

Application of Boundary Characteristic Orthogonal Polynomials on Vibration of Circular Plates with Circular Eccentric Holes

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ABSTRACT

A new approach is proposed to solve for the eigen-values and eigen-functions of circular plates with circular holes by using the Rayleigh Ritz Method. In this method, the spatial solution is expanded into separable functions in terms of polar coordinates. While trigonometric functions are used along the circumferential direction, the Boundary Characteristic Orthogonal Polynomials build the radial shape functions. Written in terms of the assumed functions, the potential and kinetic energies are modified in order to account for the holes. Although the proposed approach is applicable for plates with different boundary conditions and different hole shapes, the free vibration of a clamped circular plate with circular holes is considered in the present study. The edges of the holes are free. Four different case studies are carried out. The results of the Rayleigh Ritz Method are compared with those available in the literature.

INTRODUCTION

There are numerous engineering structures that include plates with or without perforations. Many studies have been carried out on the vibration of circular plates, annular plates, and plates of different shapes with different boundary conditions [1]. The study of circular plates with perforations at the center is the choice of most of the studies since these are used

in aerospace structures, ship structures, and many other industrial applications [1,2].

Compared with the studies on annular plates, there are very few studies on plates with multiple eccentric perforations. The Finite Element Method [2] or the Bessel's functions satisfying the boundary conditions [3,4] are used to study the free vibration response of plates with eccentric holes. Finite element analysis was carried out on a circular plate with eccentric hole considering the effects of the eccentricity, the hole size, and the boundary conditions on the mode shapes of [5]. The Rayleigh Ritz Method, where polynomials are employed as radial admissible functions, are applied in order to obtain the frequencies of circular plates with circular [6,7] and rectangular [8] perforations. There are no reported analytical solutions on the free vibration response of a plate with multiple perforations. The Boundary Integral Equation Method is proposed as a semi-analytical solution to the free vibration of a circular plate with circular perforations [9,10].

The simplicity of satisfying the geometric boundary conditions, especially when the polynomials are used as admissible functions in the Rayleigh Ritz Method, has attracted the attention of many researchers [11,12]. In the present study, circular plates with circular holes are studied by using the Boundary Characteristic Orthogonal Polynomials, which were originally proposed by Bhat [13] and were also used by Rajalingam et al. [14,15] and

Chakraverthy et al. [16]. The polynomials are employed as the assumed deflection functions in the Rayleigh Ritz Method. Two well-known cases of a full circular plate and an annular plate are first studied, where the analytical solutions are available, for comparison. Circular plates with one and two eccentric perforations are considered and the natural frequencies are compared with those obtained by BIEM method [9]. Moreover, the mode shapes of the last two cases are presented.

METHOD OF SOLUTION

The free vibration response of a circular plate $w(r, \theta, t)$ can be expressed in the following form:

$$w(r, \theta, t) = W(r, \theta)e^{j\omega t} \quad (1)$$

where the deflection shape $W(r, \theta)$ is defined in the polar coordinates. The maximum kinetic energy and the maximum strain energy of the deformed circular plate can be written as:

$$T_{\max} = \frac{1}{2} \rho h \omega^2 \int_A W^2 dA \quad (2)$$

$$U_{\max} = \frac{D}{2} \int_A \left(\left(\frac{\partial^2 W}{\partial r^2} + \frac{1}{r} \frac{\partial W}{\partial r} + \frac{1}{r^2} \frac{\partial^2 W}{\partial \theta^2} \right)^2 - 2(1-\nu) \frac{\partial^2 W}{\partial r^2} \left(\frac{1}{r} \frac{\partial W}{\partial r} + \frac{1}{r^2} \frac{\partial^2 W}{\partial \theta^2} \right) + 2(1-\nu) \left(\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial W}{\partial \theta} \right) \right)^2 \right) dA \quad (3)$$

In the Eq. (3) the flexural rigidity is defined as:

$$D = \frac{Eh^3}{12(1-\nu^2)}$$

The deflection is assumed in the form:

$$W(r, \theta) = \sum_{m=1}^M \phi_m(r) \sum_{n=0}^N (A_{mn} \cos n\theta + B_{mn} \sin n\theta) \quad (4)$$

