

On the Time-Dependent Reliability of Non-Monotonic, Non-Repairable Systems

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ABSTRACT

The system response of many engineering systems depends on time. A random process approach is therefore, needed to quantify variation or uncertainty. The system input may consist of a combination of random variables and random processes. In this case, a time-dependent reliability analysis must be performed to calculate the probability of failure within a specified time interval. This is known as cumulative probability of failure which is in general, different from the instantaneous probability of failure. Failure occurs if the limit state function becomes negative at least at one instance within a specified time interval. Time-dependent reliability problems appear if for example, the material properties deteriorate in time or if random loading is involved which is modeled by a random process. Existing methods to calculate the cumulative probability of failure provide an upper bound which may grossly overestimate the true value. This paper presents a methodology to accurately estimate the cumulative probability of failure for non-monotonic in time systems in the presence of an input random process using an equivalent time-invariant “composite” limit state. Dynamics or vibration systems are commonly non-monotonic in time. Input stationary or non-stationary random processes are represented using autoregressive moving average modeling. Multiple examples demonstrate the accuracy of the proposed method and compare it with existing methodologies.

1. INTRODUCTION

In today's technological world, reliability plays a significant role in decision making for both consumers and manufacturers of products such as automobiles, appliances, etc. A less reliable product may be cheaper to acquire, but can be very expensive over its lifetime for the manufacturer due to warranty problems, for example. Reliability is therefore, becoming an important component of the product

lifecycle cost and must be addressed in the product design process [1,2,3,4,5,6] as early as possible.

Reliability is influenced by variability and uncertainty during the design of a product and during its lifecycle. Stochastic operating conditions (loading, environmental conditions, etc.), and degradation in the components (e.g. material properties) as time goes on, cause the products to deliver inconsistent performance and sometimes to even fail prematurely. As a result, we may have high warranty cost, customer annoyance and lower resale value, or even a catastrophic injury or loss of life. It is important therefore, to maintain good reliability over a long time period in order to maximize the product value over its lifecycle.

Reliability is defined as the probability that the system will perform its intended function successfully for a specified interval of time, under stated operating and environmental conditions. It is therefore, related to product functionality over time which is determined by the so-called “hard” and “soft” failures [1, 2]. In a hard failure the system loses functionality due to a complete breakdown of one or more of its components, while in a soft failure the system is functional but one or more performance measures are out of conformance. The reliability associated with the hard failure is important for non-repairable systems where the replacement or repair of a failed component is not possible. In this case, the failed system is removed permanently from the population. In contrast, repairable systems [7] consist of multiple components which can be repaired or replaced if failed keeping therefore, the original system in the population. In this research, we use time-dependent reliability concepts associated with the so-called first-passage failure of non-repairable systems.

Figure 1 shows our notation for the input and output of a system. The input random vector $\mathbf{X} = [\mathbf{X}_v, \mathbf{X}_p(t)]$ consists of a

vector of random variables $\mathbf{X}_v$ and a vector of random processes $\mathbf{X}_p(t)$. An output random process $S(\mathbf{d}, \mathbf{X}, t)$ is obtained when $\mathbf{X}$ is passed through the system transfer function (TF). The vector of constant input parameters is denoted by $\mathbf{d}$.

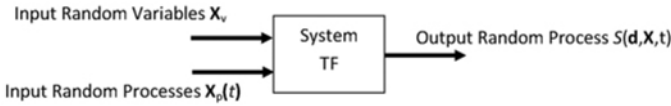


Figure 1. Schematic of input-output relations

A random input variable is quantified using its probability density distribution (PDF). If the PDF is not available, experimental data can be used to estimate it. A random input process must be however, characterized under certain assumptions (e.g. ergodicity and/or stationarity). The ergodicity assumption is common in order to avoid the need of collecting data for many sample functions of the random process in order to estimate its statistics. In this case, the statistics of the ergodic random process are determined from the temporal statistics of a single sample function.

Reliability depends on how failure is defined. First-passage reliability addresses the occurrence of the first failure. This is important for non-repairable systems or components where their replacement or repair is not possible, resulting in their removal from the population. First-passage reliability [2, 3, 8,9,10] has found significant applications in structural engineering in the presence of random input processes such as earthquakes or wind loads, or in non-repairable components such as automotive brake pads, wipers, tires etc.

