

An Expeditious High Fidelity ABAQUS-Based Surrogate Tire Model for Full Vehicle Durability Analysis in ADAMS

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04/12/2011Hamid Ansari Ardeh, Makarand Datar, Manoj kumar Jagadeesan and Dan Negrut
Univ of Wisconsin Madison

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ABSTRACT

This paper discusses an approach to construct a high fidelity surrogate tire model using a two-phase optimization-based algorithm that draws on data generated by off-line nonlinear ABAQUS tire simulations. It subsequently describes the process of Simulink-based interfacing of the resulting surrogate model to a full ADAMS vehicle model to enable accurate and expeditious durability studies. The two-phase surrogate model construction relies on an identification method that draws on the Instantaneous Center Manifold (ICM) theory. In the proposed method, a generally forced non-autonomous nonlinear structural system is represented as a sequence of harmonically excited autonomous nonlinear systems. The close-form solution of each of these systems is produced using the ICM theory. The first phase of the surrogate model construction uses an optimal Orthogonal Matching Pursuit (OMP) algorithm to unify all ICMs used to approximate the reaction force of the tire at its spindle. In the second phase, Quadratic Programming (QP) is used to form a single, closed-form mathematical model that is subsequently used as a black box surrogate model. While the method can produce surrogate models with a desired level of accuracy, higher accuracy requires more training data. This data can be a combination of experimental data and off-line simulation results. For a tire model, the proposed algorithm is validated against ABAQUS simulation results. The validated surrogate tire model is subsequently interfaced through Simulink to ADAMS for simulation typically used in the durability analysis of a High-Mobility Multipurpose Wheeled Vehicle (HMMWV). The surrogate model turns out to be more than 1,000 times faster than the ABAQUS simulation in producing tire forces while still maintaining a good level of accuracy.

INTRODUCTION

Over the last decade, CAD/CAE-enabled virtual prototyping has been widely embraced in engineering design. This trend was motivated by economic factors, i.e., reduced cost and time to market due to partial replacement of testing with computer simulation. The effectiveness of virtual prototyping can be hampered though by the presence of subsystems whose simulation is costly. Such an example is a tire sub-system in a ground vehicle model. Computationally expensive sub-systems can limit the scope of a design of experiments study or make such a study outright impossible. This observation motivates the need for approaches in which such computationally expensive components of a dynamic system are identified and replaced with faster surrogate models. Three requirements should be met by such an approach: (i) the surrogate model should be obtained expeditiously and preferably in a way that is not sub-component specific, and (ii) the surrogate model should not modify the response of the system, and (iii) the process of producing the surrogate model should yield more accurate surrogate models when the training data set increases.

In the context of ground vehicle simulations, the tire can be one of the sub-systems that are challenging to model and responsible for long vehicle-level simulation times. These two aspects are further exacerbated for off-road vehicle simulations. Advanced commercial-off-the-shelf simulation packages are available for analyzing these scenarios [1, 2]. When exercised in off-road conditions in the presence of short wavelength obstacles, the tires are usually represented by means of lumped mass or finite element models[3, 4]. While providing for good accuracy, they are computationally very demanding. In cases like this or when dealing directly with process measurement data in the absence of an actual

computer model, surrogate modeling methods become facilitators for design space exploration, prototyping and optimization [5, 6]. Surrogate models are fast-running approximations of complex time-consuming computer simulations. They are used to reduce the complexity of systems for which the governing equations are unavailable or computationally expensive to deal with. They mimic the complex behavior of the underlying simulation model and fill the gap between the numerical or experimental and the analytical models [5,6,7]. Surrogate modeling can rely on established theories in each context to find a suitable model. To this end, it usually draws on machine learning theory to produce mathematical methods that are subsequently used to construct the model that.

Several methods have been established to develop structural models based on input-output measurements performed on the real structure or the structural model. They are known as structural system identification methods and a comprehensive overview of these methods can be found in [8,9,10]. Modal analysis is probably the most popular approach for linear system identification in structural dynamics [11,12,13]. Its wide acceptance stems from its generality, which can be traced back to the simple observation that for linear structural systems the general system response can be easily generated using simple building blocks. For free response, the building blocks are mode shapes. The linear combination idea works whenever the principle of superposition is applicable, for linear systems in particular. For nonlinear systems, one needs to find alternative approaches that do not rely on the principle of superposition. Such an approach is outlined in [14]. Its first salient attribute is that it addresses the problem of representing the general forcing term using simpler terms without relying on the principle of superposition to retrieve the total response of the system. Each system excited by a simpler forcing term admits a solution that is subsequently found using the concept of Instantaneous Center Manifold (ICM). This paper utilizes the ICM theory to construct a surrogate tire model that can closely approximate the accuracy of a reference FE tire model at a fraction of the computational price. The two-phase algorithm proposed constructs and unifies ICMs and then forms a close-form mathematical model as a surrogate of the reference system.

