

Generalized Probability Distributions for Accelerated Life Modeling

2011-01-0796

Published
04/12/2011D. Gary Harlow
Lehigh University

Copyright © 2011 SAE International

doi:[10.4271/2011-01-0796](https://doi.org/10.4271/2011-01-0796)

ABSTRACT

Time dependent mechanical breakdown of materials has been a long-standing concern for structural components, and it is especially exacerbated by randomness in material properties, loading conditions, and other sources of uncertainty. A generalized phenomenological cumulative distribution function for lifetime, which is consistent with fundamentals of reliability theory, is presented. The form of the cdf includes a kernel that incorporates functions for time dependent loading, mechanical breakdown, and statistical behavior. The formalism is applicable for a variety of loading conditions including creep rupture, rate dependent tensile loading, and fatigue. Analysis of failure under stress histories such as variable stress amplitude or step-stress loadings is manageable as well. The model is a generalization of the classical accelerated life model in which covariates are included to account for physical attributes that directly influence component lifetime. Specific examples will be provided for fatigue failure. The procedure is well suited to explicitly relate statistical characteristics to classical stress-life analyses.

INTRODUCTION

Materials degradation is a principal cause for reduction in reliability and safety of engineered components and structures. The demand for making physically based probabilistic and statistical analyses in life predictions and prognosis for critical components and structures is increasing. Due to the complexity of the micromechanics of materials, the design service objectives, and loading conditions, these analyses require more advanced modeling than what has been commonly used. Typically, experientially and empirically based methodologies have been employed. The problem is exacerbated when accelerated life testing and modeling are

used to assess and predict degradation of components and structures under typical conditions. Often methodologies make use of parametric representations of statistical fits to experimental data that are subsequently cast into a probabilistic framework. Also, a specific cumulative distribution function (cdf) is frequently assumed to characterize observed data. The choice is usually based on mathematical and computational expediency. Whereas, cdfs that are consistent with fundamental mechanistic properties are more challenging to use. A generalized phenomenological cdf for lifetime, which is consistent with fundamentals of reliability theory, is presented. The form of the cdf includes a kernel that incorporates functions for a variety of loading conditions and mechanistically based material breakdown. The statistical behavior can be incorporated by a range of functional forms. The formalism is specifically applicable for creep rupture, rate dependent tensile loading, step-stress load programs, and fatigue. The model can also be viewed as a generalization of the classical accelerated life model in reliability in which covariates are included to account for physical attributes that directly influence component lifetime. Specific examples will be provided for fatigue failure. The procedure is well suited to explicitly relate statistical characteristics to classical stress-life analyses.

CDF FORMULATION

Let T be a continuous, non-negative random variable (rv) for lifetime. There are several ways in which its cdf $F(t)$ can be represented, and these can be found in any standard textbook on reliability; see [1], for example. One of the more popular representations uses the hazard rate $h(t)$ which is given by

$$h(t) = \frac{f(t)}{1 - F(t)},$$

where $f(t)$ is its probability density function (pdf) for T . Roughly, $h(t)$ is the rate at which damage accumulates. It is easy to show that

$$F(t) = 1 - \exp \left[- \int_0^t h(s) ds \right], t \geq 0. \quad (2)$$

The integral in [equation \(2\)](#) is the cumulative hazard function which totals the damage accumulated up to time t . Obviously, [equation \(2\)](#) is descriptive of a very broad class of cdfs.

The natural generalization of [equation \(2\)](#) lifetime analysis is given by

$$F(t|L) = 1 - \exp \left\{ -\Psi \left[\int_0^t \kappa(L(s)) ds \right] \right\}, t \geq 0, \quad (3)$$

where $L(t)$ is the time dependent load program for the component or structure, $\kappa(x)$ is the mechanistically based breakdown rule, and $\Psi(x)$ is the hazard function. The fundamentals of reliability are still apparent in [equation \(3\)](#). One of the first papers to use this cdf for mechanistic modeling was [2], and an excellent summary of its features for mechanistic modeling is contained in [3]. It should be noted that [equation \(3\)](#) is a very wide-ranging form for a cdf. It has been studied extensively in reliability theory for accelerated life testing [4].