Most of repairable systems [7] consist of multiple components where it may be possible to repair or replace a failed component while keeping the original system in the population. An automobile is an example of a repairable system where failed components such as brakes and tires are typically replaced. The reliability associated with repairable systems is known as repairable system reliability where the Rate-Of-Occurrence-Of-Failure (ROCF) or “repair rate” provides the rate at which failures occur during the system operation time. Terms such as failure rate or hazard rate are avoided for repairable systems because they mostly apply to the first failure of non repairable components or systems.

Figure 2 shows a sample function of a random process describing the state (up or down) of a system. The sample function indicates a sequence of up and down times through the lifecycle of one of many systems. The statistical quantification of the time to the first failure of the system determines the first-passage reliability. After a failure, the system experiences a down time in order to be repaired, followed by an uptime before the system fails again. This cycle continues multiple times throughout the system's

lifecycle. The time between failures is statistically quantified using multiple but similar systems, to determine what is called Mean-Time-Between-Failures (MTBF). The MTBF metric is commonly used in evaluating the reliability of a repairable system.

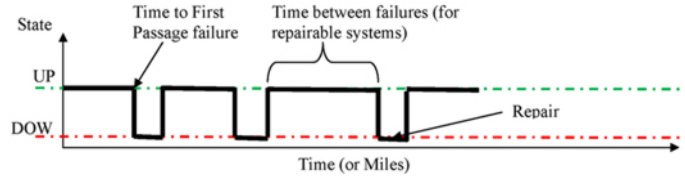


Figure 2. State definition for a repairable system

If the time to first failure (or first-passage failure) is of interest, the statistical average of the times to first occurrence of failure from multiple sample functions determines the “Mean-Time-To-Failure (MTTF)”. In this paper, some relevant time-dependent reliability techniques are addressed which are associated with the first-passage failure. A new method is also proposed and demonstrated with examples.

2. TIME-DEPENDENT RELIABILITY

Let $S(\mathbf{d}, \mathbf{X}, t)$ be a random process and $S_{\text{threshold}}$ be an allowable threshold. A limit state is defined as

$$g(\mathbf{d}, \mathbf{X}, t) = S_{\text{threshold}} - S(\mathbf{d}, \mathbf{X}, t) \quad (1)$$

where $g(\mathbf{d}, \mathbf{X}, t) < 0$ indicates failure. Denoting the time interval of interest by $t_L = [t_{\min}, t_{\max}]$, the probability of failure within this time interval is defined as

$$p_f^c(t_{\min}, t_{\max}) = P(\exists t \in [t_{\min}, t_{\max}], g(\mathbf{d}, \mathbf{X}, t) < 0) \quad (2)$$

where failure occurs if the limit state function becomes negative at least at one instance within the time interval. This is known as *cumulative* probability of failure [3, 9, 11,12,13]. Figure 3 shows the first passage failure using a sample function of a random process.

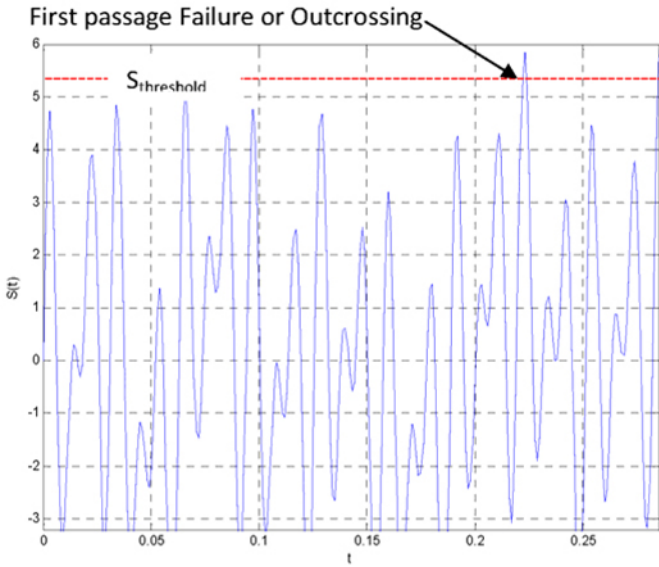


Figure 3. First Passage failure in a random process

The definition of cumulative probability of failure in Eq. (2) is different from the *instantaneous* probability of failure

$$p_f^i(t) = P(g(\mathbf{d}, \mathbf{X}, t) < 0) \quad (3)$$

which is calculated in a quasi-steady sense, by fixing time t and replacing the random process by a random variable. As a result, it does not account for possible failures before time t . The conceptual difference between the cumulative and the instantaneous probabilities of failure is very important. The calculation of cumulative probability of failure is not as straightforward and may or may not be directly related to the instantaneous probability of failure depending upon whether the limit state is monotonic or non-monotonic. This paper discusses this issue in detail.