ICM APPROACH FOR IDENTIFICATION OF FORCED NONLINEAR STRUCTURAL SYSTEMS

In what follows it is assumed that the continuous system has been spatially discretized, using a method such as finite element, smoothed particle hydrodynamics, boundary element, etc., and the resulting dynamical system is governed by a set of Equations of Motions (EOM) of the form

$$\ddot{\mathbf{x}}(t) + \mathbf{f}(\mathbf{x}(t), \dot{\mathbf{x}}(t)) = \mathbf{f}_e(t, \mathbf{x}_e, \dot{\mathbf{x}}_e) \quad , \quad (1)$$

where $\mathbf{x}$, $\dot{\mathbf{x}}$ and $\ddot{\mathbf{x}}$ are the displacement, velocity and acceleration vectors respectively; $\mathbf{f}$ is a generic nonlinear autonomous restoring force vector; and $\mathbf{f}_e(t, \mathbf{x}_e, \dot{\mathbf{x}}_e)$, the forcing term, is a function of states of the system. Such forcing terms are said to be non-ideal sources of energy [15]. In [14], we proposed to construct the solution of Eq.(1) by solving at each time step an equivalent, harmonically-excited, autonomous systems. To that end, at each time step, the system in Eq.(1) will be replaced by the equivalent system

$$\ddot{\mathbf{x}}(t) + \mathbf{f}(\mathbf{x}(t), \dot{\mathbf{x}}(t)) = \mathbf{f}_{he}(\mathbf{x}_{he}, \dot{\mathbf{x}}_{he}) \quad , \quad (2)$$

where $\mathbf{f}_{he}$ is the autonomous excitation vector needed to apply harmonic motions to a subset of coordinates $\mathbf{x}_e \subset \mathbf{x}$ ($\mathbf{x}_e$: Excited coordinates). Hence, to solve Eq.(1), first, we need to represent the set $\mathbf{x}_e \subset \mathbf{x}$ based on the harmonic motions $\mathbf{x}_{he}$ at each time step. Second, we have to be able to solve Eq.(2) for a harmonic excitation. The proposed ICM-approach addresses these two problems.

As discussed in [14], the first step cannot be carried out using decomposition-based representation methods that rely on representing the history of the original signal in either the time or frequency domain. This is because the principle of superposition would be needed later to assemble the overall response subcomponents. This problem can be resolved by using the analytic representation of $\mathbf{x}_e$ [14]. Representing $\mathbf{x}_e$ in analytic form ($\tilde{\mathbf{x}}_e$)

$$\tilde{\mathbf{x}}_e(t) = \mathbf{x}_e(t) + i\hat{\mathbf{x}}_e(t) = \mathbf{a}(t) \exp[i\phi(t)] = \mathbf{a}(t) \exp[i\omega(t)] \quad , \quad (3)$$

in which $\mathbf{a}(t)$ is the instantaneous envelop vector, $\phi(t)$ is the instantaneous phase vector, $\omega(t)$ is the instantaneous frequency vector

$$\omega(t) = \frac{d}{dt} \phi(t) \quad (4)$$

and $\hat{\mathbf{x}}_e$ is the Hilbert transform of $\mathbf{x}_e$. Then $\mathbf{x}_e$ can be rewritten in form of

$$\mathbf{x}_e(t) = \text{Re}\{\mathbf{a}(t) \exp[i\phi(t)]\} \quad . \quad (5)$$

Therefore, at any fixed time $t = t_i$,

$$\mathbf{x}_e^{(i)} = \mathbf{x}_e(t_i) = \mathbf{a}_i \cos(\phi_i) = \mathbf{a}_i \cos(\omega_i t_i) \quad (6)$$

Therefore, a set of generally excited non-autonomous EOMs of the form in Eq.(1) can be represented as a sequence of autonomous EOMs excited by harmonic motions as outlined in of Eq.(2).