Substituting Eq. (4) in Eqs. (2) and (3), the Rayleigh quotient is defined as:

$$\omega^2 = \frac{U_{\max}}{T_{\max}^*} \quad (5)$$

where $T_{\max} = \omega^2 T_{\max}^*$. Applying the condition of stationarity of the eigen-values with respect to the arbitrary constants A_{ij} and B_{ij} .

$$\frac{\partial \omega^2}{\partial A_{ij}} = 0 \quad \text{and} \quad \frac{\partial \omega^2}{\partial B_{ij}} = 0 \quad (6)$$

results in an eigen-value problem in the form:

$$[\mathbf{K}] \begin{Bmatrix} \{A_{ij}\} \\ \{B_{ij}\} \end{Bmatrix} - \omega^2 [\mathbf{M}] \begin{Bmatrix} \{A_{ij}\} \\ \{B_{ij}\} \end{Bmatrix} = 0 \quad (7)$$

With R as the outer radius of the plate, the following dimensionless values are defined:

$$\lambda^4 = \Omega^2 = \frac{\rho h R^4 \omega^2}{D} \quad (8)$$

GENERATION OF BOUNDARY CHARACTERISTIC ORTHOGONAL POLYNOMIALS

The first member of the Boundary Characteristic Orthogonal Polynomial set is constructed to satisfy the geometrical boundary conditions of the plate. For plate with an outer-edge clamped the boundary conditions are $\phi(R) = \phi'(R) = 0$, and the first member can be constructed as

$$\phi_1(r) = d_1(r^2 - R^2)^2 \quad (9)$$

such that the displacement and the slope vanish at the outer edge of the plate.

The Gram-Schmidt process has been used to generate the higher order members of the Boundary Characteristic Orthogonal Polynomial set as:

$$\phi_2(r) = d_2(r - b_1)\phi_1(r) \quad (10)$$

$$\phi_m(r) = d_m[(r - b_{m-1})\phi_{m-1}(r) - c_{m-1}\phi_{m-2}(r)] \quad \forall m \geq 3 \quad (11)$$

where,

$$b_{m-1} = \int_{R_i}^R r^2 \phi_{m-1}^2(r) dr \quad (12)$$

$$c_{m-1} = \int_{R_i}^R r^2 \phi_{m-1}(r) \phi_{m-2}(r) dr \quad (13)$$

The set of Boundary Characteristic Orthogonal Polynomials is normalized such that in Eqs. (9), (10), (11) the constants d_m can be identified by using the following equation:

$$\int_{R_i}^R \phi_m^2(r) r dr = 1 \quad (14)$$

resulting in ortho-normal functions. It should be mentioned that R_i refers to the inner radius of the plate, and, if there is no hole at the center, it is equal to zero.

NUMERICAL RESULTS

Integrations in the energies equations have been performed for $0 \leq \theta \leq 2\pi$ and $0 \leq r \leq R$ for the full circular plate. The domain of integration for the annular plate is $0 \leq \theta \leq 2\pi$ and $R_i \leq r \leq R$. The deflection shape function can be expressed as:

$$W(r, \theta) = \sum_{m=1} \phi_m(r) \sum_{n=0} (A_{mn} \cos n\theta) \quad (15)$$

for the case studies of full circular plate and annular plate. Moreover, the following identity is used in all the integrations:

$$\int_0^{2\pi} \cos n\theta \cos m\theta d\theta = 0 \quad m \neq n \quad (16)$$

When there is an eccentric hole on the plate, as shown in Figure 1, initially the potential and kinetic energies are estimated for a plate without hole. Then, the energy corresponding to the hole are calculated and subtracted from the total energy. The kinetic and potential energies of an eccentric hole are calculated in the region $r_1(\theta) \leq r \leq r_2(\theta)$ and $\theta_0 - \alpha \leq \theta \leq \theta_0 + \alpha$ as illustrated in Figure 2. It should be noted that:

$$\sin \alpha = \frac{a}{e} \quad (17)$$

$$r_1(\theta) = e \cos(\theta - \theta_0) - \sqrt{a^2 - e^2 \sin^2(\theta - \theta_0)} \quad (18)$$

$$r_2(\theta) = e \cos(\theta - \theta_0) + \sqrt{a^2 - e^2 \sin^2(\theta - \theta_0)} \quad (19)$$