LOAD PROGRAMS

Possibly because of its simplicity, the most frequently used special case of [equation \(3\)](#) is for a single constant load $L(t) = L_0$. This is most applicable to model creep or creep-rupture; however, it can be used for constant amplitude fatigue as well. In this case the cdf simplifies to the following:

$$F(t|L_0) = 1 - \exp \{ -\Psi[\kappa(L_0)t] \}, t \geq 0. \quad (4)$$

Assume that T is the rv for the lifetime with cdf given by [equation \(4\)](#), and define the rv $T^* = \kappa(L_0)T$. The cdf for T^* is

$$F_*(t) = \Pr\{T^* \leq t\} = 1 - \exp \{ -\Psi(t) \}, t \geq 0, \quad (5)$$

which does not depend on the load L_0 . Thus, the main feature of the cdf in [equation \(4\)](#) is the form of the hazard function $\Psi(x)$. Furthermore, if the mean and variance of T^* are μ and σ^2 , respectively, then the mean and variance for T are $\mu/\kappa(L_0)$ and $\sigma^2/\kappa(L_0)^2$, respectively. The coefficient of variation for both T and T^* is σ/μ , which is completely independent of the breakdown rule. Specific cases will be considered below.

Although [equation \(3\)](#) is sufficiently versatile to allow for time dependent load programs, the mathematical structure is much more complicated. For these specific applications, the key to the model is the breakdown rule. The simplest time dependent load program is $L(t) = Rt$, where R is a constant loading rate, and the model describes linearly increasing loading leading to the tensile strength. This loading will be used below as an example. There are many other types of time dependent loading that could be considered. Since there has been a significant amount of investigation using step-stress modeling, it will be investigated more thoroughly.

BREAKDOWN RULES

The breakdown rule $\kappa(x)$ is a continuous, non-negative function defined for $x \geq 0$. The breakdown rule is directly associated with the stress-life behavior. By far, the most common breakdown rules are the power law, given by

$$\kappa_p(x) = \beta x^\rho, x \geq 0, \quad (6)$$

where $\beta > 0$ and $\rho > 0$ are constants, and the exponential, given by

$$\kappa_e(x) = \phi \exp(\eta x), x \geq 0, \quad (7)$$

where $\phi > 0$ and $\eta > 0$ are constants. Other breakdown rules that incorporate temperature and humidity, for example, are listed in [5]. In [2] even more advanced breakdown rules are suggested. Consequently, it is essential to select a breakdown rule for which there is some physically motivated reason for its selection. Otherwise, the selection of the breakdown rule becomes nothing more than an empirical data reduction scheme. The examples considered herein will focus exclusively on the power law and exponential breakdown rules.

HAZARD FUNCTION

The final function in [equation \(3\)](#) to be described is the hazard function $\Psi(x)$ which primarily characterizes the variability in the model. Certainly, numerous functions are possible; however, the Weibull hazard function is frequently used, and it is given by

$$\Psi_w(x) = x^\alpha, x \geq 0, \quad (8)$$

where $\alpha > 0$ is a constant. For the remainder of this paper, the Weibull hazard function will be assumed.

POWER LAW BREAKDOWN

In order to demonstrate the contributions of the three functions in the generalized cdf, consider the Weibull hazard

function, [equation \(8\)](#), power law breakdown rule, [equation \(6\)](#), and constant load. [Equation \(3\)](#) with these functions becomes

$$F(t|L_o) = 1 - \exp \left\{ - \left[\int_0^t \beta L_o^\rho ds \right]^\alpha \right\} = 1 - \exp [- (\beta L_o^\rho t)^\alpha]. \quad (9)$$

Thus, the cdf in [equation \(9\)](#) is a Weibull cdf with a shape parameter of α which is the same parameter in the Weibull hazard function and with a load dependent scale parameter of βL_o^ρ . For a fixed load L_o , the transformation to linearize [equation \(9\)](#) for a probability plot is

$$\ln \{ -\ln [1 - F(t|L_o)] \} = \alpha \ln(t) + \alpha \ln(\beta L_o^\rho), \quad (10)$$

where the abscissa is $\ln(t)$, the ordinate is $\ln \{ -\ln [1 - F(t|L_o)] \}$, the shape parameter α is the slope, and the intercept is $\alpha \ln(\beta L_o^\rho)$; see [6]. Since only the intercept depends on L_o , the linearized probability curves are parallel with slope α . On the other hand, for a fixed percentile $p = F(t|L_o)$, [equation \(10\)](#) can be rearranged to represent a stress-life graph, *i.e.*,

$$\ln(L_o) = -\frac{\ln(t)}{\rho} + \frac{\ln \{ -\ln [1 - p] \} - \alpha \ln(\beta)}{\alpha \rho}. \quad (11)$$

Thus, for fixed percentiles, a logarithmic versus logarithmic plot is linear with slope $-1/\rho$.

