

Fuzzy Boundary Element Method with Uncertain Shear Modulus in Linear Plane Strain Elasticity

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ABSTRACT

Engineering systems are often too complex and their behavior cannot be determined using closed form or exact methods. To circumvent this problem, numerical methods, such as boundary element method, have been formulated to obtain approximate solutions to partial differential equations, which describe the behavior of the physical engineering systems. However, the inherent uncertainty in constitutive formulation causes uncertainties in the solutions obtained by numerical methods and undermines their validity. Conventional analysis does not account for material uncertainty, which is, however, accounted for in the design phase. This paper addresses the impact of uncertain shear modulus for plane strain linear elasticity problems on the numerical solutions obtained using boundary element method. The uncertainty is modeled using fuzzy approach. Matrix parameterization is developed to obtain exact worst case bounds on the solutions assuming that a correct partial membership function is given. Numerical examples are presented to illustrate the behavior of the method.

INTRODUCTION

Practicing engineers rely on various analysis techniques to predict the behavior of various engineering systems subjected to various boundary conditions. In the vast majority of cases, however, these methods do not provide an exact prediction, which undermines their reliability. The inaccuracies in most numerical techniques such as finite element method (FEM), boundary element method (BEM), discrete element method (DEM), meshfree method (MFM), and finite difference method (FDM) occur due to various errors and uncertainties, which may be inherent to an individual numerical method. These include discretization error, integration error, round-off error, uncertainty in boundary conditions, uncertainty in

system geometry, uncertainty in system or material parameters, and uncertainty in mathematical model. The latter uncertainty is hard to quantify since it requires the measure of uncertainty relating the governing equations to the real system. In this paper it is assumed that the governing equations represent the true behavior of the system. Although the verification errors for systems are hard to assess, the main uncertainty in the solutions comes from the validation errors. Uncertainties in material characteristics are inherent to the system since the material properties are obtained from either experimental data or from experience. These components usually involve a relationship between a potential, such as displacement or strain, and a flux variable, such as traction or stress. In the case of linear elasticity, shear modulus can provide the necessary relationship between these two. Shear modulus is measured using experimental techniques and it has an associated uncertainty that corresponds to the error with which it was measured. The majority of the measurement tools have a measurement tolerance, which is given by a manufacturer and potentially can be influenced by field conditions such as temperature. Although a true measurement error of the measured quantity cannot be known, the worst case measurement error, and therefore, the worst case uncertainty, is given directly from the measurement tolerances. It is thus both a natural and a convenient choice to use the given tolerances as the lower and upper bounds on the parameter uncertainty.

Uncertainty in system parameters, such as shear modulus, influences the system performance. Stochastic methods or sensitivity analyses can be performed to study the effects of parameter uncertainty on system behavior; however, they are both expensive and do not guarantee that the worst case behavior of the system has been captured. It is beneficial to obtain a computationally inexpensive accurate method, which computes the worst case behavior of the system with

uncertain parameters. Since most engineering systems are complex, it is necessary for such a method to be numerical. Currently, most engineering systems are solved using FEM, FDM, BEM, or a combination of these. In this work, boundary element method is used to obtain approximate solutions to partial differential equations.

It should be noted that approximate solutions obtained via BEM are affected by errors and uncertainties such as discretization error, integration error, round-off error, uncertainty in system geometry, uncertainty in boundary conditions, and uncertainty in system parameters. These errors or uncertainties impact the behavior of the system but currently are not considered in conventional analysis methods. Finite element based methods using fuzzy numbers ([Zadeh 1978](#), [Ross 1995](#)) and interval numbers ([Moore 1966](#), [Neumaier 1990](#)) have been used to address uncertainty in boundary conditions, geometry, and material resulting in fuzzy finite element method (FFEM) ([Muhanna and Mullen 1999](#)) and interval finite element method (IFEM) ([Mullen and Muhanna 1999](#), [Muhanna and Mullen 2001](#), [Muhanna et al. 2005](#), [Pownuk 2008](#), [Skalna and Pownuk 2008](#)). Minimum buckling load, first eigenvalue of the buckling problem, has been bounded considering material uncertainty using IFEM ([Modares et al. 2006](#)). Interval boundary element methods (IBEM) and fuzzy boundary element methods (FBEM) have been developed to obtain worst case enclosure of the solution considering discretization, truncation, and round-off errors, and uncertainty in boundary conditions, system geometry, and thermal conductivity ([Zalewski and Mullen 2008](#), [Zalewski and Mullen 2008](#), [Zalewski and Mullen 2009](#), [Zalewski et al. 2009](#), [Zalewski and Mullen 2009](#), [Zalewski et al. 2009](#), [Zalewski 2010](#), [Zalewski and Mullen 2010](#)). IBEM was also used to treat uncertainty in system parameters for transient diffusion problems ([Piasecka 2007](#)).