We can use analytical or simulation-based methods to calculate the probability of failure. The former include the well known first-order and second-order reliability methods (FORM [14], SORM [14, 15]), and multi-point approximation methods [16]. For system reliability analysis involving multiple limit states, the joint failure probability must be accounted for. Due to the difficulty in determining the joint failure probability of more than two failure modes except through MCS, approximations using first-order and second-order bounds have been developed [17, 18]. Simulation-based methods are also used for component or system reliability analysis. Among them, the Monte Carlo Simulation (MCS) method is very simple and accurate. Because its computational cost can be prohibitively high, more efficient simulation-based techniques have been proposed. From the MCS-based methods seeking to improve computational efficiency, adaptive importance sampling (AIS) techniques [19, 20] and multi-modal importance

sampling methods have been reported [21]. They use an importance sampling density function, which is gradually refined to reflect the increasing state of knowledge of the failure domain. A combination of analytical and simulation-based methods can also be used [22].

There is a class of problems (e.g. degradation phenomena) where $g(\mathbf{d}, \mathbf{X}, t)$ may be monotonically decreasing over the time interval $[t_{\min}, t_{\max}]$. For other problems, $g(\mathbf{d}, \mathbf{X}, t)$ may be increasing monotonically (Figure 4). Many existing methods to calculate the cumulative probability of failure assume monotonic behavior which may result in a large error. Accounting for non-monotonicity in the calculation of cumulative probability of failure increases the prediction accuracy but it also increases the computational cost. This paper proposes an improved method to increase the prediction accuracy for problems with non-monotonic limit states.

A limit state $g(\mathbf{d}, \mathbf{X}, t)$ is monotonically decreasing (e.g. degradation phenomena) or increasing in $[t_{\min}, t_{\max}]$ if

$$\begin{aligned} \forall t_1, t_2 \in [t_{\min}, t_{\max}], t_1 < t_2 \Rightarrow g(\mathbf{d}, \mathbf{X} = \mathbf{X}_0, t_1) \geq \dots \\ \geq g(\mathbf{d}, \mathbf{X} = \mathbf{X}_0, t_2) \text{ (decreasing)} \end{aligned} \quad (4a)$$

or

$$\begin{aligned} \forall t_1, t_2 \in [t_{\min}, t_{\max}], t_1 < t_2 \Rightarrow g(\mathbf{d}, \mathbf{X} = \mathbf{X}_0, t_1) \leq \dots \\ \leq g(\mathbf{d}, \mathbf{X} = \mathbf{X}_0, t_2) \text{ (increasing)} \end{aligned} \quad (4b)$$

for every realization $\mathbf{X}_0$ of the random process $\mathbf{X}$. This is

$$\text{true if } \frac{\partial g(\mathbf{X}, t)}{\partial t} \geq 0 \text{ or } \leq 0 \quad \text{for all sample realizations of } \mathbf{X}.$$

For a monotonically degrading problem (see Equation 4a), the “instantaneous” failure region at time t_2 should be either the same or greater than that at time t_1 . In this case,

$$p_f^c(t_1, t_2) = p_f(t_1 \cup t_2) = p_f^i(t_2) \quad (5)$$

or in general,

$$p_f^c(t_{\min}, t_{\max}) = p_f(t_{\min} \cup \dots \cup t_{\max}) = p_f^i(t_{\max}). \quad (6)$$

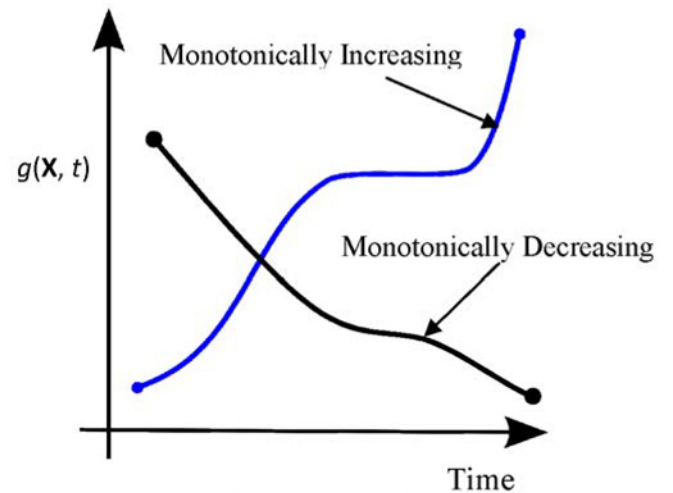
Therefore, for monotonically degrading systems, we have