$$\begin{aligned} t \rightarrow \{t_i\} \equiv \mathbf{x}_e \rightarrow \{\mathbf{x}_e^{(i)}\} \\ \ddot{\mathbf{x}}(t) + \mathbf{f}(\mathbf{x}(t), \dot{\mathbf{x}}(t)) = \mathbf{f}_e \quad t \quad \mathbf{x}_e, \dot{\mathbf{x}}_e \rightarrow \{\ddot{\mathbf{x}}(t) + \mathbf{f}(\mathbf{x}(t), \dot{\mathbf{x}}(t)) \\ = \mathbf{f}_{he}(\mathbf{x}_{he}^{(i)}, \dot{\mathbf{x}}_{he}^{(i)})\} \end{aligned} \quad (7)$$

Before solving the Eq.(2), these equations are represented into state-vector form by defining (where T denotes a transpose):

$$\mathbf{y} = \dot{\mathbf{x}}, \quad \mathbf{z} = [\mathbf{x}^T \quad \mathbf{y}^T]^T \quad (8)$$

Equation (2) is equivalently rewritten as

$$\dot{\mathbf{z}} = \mathbf{g}(\mathbf{z}), \quad \mathbf{g}(\mathbf{z}) = [\mathbf{y}^T (\mathbf{f}_{he} - \mathbf{f})^T]^T \quad (9)$$

Suppose next that the equilibrium solution (a fixed point) of Eq.(9) is $\bar{\mathbf{z}}$ (i.e., $\dot{\bar{\mathbf{z}}} = \mathbf{g}(\bar{\mathbf{z}}) = \mathbf{0}$) and let $\mathbf{T}_g = \mathbf{Dg}(\bar{\mathbf{z}})$ be the Jacobian of the $\mathbf{g}$ at the equilibrium solution $\bar{\mathbf{z}}$. Starting with this setup, if $\mathbf{g}$ is a homeomorphism of C^r and for all eigenvalues of $\mathbf{T}_g$ (possibly complex and/or multiple), $s_1, \dots, s_{2n}$, the condition

$$s_i \neq s_1^{m_1} \dots s_1^{m_{2n-1}} \quad (10)$$

¹This relation implies that $|s_i| \neq 0, 1$ for any i . One can show that this leads to non-resonance conditions among eigenvalues which is not considered in the Hartman-Grobman Theorem which is valid only for hyperbolic fixed points.

holds for all non-negative integers $m_1, \dots, m_{2n}$, then by Sternberg's Theorem [16, 17], the linearized (defined by $\mathbf{T}_g$) and the original ($\mathbf{g}$) problems are topologically conjugate; i.e., close to the equilibrium solution ($\bar{\mathbf{z}}$), the dynamics of $\mathbf{g}$ can be arbitrary precisely approximated by the dynamics of $\mathbf{T}_g$. Assuming that $\mathbf{g}$ and $\mathbf{T}_g$ satisfy the mentioned conditions, by the Primary Decomposition Theorem [18], $\mathbf{R}^{2n}$ can be represented as a direct sum of three subspaces denoted by $\mathbf{E}^s$, $\mathbf{E}^u$ and $\mathbf{E}^c$ [19]

$$\mathbf{R}^{2n} = \mathbf{E}^s \oplus \mathbf{E}^c \oplus \mathbf{E}^u, \quad (11)$$

which are defined as follows:

$$\begin{aligned} \mathbf{E}^s &= \text{span}\{\mathbf{e}_1, \dots, \mathbf{e}_s\} \\ \mathbf{E}^u &= \text{span}\{\mathbf{e}_{s+1}, \dots, \mathbf{e}_{s+u}\}, \quad s+u+c=2n \\ \mathbf{E}^c &= \text{span}\{\mathbf{e}_{s+u+1}, \dots, \mathbf{e}_{s+u+c}\} \end{aligned} \quad (12)$$

Here $\{\mathbf{e}_1, \dots, \mathbf{e}_s\}$ are the generalized eigenvectors of $\mathbf{T}_g$ corresponding to the eigenvalues of $\mathbf{T}_g$ having negative real part; $\{\mathbf{e}_{s+1}, \dots, \mathbf{e}_{s+u}\}$ are the generalized eigenvectors of $\mathbf{T}_g$ corresponding to eigenvalues of $\mathbf{T}_g$ having positive real part; and $\{\mathbf{e}_{s+u+1}, \dots, \mathbf{e}_{s+u+c}\}$ are the generalized eigenvectors of $\mathbf{T}_g$ corresponding to eigenvalues of $\mathbf{T}_g$ having zero real part [18]. The $\mathbf{E}^s$, $\mathbf{E}^u$ and $\mathbf{E}^c$ subspaces are referred to as the stable, unstable and center invariant manifolds since the solution of Eq.(9) with initial conditions entirely contained in either $\mathbf{E}^s$, $\mathbf{E}^u$ or $\mathbf{E}^c$ remains in that particular manifold for $t > 0$ [18,19,20]. Moreover, solutions starting in $\mathbf{E}^s$ approach $\mathbf{z} = \mathbf{0}$ asymptotically as $t \rightarrow \infty$ and solution starting in $\mathbf{E}^u$ approach $\mathbf{z} = \mathbf{0}$ asymptotically as $t \rightarrow -\infty$.