There are two manners in which uncertainty can be characterized: aleatory and epistemic. The aleatory uncertainty, which comes from *alea* or rolling of dice, is a random type of uncertainty that is truly stochastic in nature. Epistemic uncertainty characterizes an uncertainty which occurs due to the lack of knowledge, such as uncertainty in shear modulus of the material. Such uncertainty can be decreased or in some cases eliminated due to increased resources and/or better understanding of the behavior. There are many methods that are used to quantify each type of uncertainty. These include probabilistic methods ([Papoulis 1965](#)), evidence theory ([Dempster 1967](#), [Shafer 1976](#), [Mourelatos and Zhou 2006](#)), probability boxes ([Ferson et al. 2003](#)), interval numbers ([Moore 1966](#), [Neumaier 1990](#)), and clouds ([Neumaier 2004](#)). The major difficulty in characterizing uncertainty with probabilistic methods is that random variables are characterized by a cumulative distribution function (CDF), which in practice may not be known or even obtainable. This leads to a necessity to assume a CDF, which itself is than uncertain. Evidence theory and

probability boxes, or p-boxes, attempt to alleviate this problem as they provide worst case bounds on CDF. However, validation of such bounds can be an extensive process as the assessment of uncertainty on the CDF is difficult. Interval methods provide the worst case bounds on the solution; however, they don't allow the analyst to consider the degree of uncertainty associated with the interval quantity, such as shear modulus. This degree of uncertainty can be based upon the accuracy of the measurement tool and is a characteristic of the uncertainty. Cloud numbers consider a certain probability of occurrence of a variable in a fuzzy set. They require the knowledge of the CDF, which again may be unknown. Fuzzy approaches attempt to alleviate the above shortcomings as they consider the worst case uncertainty, and therefore provide a guaranteed worst case solution, but consider the degree of uncertainty based on the degree of belief of the input data, such as laboratory experiments and expert opinions. Fuzzy approach, like interval methods, is computationally inexpensive and directly provides a reliable worst case solution.

This paper addresses the treatment of uncertainty in shear modulus for linear elastic plane strain problems using fuzzy numbers. A fuzzy number is a convex set whose bounds vary dependent on the level of confidence in the correctness of these bounds. Bounds of the fuzzy number can be obtained from different sources such as measurement tolerance and expert opinions. It should be noted that the level of uncertainty of these sources varies with the level of confidence in the sources themselves. For example, the quality of the strain gauge and the quality of its adhesion will most likely affect the certainty of its measurement. In this paper, it is assumed that all experts are knowledgeable in the field in question and they behave rationally. It is also assumed that the more resources are used, the more expensive the measurement tools, and the better the quality control is, the better the quality of the uncertainty. Therefore, all sources give a smaller value of a lower bound if their certainty is low and a larger value of a lower bound if their certainty is high. Also all sources give a larger value of an upper bound if their certainty is low and a smaller value of an upper bound if their certainty is high. Hence, the difference between the upper and lower bounds decreases monotonically with increased certainty of the sources. The enclosure of the true worst case shear modulus uncertainty bounds using fuzzy numbers is assumed and its validation is outside the scope of the paper. Hence, the paper obtains the worst case response of the system provided that the worst case uncertainty of the shear modulus is provided as an input.

The following presents a formulation of fuzzy boundary element method to treat uncertainty in linear elastic plane strain problems with homogeneous materials. The uncertainty in shear modulus is characterized using fuzzy approach, which treats the uncertainty as a bounded quantity varying dependent on the degree of belief of the input bounds. A

fuzzy matrix parameterization is developed, which allows computation of the exact solution, per given mesh, and does not result in any overestimation commonly present in interval type approaches. Numerical examples illustrate the behavior of the method and show that the bounds are exact and their quality is independent of the mesh size.