$$\forall t \in [t_{\min}, t_{\max}], p_f^c(t_{\min}, t) = p_f^i(t), \quad (7)$$

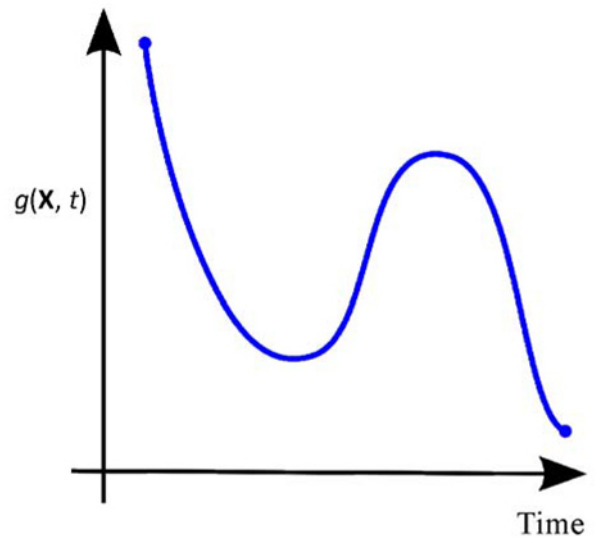
indicating that the cumulative probability of failure is equal to the instantaneous probability of failure at the final time [9, 12]. Similarly for monotonically improving problems, the cumulative probability of failure is equal to the instantaneous probability of failure at the initial time [9, 12]; i.e.

$$\forall t \in [t_{\min}, t_{\max}], p_f^c(t_{\min}, t) = p_f^i(t_{\min}). \quad (8)$$

It is shown in section 2.1 that Equations (7) and (8) are also valid for non-monotonic problems if the limit states are locally monotonic. In order to understand this local monotonicity, we must view the limit states in the standard normal U-space.



a). Monotonic in time



b). Non-monotonic in time

Figure 4. Illustration of monotonicity in time

2.1. MONOTONICITY IN STANDARD NORMAL SPACE

Figure 5 shows some instantaneous limit states of a hypothetical problem at instants t_1 , t_2 , and t_3 in the standard normal U-space. According to the definition of monotonicity in Equations (4a) and (4b), the instantaneous limit states in the U-space must not cross each other as time progresses. In this case, the new failure region is completely enclosed by the old failure region (improving system with increasing reliability with time), or the old failure region is completely enclosed by the new failure region (degrading system with reducing reliability with time). Figure 5a shows a degrading system. If the instantaneous limit states cross each other, the system exhibits local areas (sub-domains of U-space) with improving or degrading reliability (see Figure 5b). In this

case, although the system is non-monotonic, it experiences local monotonicity.

Most of vibratory systems with response similar to that of [Figure 3](#), are non-monotonic because they exhibit multiple peaks and valleys in the time-domain. As time increases, their instantaneous limit states may be approaching the origin of the U-space, and then they may be reversing direction moving away from the origin, or vice versa. Also, the instantaneous limit states may be crossing each other. In both cases, the system has a non-monotonic behavior. [Figure 5b](#) for example, shows non-monotonicity when the instantaneous limit states indicate degradation from time t_1 to t_2 and then improvement from t_2 to t_3 .

Most practical dynamic or vibratory systems are non-monotonic. However, if their instantaneous limit states cross each other as in [Figure 5c](#), they can be treated as “locally” monotonic. We should therefore, identify the local area of interest and check for local monotonicity in that area. The area of interest should be around the MPP, or if we have multiple MPPs, around the dominant (or significant) MPPs. For example, [Figure 5c](#) shows “local” monotonic degradation in the region closest to the origin, and non-monotonic behavior in the low-right region of the U-space where the effect of non-monotonicity would be insignificant to the cumulative probability of failure.

