosure of the true solution by the given uncertainty bounds is assumed and its validation is outside the scope of the paper.

This work presents the formulation of fuzzy boundary element method, which accounts for the uncertainty in thermal conductivity of a homogeneous material for linear steady state heat transfer problems. The conductivity

uncertainty is handled via fuzzy approach, which treats the uncertainty as bounded quantity varying dependent on the level of confidence in the input bounds. A fuzzy matrix parameterization is developed, which allows computation of the exact solution, for a given mesh, and does not result in any overestimation commonly present in interval type solutions. Numerical examples illustrate the behavior of the method and show that the bounds are exact independent of the mesh size.

FUZZY BOUNDARY ELEMENT METHOD (FBEM)

BOUNDARY ELEMENT ANALYSIS OF LAPLACE EQUATION

The following section provides a review of the formulation for a two dimensional boundary element formulation of the Laplace equation, which describes the behavior of the steady state heat conduction in homogeneous bodies (Brebbia 1992):

$$\begin{aligned}\nabla \cdot (k \nabla u) &= 0 \text{ in } \Omega \\ u &= \hat{u} \text{ on } \Gamma_1, \quad k \frac{\partial u}{\partial n} = q = \hat{q} \text{ on } \Gamma_2\end{aligned}\quad (1)$$

Where k is the thermal conductivity of the material, Ω is the domain of the system, Γ is the boundary of the system and $\hat{u}$ and $\hat{q}$ are the values of temperature u and heat flux q at the boundary. Eq. (1) is written in a weak form or weighted residual form with a test function w . For convenience thermal conductivity is set equal to one, yielding:

$$\int_{\Omega} \nabla^2 u w d\Omega = \int_{\Gamma_2} (q - \hat{q}) w d\Gamma - \int_{\Gamma_1} (u - \hat{u}) \frac{\partial w}{\partial n} d\Gamma \quad (2)$$

Eq. (2) is integrated by parts twice. Also $u^* = w$ and $q^* = \partial u^* / \partial n$ are considered yielding:

$$\int_{\Omega} \nabla^2 u^* u d\Omega = - \int_{\Gamma_2} \hat{q} u^* d\Gamma - \int_{\Gamma_1} q u^* d\Gamma + \int_{\Gamma_2} u q^* d\Gamma + \int_{\Gamma_1} \hat{u} q^* d\Gamma \quad (3)$$

or:

$$u(\xi) + \int_{\Gamma_2} u q^* d\Gamma + \int_{\Gamma_1} \hat{u} q^* d\Gamma = \int_{\Gamma_2} \hat{q} u^* d\Gamma + \int_{\Gamma_1} q u^* d\Gamma, \xi \in \Omega \quad (4)$$

where ξ is a source point or a point at which a concentrated heat source is applied. u^* is the Green's function for Laplace equation:

$$\nabla^2 u^* + \delta(x - \xi) = 0 \quad (5)$$

The solution to Eq. (5) for a two-dimensional isotropic domain is:

$$u^* = -\frac{1}{2\pi} \ln(r) \quad (6)$$

$$q^* = -\frac{1}{2\pi r^2} (x - \xi) \cdot n \quad (7)$$

where $r = |x - \xi|$ is the distance between the source point ξ and any field point x , which is a point at which the response to the concentrated heat source is measured. For notational convenience, the boundary is taken along x . Rewriting Eq. (4) before the application of boundary conditions:

$$u(\xi) + \int_{\Gamma_x} q^*(x, \xi) u(x) d\Gamma_x = \int_{\Gamma_x} u^*(x, \xi) q(x) d\Gamma_x, \xi \in \Omega \quad (8)$$

Integrating Eq. (8) such that the source point, ξ , is included on the circular boundary of radius ε , as $\varepsilon \rightarrow 0$, results in the left side integral vanishing. For constant elements the

right side integral is $-\frac{1}{2} u(\xi)$. Eq. (8) is rewritten as:

$$\frac{1}{2} u(\xi) + \int_{\Gamma_x} q^*(x, \xi) u(x) d\Gamma_x = \int_{\Gamma_x} u^*(x, \xi) q(x) d\Gamma_x, \xi \in \Gamma \quad (9)$$

The boundary integral equation is discretized into boundary elements consisting of nodes and polynomial shape or interpolation functions. The boundary conditions consist of either a known temperature or flux at the nodes. In this work, for convenience one-noded boundary elements with constant shape functions are used. This results in the following discretization:

$$u(x) = u_i \Phi \quad (10)$$

$$q(x) = q_i \Phi \quad (11)$$

where u_i and q_i are the vectors of nodal values of u and q , respectively, at node i , and Φ is the vector of constant shape functions. Discretization of Eq. (9) results in:

$$\frac{1}{2} u_i + \sum_i u_i \int_{\Gamma_x} q^*(x, \xi) \Phi(x) d\Gamma_x = \sum_i q_i \int_{\Gamma_x} u^*(x, \xi) \Phi(x) d\Gamma_x \quad (12)$$

or in a matrix form:

$$Hu = Gq \quad (13)$$

where matrix H is singular. Eq. (13) is rearranged based on boundary conditions and solved as:

$$Ax = f \quad (14)$$

FUZZY SETS

In this paper uncertainty in material thermal conductivity is treated using fuzzy numbers (Zadeh 1978, Ross 1995). A fuzzy set is a convex set consisting of interval sets, which vary monotonically in width dependent on the level of presumption which is bounded between 0 and 1 (Ross 1995). The highest level of presumption is level 1 and the lowest level of presumption is level 0. The lower bounds of interval sets increase monotonically with increasing level of presumption while the upper bounds decrease monotonically with increasing level of presumption (Figure 1).

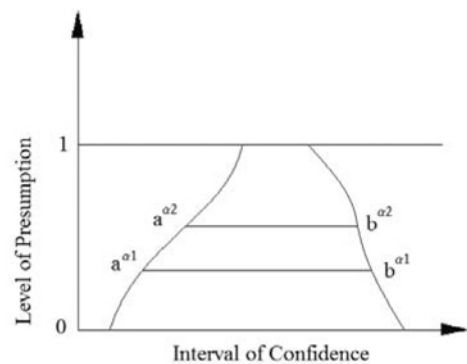


Figure (1). Fuzzy membership function

For every level of presumption, or α cut, there is an associated interval of confidence, which is a convex set of real numbers bounded between the lower and upper limit and is determined from the α cut such that