Suppose next that the equilibrium solution $\bar{\mathbf{z}}$ is not unstable, in other words

$$\bar{\mathbf{z}} \in \mathbf{E}^{s \oplus c} \equiv \mathbf{E}^s \oplus \mathbf{E}^c \equiv \text{span}\{\mathbf{e}_1, \dots, \mathbf{e}_s, \mathbf{e}_{s+1}, \dots, \mathbf{e}_{s+u}\} \quad (13)$$

Since for $\mathbf{z}_0 \in \mathbf{E}_s$ the solution $\bar{\mathbf{z}}$ will vanish after a long enough time, from this point on, the emphasis of the discussion is on the solutions that start from $\mathbf{z}_0 \in \mathbf{E}_c$ and can be expressed as [14, 18, 21]

$$\bar{\mathbf{z}} = \mathbf{e}_{s+1} \mathbf{e}_{s+1}^T \mathbf{z}_0 e^{s_1 t} + \dots + \mathbf{e}_{s+c} \mathbf{e}_{s+c}^T \mathbf{z}_0 e^{s_{s+c} t} \quad (14)$$

One way to construct the solution $\bar{\mathbf{z}}$ as a close-form solution of the system defined by Eq.(14) is to find the eigenvalues $\{s_{s+1}, \dots, s_{s+u}\}$ and the eigenvectors $\{\mathbf{e}_{s+1}, \dots, \mathbf{e}_{s+u}\}$. This approach requires the solution of an eigenvalue problem in a vector space $\mathbf{R}^{2n}$. It is also conditioned on first, having the governing EOM analytically defined, and second, being able to find the Jacobian of the EOM at each time step. The approach to construct the solution $\bar{\mathbf{z}}$ herein is different and regards this quantity as an unknown function. A function space $\mathbf{E}^c$ is first defined as the space of all C^l continuous functions $\bar{\mathbf{z}}$. This way $\mathbf{E}^c$ will include all possible equilibrium solutions $\bar{\mathbf{z}}$. The next step is to find a close form

representation of the functions in E^c . The following theorem proved [14] is used to this end.

Theorem A. Let $\mathbf{g}$ be a C^r vector field on $\mathbf{R}^{2n}$ vanishing at $\bar{\mathbf{z}}$ (i.e., $\mathbf{g}(\bar{\mathbf{z}}) = \mathbf{0}$). Then, if $\bar{\mathbf{z}}$ is marginally stable ($\bar{\mathbf{z}} \in E^c$), it is locally a generalized Fourier series of an Almost Periodic Function. That is, it belongs to the space of almost periodic functions B (for definition and properties see [22,23,24]).

With $E^c \subset B$, E^c is defined as follows:

Definition: Let $\mathbf{g}$ be a C^r vector field on $\mathbf{R}^{2n}$ and E^c be the center manifold corresponding to $\mathbf{g}$ close to its equilibrium solution $\bar{\mathbf{z}}$. Then, the space of all functions defined by $\Upsilon: E^s \rightarrow E^s$ equipped with the inner product

$$\langle f, g \rangle = M_t \{ f(t) \overline{g(t)} \}, \quad \|f\| = \langle f, g \rangle^{1/2}, \quad (15)$$

in which

$$M_t \{ f(t) \} = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T f(x+t) dt, \quad (16)$$

is denoted by E^c and called the instantaneous center manifold (ICM).

Essentially, the instantaneous center manifold E^c is a function space that is defined based on E^c in conjunction with a metric induced by inner product of Eq.(15). It is equivalent to the space of all marginally stable solutions of the vector field $\mathbf{g}$ in a vicinity of its equilibrium solutions. This way, for any fixed moment $t \in \mathbf{R}^+$, $E_t^c \equiv E^c$, that is, E^c represents a function space instantaneously equivalent to the center manifold of the vector field. Referring back to the definition of the function family Υ , it becomes apparent that $E^c \subseteq B$ since $E^c \subseteq \mathbf{R}^{2n}$ by adding a metric becomes an incomplete Hilbert space [23, 24], and B , which is defined by the same inner product, contains all possible equilibrium solutions $\bar{\mathbf{z}} \in E^c$.