To check for local monotonicity for $t \in [t_{\min}, t_{\max}]$, we must first identify all significant Most Probable Points (MPPs) [14, 23, 13], and then examine if the $\left[\frac{\partial g(\mathbf{d}, \mathbf{X}, t)}{\partial t} \right]$ is either positive (increased reliability) or negative (degrading reliability) at these MPPs for $t \in [t_{\min}, t_{\max}]$. If the sign does not change with time, the problem can be treated as locally monotonic allowing the use of the simpler solutions of [Equations \(7\) and \(8\)](#). Note that because each instantaneous limit state has its own MPPs, the locus of each significant MPP as a function of time may be needed.

<figure 5 here>

3. CALCULATION OF PROBABILITY OF FAILURE

[Equations \(7\) and \(8\)](#) show that for monotonic or “locally” monotonic problems, an instantaneous probability of failure is equal to the cumulative probability of failure. However, [Equations \(7\) and \(8\)](#) may result in a large error for non-monotonic problems. As we have mentioned, the instantaneous limit states of non-monotonic problems can cross each other with trend reversals at multiple locations in

the U-space. To account for this, the composite limit state concept is used [3]. The composite limit state includes the segments of the instantaneous limit states which are closest to the origin of the standard normal space. [Figure 7](#) illustrates the composite limit state for a hypothetical example in grey.

Let the time interval $[t_{\min}, t_{\max}]$ being discretized using a time step Δt , so that $t = i \cdot \Delta t$ where $i = 0, 1, 2, \dots, I$ with $t \in [t_{\min}, t_{\max}]$. The first-passage cumulative probability of failure is then equal to

$$p_f^c(t) = p_f(t_{\min} \cup \dots \cup t_{\max}) = P \left\{ \bigcup_{i=1}^I (g(\mathbf{d}, \mathbf{X}, t_i) \leq 0) \right\}, \quad (9)$$

where failure occurs if the time-dependent limit state becomes negative (i.e. $g(\mathbf{d}, \mathbf{X}, t) < 0$) for the first time.

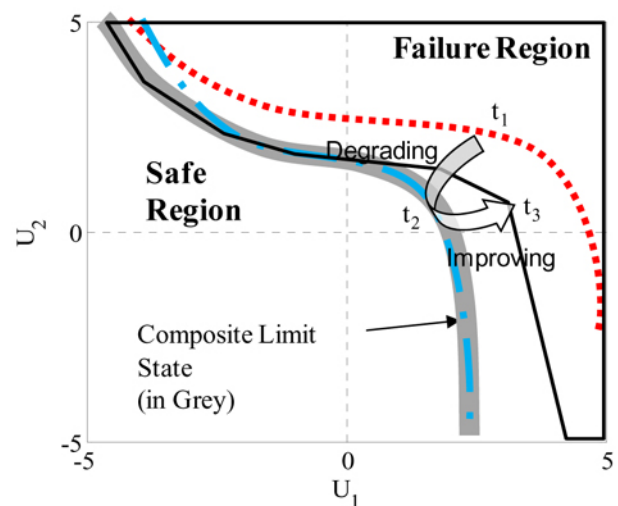


Figure 6. Composite limit state for a non-monotonic problem

MCS can be used to accurately calculate the cumulative probability of failure based on [Equation \(9\)](#) if a large number of sample functions is utilized. This may however, increase the required computational cost. In order to efficiently

calculate the cumulative probability of failure p_f^c many analytical approaches have therefore, been proposed. Rice [24] introduced an out-crossing rate approach to calculate the time-variant probability that a load exceeds a given threshold, known as the barrier crossing problem. Rice's method assumes that the system is linear resulting in a stationary output Gaussian process under the additional assumption of Poisson distributed and statistically independent out-crossings. The assumption of Poisson distributed out-crossings is commonly used in the literature [25, 26, 27, 28].