The popularity of this approach stems from the ease of graphical analysis coupled with linear regression. For each fixed L_o , if lifetime data are linear on Weibull probability paper, then the Weibull hazard function is applicable. Furthermore, the slope of the linear approximations, assumed to be parallel, is an estimate for the Weibull shape parameter α . Similarly, if the percentile curves for stress-life are linear and parallel on logarithmic versus logarithmic graph paper, then the power law breakdown rule is appropriate. The constant slope yields an estimate for ρ . Conversely, if percentile curves for stress-life data can be assumed to be linear on logarithmic versus logarithmic graph paper, then the breakdown rule is necessarily inferred. The other parameter β in the model can be estimated from any applied load and its corresponding percentile. Although this is primarily an empirical methodology, and consequently constrained by the available experimental data, it is well grounded in reliability theory. The results from this model are appropriate for making interpolations within the range of the supporting data,

but its usefulness for extrapolations or predictions well beyond the data is limited.

It should be noted that for linearly increasing loading the cdf is more complicated. Let R be the constant loading rate, then

$$F(t|L_o) = 1 - \exp \left\{ - \left[\int_0^t \beta (Rs)^\rho ds \right]^\alpha \right\} = 1 - \exp \left[- \frac{\beta^\alpha R^{\alpha \rho} t^{\alpha(\rho+1)}}{(\rho+1)^\alpha} \right]. \quad (12)$$

The function is still a Weibull cdf, but the shape parameter is $\alpha(\rho+1)$ which combines the exponents from the Weibull hazard function and the power law breakdown rule. The scale parameter is chiefly dependent on $R^{\alpha \rho}$. For fixed percentiles, a plot of loading rate R versus lifetime t on logarithmic versus logarithmic graph paper is linear with a slope of $-(\rho+1)/\rho$. For appropriate data, once a graphical estimate for ρ from the loading rate versus life plots is obtained, then α can be estimated from the Weibull probability graph. The final parameter β can be estimated from any applied loading rate and its corresponding percentile.

Obviously, if the loading were much more intricate, then the analysis would become analytically intractable. A numerical solution would be rather involved as well. For complex loading histories it may be better to resort to standard simulation methods to analyze the model.

EXPONENTIAL BREAKDOWN

The use of the exponential breakdown rule, rather than the power law, with constant load poses little difference. Thus,

$$F(t|L_o) = 1 - \exp \left\{ - \left[\int_0^t \phi \exp(\eta L_o) ds \right]^\alpha \right\} = 1 - \exp \{ - [\phi \exp(\eta L_o) t]^\alpha \}. \quad (13)$$

Again, [equation \(13\)](#) is a Weibull cdf with shape parameter α and scale parameter $\phi \exp(\eta L_o)$ which is load dependent. Using the linearization scheme above, it is easy to show that for fixed percentiles, a linear versus logarithmic plot is linear with slope $-1/\eta$.

When linearly increasing loading is considered, the cdf is more problematic. Thus,

$$F(t|L_o) = 1 - \exp \left\{ - \left[\int_0^t \phi \exp(\eta R s) ds \right]^\alpha \right\} = 1 - \exp \left\{ - \left[\frac{\phi (\exp(\eta R t) - 1)}{R \eta} \right]^\alpha \right\}. \quad (14)$$

Typically, t is sufficiently large that $\exp(\eta R t)$ is much larger than one, so that it is assumed that the cdf is essentially

$$F(t|L_o) = 1 - \exp \left\{ - \frac{\phi^\alpha \exp(\alpha \eta R t)}{R \eta} \right\}. \quad (15)$$

The simplification in [equation \(15\)](#) allows for graphical analysis similar to the above cases. There are cases, however, when the loading rate is sufficiently small that $\exp(\eta R t)$ is not much greater than one [7]. In these instances the simplification is not suitable. Consequently, care must be exercised for this breakdown rule.