CONSTRUCTING ICMs

It is shown in [23] that if H is an incomplete Hilbert space and $(b_v)_{v \in \mathbf{R}}$ is an orthonormal system for H , then the following statements are equivalent:

a. $z = \sum_{i \in \mathbf{I}} \langle z, b_i \rangle b_i$ for all $z \in H$

b. Parseval's equation

$$\|x\|^2 = \sum_{i \in \mathbf{I}} |\langle x, b_i \rangle|^2 \quad (17)$$

holds for each $z \in H$.

in which $b_i \in (b_v)_{v \in \mathbf{R}}$.

Statement (a) indicates that constructing the space E^c is equivalent to establishing an orthonormal system for this space. Statement (b) indicates that any function in the space can be represented as a linear combination of the element of the orthonormal system and this representation is preservative with respect to instantaneous power. The latter also implies how we can recognize the elements of the orthonormal system of pre-Hilbert space that also belong to the orthonormal system of the space E^c . It is shown in [14] that if $(b_c)_{c \in \mathbf{I}}$ is an orthonormal system for E^c and if

$$(b_v)_{v \in \mathbf{R}} = \{b_v(t) | b_v(t) = \exp(ivt), v \in \mathbf{R}\} \quad (18)$$

is an orthonormal system for B (see proof in [24]), then

$$(b_c)_{c \in \mathbf{I}} \subset (b_v)_{v \in \mathbf{R}}, \quad \mathbf{I} \in \mathbf{R} \quad (19)$$

Therefore we need to find a orthonormal system $(b_c)_{c \in \mathbf{I}}$ such that for all $z \in E^c$

$$z = \sum_{c \in \mathbf{I}} \langle z, b_c \rangle b_c \quad (20)$$

Equation (20) is sometimes called the sparse approximation problem (SAP). Since, the value of n that controls the dimension $2n$ of the space E^c and also $\mathbf{I}$, is generally unknown, solving the SAP will lead to an optimization problem. Before formally stating the optimization problem, some remarks are in order. First, since E^c is defined based on E^c , it inherits the property that it is defined in a vicinity of the equilibrium solution. This restricts each E^c to its corresponding equilibrium solution. Second, it is shown in [18, 25] that the solution of system defined by Eq.(2) can be uniquely determined knowing the excited coordinate's states. And finally, as mentioned before, excited coordinates, which are considered to be harmonic, are also parts of the solution. Therefore, if $z_e = [x_e \dot{x}_e]^T \in \mathbf{z}_e$ is one of excited coordinate sets, then

$$\begin{aligned} \tilde{x}_e &= x_e + i\hat{x}_e = c_e \cos(\omega_e t) + ic_e \sin(\omega_e t) = \\ &= c_e \exp(i\omega_e t), \quad c_e, \omega_e \in \mathbf{R}^+ \end{aligned} \quad (21)$$

which, considering that if any function in E is of the form defined by Eq.(18), it is also a member of $(b_v)_{v \in \mathbf{R}}$ (because of

orthogonality of the system $(b_c)_{c \in \mathbf{I}}$. This means that analytic representation of all excited coordinates of marginally stable equilibrium solutions of $\mathbf{g}$ are also in the orthonormal system $(b_c)_{c \in \mathbf{I}}$. These points suggest that every E^c can be uniquely constructed using the states associated with the excited coordinates of its corresponding equilibrium solution. Considering the fact that all $z \in E^c$ can be expressed as in Eq.(14) and presuming that the orthonormal system $(b_c)_{c \in \mathbf{I}} \subset (b_v)_{v \in \mathbf{R}}$ is known, one way to construct E^c is to reproduce $(b_v)_{v \in \mathbf{R}}$ using the states associated with the excited coordinates. To this end, a mapping $\Phi: \tilde{\mathbf{x}}_e \rightarrow \mathbf{B}$ was defined in [14] as

$$\Phi(\tilde{x}_e) = \Phi(x_e, \hat{x}_e) = \left\{ \exp[i\gamma \text{atan}(\hat{x}_e / x_e)] \mid x_e \in \mathbf{x}_e, \gamma \in \mathbf{R} \right\} \quad (22)$$