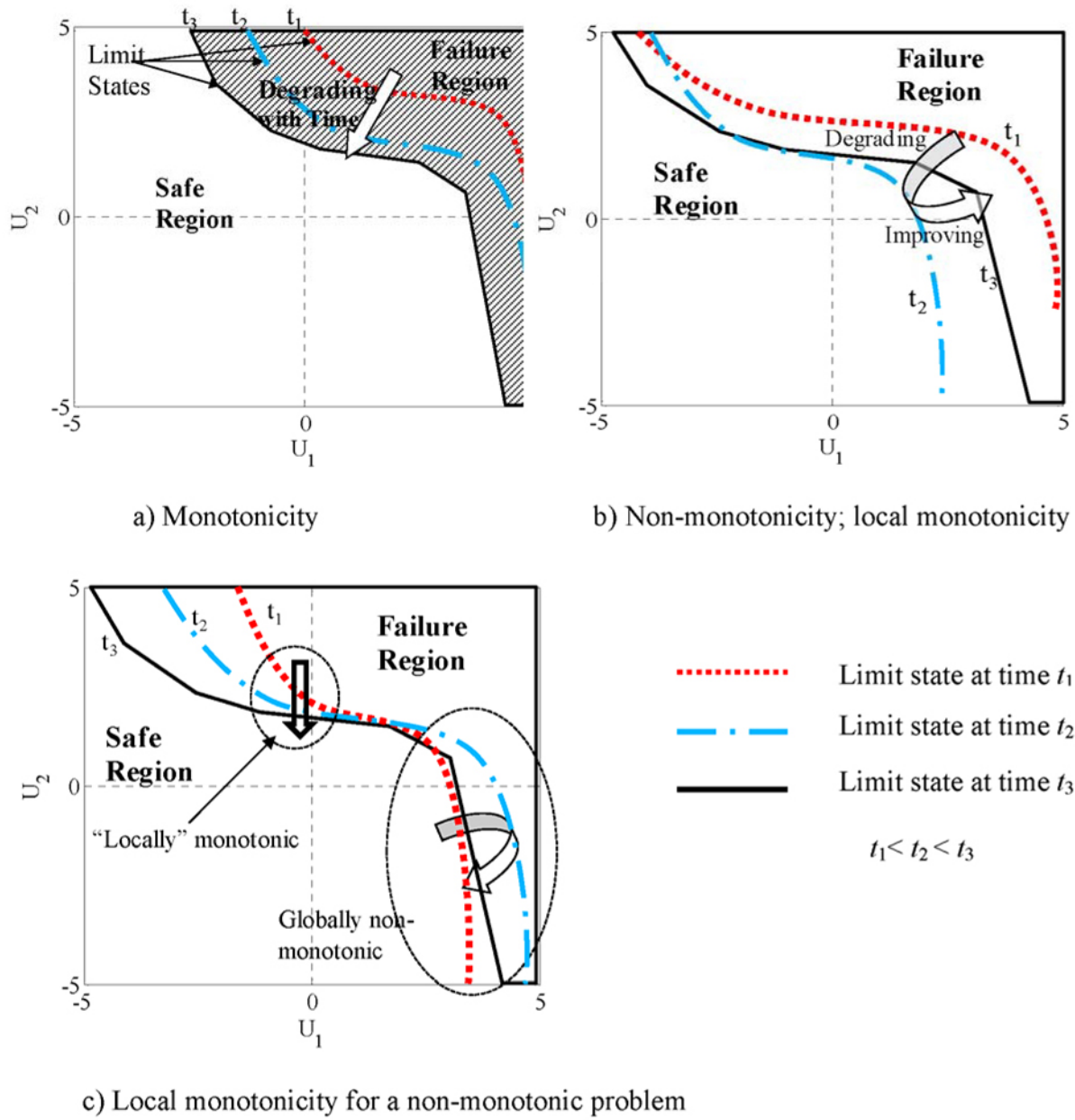


Figure 5. Non-monotonic and locally monotonic behavior in U -space

This assumption is more appropriate for repairable systems where a component is repaired and put back into the population. The out-crossing rate is defined [29] as

$$v^+(t) = \lim_{\Delta\tau \rightarrow 0, \Delta\tau > 0} \frac{P[g(\mathbf{d}, \mathbf{X}, t) > 0 \cap g(\mathbf{d}, \mathbf{X}, t + \Delta\tau) \leq 0]}{\Delta\tau} \quad (10)$$

If $N^+(t_{\min}, t)$ is the number of up-crossings in the time interval $[t_{\min}, t]$, Ditlevsen and Madsen [21] calculated the mean number of out-crossings as

$$E[N^+(t_{\min}, t)] = \int_{t_{\min}}^t v^+(t) dt \quad (11)$$

Under the assumption of independent and Poisson-distributed out-crossings, the cumulative probability of failure is given by [30]

$$p_f^c(t_{\min}, t) = 1 - \exp(-E[N^+(t_{\min}, t)]) \quad (12)$$