Again, complex loading schemes make the analysis virtually impossible to solve analytically. Numerical solutions are equally challenging. Under these conditions, alternative methods of solution are to be investigated.

FATIGUE LIFE PREDICTION: CONSTANT AMPLITUDE LOADING

An extensive stress-life data base was developed in [8]. About 1000 specimens, for 2024-T4 aluminum alloy were tested. The main concern was the effect of notch configurations on fatigue life, and they tested four different conditions. The specific case considered herein is for a series of fatigue tests for rectangular specimens with dimensions of 110 mm long, 52 mm wide, and 1 mm thick, and with a center cut notch of 10 mm in length with a theoretical elastic stress concentration factor of 3.8. The notches were made by drilling in jigs, and burrs were removed by light stoning. The fatigue testing was conducted in a typical laboratory environment, *i.e.*, temperatures of 295-297 K and relative humidities of 50-56%. All of the constant amplitude tests were performed at a frequency of 30 Hz on a single machine with a single operator. Thus, experimental error was minimized.

A statistical summary of the fatigue life data from [8] and considered in this paper is given in [Table I](#). Testing 252 specimens for one condition is an extensive undertaking, and it should be noted that the sample sizes for each $\Delta\sigma$ are quite large. For stress amplitudes $\Delta\sigma$ of 147 MPa or greater the sample coefficients of variation are essentially the same, and they are reasonably small. For 118 MPa, however, the scatter

slightly increases. Indeed, this is the dominant challenge for life prediction. Fatigue lives become very large and their scatter increases as the applied stress is lowered to reflect normal operating conditions. Consequently, fatigue life predictions cannot be based solely on simple empirical stress-life data analyses.

The scatter in the data, for each given $\Delta\sigma$, is shown on a two-parameter Weibull probability graph in [Fig. 1](#). The lines are the maximum likelihood estimates (MLEs) for the Weibull cdfs. This paper was chosen to assess the feasibility of using a Weibull hazard function to describe the data. Clearly, the graph demonstrates that the data for each $\Delta\sigma$ is well characterized by a two-parameter Weibull cdf. This conclusion can also be established by the Kolmogorov-Smirnov (K-S) and Anderson-Darling (A-D) goodness-of-fit tests. The lines are not exactly parallel. This was anticipated because the coefficients of variation given in [Table I](#) are not the same. In fact, the slopes gradually diminish as $\Delta\sigma$ decreases which imply that the scatter is increasing. Nevertheless, the lines are not too far from parallel. It may be reasonably acceptable to use the Weibull hazard function for this data, at least as an engineering approximation.

The characteristic stress-life curve for fatigue data is usually determined from data medians. [Figure 2](#) shows the data from [Fig. 1](#) plotted in the traditional logarithm versus logarithm format. As the stress amplitude is reduced, the increased scatter is apparent. The diamonds represent the medians at each stress, and the line is the least squares fit through the medians. The fit is quite good with a correlation coefficient of 0.996. It appears that the power law breakdown rule would be acceptable. The only disclaimer is that other percentile curves would not be parallel to the line through the medians. The exponential breakdown rule is inappropriate for these data. Overlooking the discrepancies between the fatigue data and assumptions for the model of [equation \(9\)](#), the parameters for the model are estimated as $\hat{\alpha} = 8.3$; $\hat{\rho} = 4.8$; $\hat{\beta} = 3.0 \times 10^{-16} \text{ cyc}^{-1} \text{ MPa}^{-\rho}$.

This example reveals several important observations. In virtually every fatigue testing program, statistical scatter increases as the applied load is decreased. The scatter increases dramatically as the applied load approaches the actual operating conditions. Thus, accurate modeling must be able to allow for stress dependent variability. For the models considered thus far that is equivalent to having a hazard function that is stress dependent as well as the breakdown rule, which is the case with the above model. Furthermore, the same form of the cdf may not be appropriate for all applied loads. Likewise, the breakdown rule may require more complexity because the percentile curves typically are not linear or parallel. A more complicated set of fatigue data that necessitates advanced modeling is described in [9].