FUZZY BOUNDARY ELEMENT METHOD (FBEM)

BOUNDARY ELEMENT FORMULATION FOR THE ELASTICITY PROBLEM

Boundary Element Formulation of the Navier's Equation

A detailed boundary element formulation for the behavior of isotropic and homogeneous bodies can be found in (Brebbia and Dominguez 1992). The following section reviews the plane strain boundary element formulation for the elasticity problem. The elasticity problem can be described using Navier's equation as:

$$\left. \begin{aligned} \sigma_{ij,j} + b_i &= 0 \text{ in } \Omega \\ u_i &= \hat{u}_i \text{ on } \Gamma_1, t_i = \hat{t}_i \text{ on } \Gamma_2 \\ \bigcup_{i=1}^2 \Gamma_i &= \Gamma \text{ and } \bigcap_{i=1}^2 \Gamma_i = 0 \end{aligned} \right\} \quad (1)$$

where Ω is the domain of the system, Γ is the boundary of the system, σ_{ij} is the stress tensor, b_i is the body force vector, u_i is the displacement vector with a forced boundary condition $\hat{u}_i$ on Γ_1 , and t_i is the traction vector with a natural boundary condition $\hat{t}_i$ on Γ_2 . Since the exact solution to Eq. (1) cannot be generally obtained, numerical methods are used to obtain approximate solutions. For Navier's equation the starting point of the boundary element formulation is Betti's reciprocal theorem:

$$\int_{\Gamma} b_i^* u_i d\Gamma + \int_{\Gamma} t_i^* u_i d\Gamma = \int_{\Gamma} u_i^* b_i d\Gamma + \int_{\Gamma} u_i^* t_i d\Gamma \quad (2)$$

Equilibrium equation $\sigma_{ij,j}^* = -b_i^*$ is substituted into Eq. (2) yielding:

$$-\int_{\Gamma} \sigma_{ij,j}^* u_i d\Gamma + \int_{\Gamma} t_i^* u_i d\Gamma = \int_{\Gamma} u_i^* b_i d\Gamma + \int_{\Gamma} u_i^* t_i d\Gamma \quad (3)$$

In order to decrease the dimension of the integral equation, Eq. (3), the weighted residual function is set to be the Green's function, which is obtained by applying a point load in direction a_i . This can be expressed as:

$$\sigma_{ij,j}^* = -\delta(x - \xi) a_i \quad (4)$$

where ξ is a source point at which a concentrated force in direction a_i is applied and x is a field point at which the response of this concentrated force is measured. The resulting fundamental solution is:

$$u_i^* = u_{ji}^* a_j, t_i^* = t_{ji}^* a_j \quad (5)$$

where u_{ji}^* and t_{ji}^* are i components of the displacements and tractions, respectively, due to a concentrated force in the j direction, and a_j is a unit vector in the direction of the applied concentrated force. The kernel functions u_{ji}^* and t_{ji}^* , or u_{ij}^* and t_{ij}^* are given as:

$$u_{ij}^* = \frac{1}{8\pi(1-\nu)G} \left[(4\nu-3)\ln(r)\delta_{ij} + \frac{(\bar{x}-\bar{\xi}) \cdot \bar{i}}{r} \cdot \frac{(\bar{x}-\bar{\xi}) \cdot \bar{j}}{r} \right],$$

$$t_{ij}^* = \frac{-1}{4\pi(1-\nu)r} \left\{ \left[(1-2\nu)\delta_{ij} + 2 \frac{(\bar{x}-\bar{\xi}) \cdot \bar{i}}{r} \cdot \frac{(\bar{x}-\bar{\xi}) \cdot \bar{j}}{r} \right] \cdot \frac{(\bar{x}-\bar{\xi}) \cdot \bar{n}}{r} - (1-2\nu) \left[\frac{(\bar{x}-\bar{\xi}) \cdot \bar{i}}{r} n_y - \frac{(\bar{x}-\bar{\xi}) \cdot \bar{j}}{r} n_x \right] \right\} \quad (6)$$

where G is the shear modulus and ν is Poisson's ratio. Substituting Eq. (4-5) into Eq. (3) and interchanging indices in all integral terms yields:

$$u_i(\xi) a_i + \int_{\Gamma} t_{ij}^* a_i u_j d\Gamma = \int_{\Gamma} u_{ij}^* a_i b_j d\Gamma + \int_{\Gamma} u_{ij}^* a_i t_j d\Gamma, \xi \in \Omega \quad (7)$$

Constant a_i coefficients are cancelled out and for convenience the body force in Eq. (7) is assumed to be zero. This results in:

$$u_i(\xi) + \int_{\Gamma} t_{ij}^* u_j d\Gamma = \int_{\Gamma} u_{ij}^* t_j d\Gamma, \xi \in \Omega \quad (8)$$

Eq. (8) is integrated such that the source point, ξ , is enclosed by a half-circular boundary of radius ϵ as $\epsilon \rightarrow 0$. This results