Additionally, a kernel $\mathbf{K}: \tilde{\mathbf{x}}_e \times \tilde{\mathbf{x}}_e \rightarrow \mathbf{R}$ is defined as

$$\mathbf{K}(\tilde{x}_e, \tilde{x}_e') = \langle \Phi(\tilde{x}_e), \Phi(\tilde{x}_e') \rangle^2 \quad (23)$$

²It's easy to show that $\mathbf{K}$ is a semi-positive kernel.

in which $\langle, \rangle$ is defined by Eq.(15). This way, one has that

$$(b_\gamma)_{\gamma \in \mathbf{R}} = \Phi(\tilde{\mathbf{x}}_e) = \bigcup_{e \in e} \left\{ e^{i\gamma \omega_e t} \mid \gamma \in \mathbf{R} \right\} \quad (24)$$

which makes it a redundant orthonormal system for $\mathbf{B}$ and therefore $(b_v)_{v \in \mathbf{R}} \supseteq (b_\gamma)_{\gamma \in \mathbf{R}}$.

The problem of constructing E^c , that is, the SAP, is stated based on the states of the excited coordinates of its corresponding equilibrium solution to lead to the following:

Constructing ICM - Let $\mathbf{g}$ be a C^r vector field on $\mathbf{R}^{2n}$ and E^c be the instantaneous center manifold corresponding to $\mathbf{g}$ close to its equilibrium solution $\tilde{\mathbf{z}}$. The index set $\mathbf{I}$, that is, the solution of the SAP

$$\min_{|\mathbf{I}|} \left| \sum_j \left\| z_j \right\|^2 - \sum_{i \in \mathbf{I}} \left| \langle z_j, b_i \rangle \right|^2 \right|, \quad b_i \in (b_\gamma)_{\gamma \in \mathbf{R}}, z_j \in \tilde{\mathbf{Z}} \quad (25)$$

uniquely determines the orthonormal system $(b_i)_{i \in \mathbf{I}} \subseteq (b_\gamma)_{\gamma \in \mathbf{R}}$ of the space E^c .

This can be interpreted as finding the appropriate set of elements from the orthonormal set $(b_\gamma)_{\gamma \in \mathbf{R}}$ that minimizes the error defined in Eq.(25) for all states of the equilibrium solution.

A TWO PHASE ALGORITHM FOR CONSTRUCTING AN ICM-BASED SURROGATE MODEL

Solving the problem stated by Eq.(25) generates a different ICM for each equilibrium solution. This is the case because first of all, the optimal set of basis functions for each equilibrium solution may be different as the frequencies stem from the eigenvalues of the tangent vector field, which are changing in time [18]. Therefore it is not expected to have the same frequencies, which can be directly related to the optimal basis, in all equilibrium solutions.

Secondly, the coefficient set

$$c_{ij} = \{ \langle Z_j, b_i \rangle \mid b_i \in (b_\gamma)_{\gamma \in \mathbf{R}}, z_j \in \tilde{\mathbf{Z}} \} \quad (26)$$

of each equilibrium solution can also be different. In this sense, it is known that the solutions of a nonlinear oscillatory system are amplitude-dependent [9, 14, 15, 21, 26,27,28,29]. Then, since the elements of the orthonormal system do not depend on the amplitude of the excitations, the coefficient of each basis function in the optimal set have to do.

Finding therefore a single close-form solution that can operate as a mathematical model (a surrogate model) depends on whether one can find a way to unify all ICMs obtained by solving Eq.(25) for a set of different equilibrium solutions at the levels of both the optimal basis set and the coefficients set. In this regard, the first step is to unify all optimal basis sets. This is not possible at the frequency level because ICMs are expected to have different frequency content. The alternative left is to unify the close-form solutions at another level. That is, in fact, the reason for defining the orthonormal system $(b_\gamma)_{\gamma \in \mathbf{R}}$ based on the dimensionless factor γ

$$\gamma = \omega_\gamma / \omega_e \quad (27)$$

where ω_γ represents the eigenvalues of the tangent vector fields. This way, it becomes possible to unify the optimal basis sets at the γ -level while having different optimal basis sets for each equilibrium solution. In this regard, and also to facilitate the optimization problem of constructing the ICMs, γ is discretized by assuming

$$\gamma = k / C_{FR}, \quad k \in \mathbf{N}, \quad (28)$$

where C_{FR} is a constant called the frequency resolution constant. Based on Eq.(27) one can write

$$\gamma \Delta \omega_e = \Delta \omega_\gamma \quad (29)$$