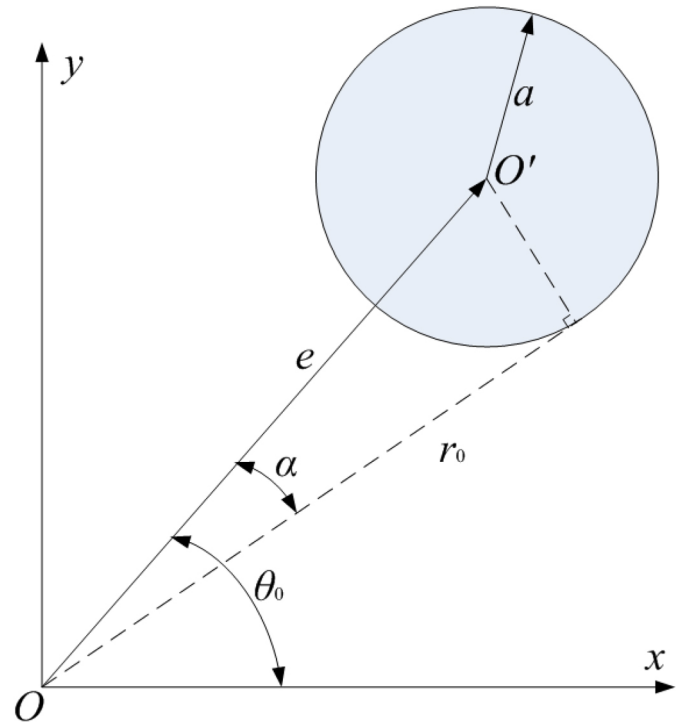


Figure 1. Geometry and coordinates of the hole

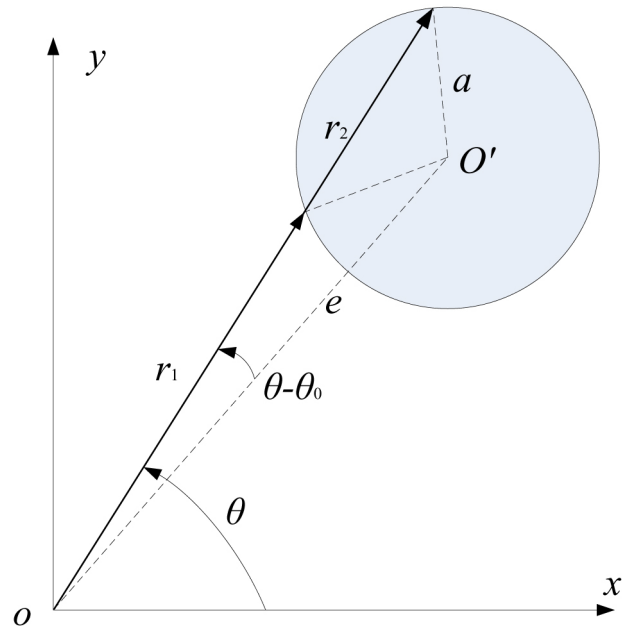


Figure 2. Integral boundaries of the hole

CASE STUDY 1: FULL CIRCULAR PLATE WITH CLAMPED OUTER EDGE

The dimensionless natural frequencies by using the Rayleigh Ritz Method are compared with the exact solution in Table 1. The Rayleigh Ritz Method is in good agreement with the exact solution.

Table 1. Dimensionless natural frequencies Ω of a full circular plate with a clamped boundary

number of radial nodes	$n=0$		$n=1$	
	Ref [1]	Present Method	Ref [1]	Present Method
0	10.216	10.216	21.26	21.260
1	39.771	39.771	60.82	60.829
2	89.104	89.104	120.08	120.079
3	158.183	158.185	199.06	199.093
4	247.005	247.274	297.77	298.108
	$n=2$		$n=3$	
	Ref [1]	Present Method	Ref [1]	Present Method
0	34.88	34.877	51.04	51.030
1	84.58	84.583	111.01	111.022
2	153.81	153.816	190.30	190.329
3	242.71	242.942	289.17	289.402
4	351.38	354.621	407.72	423.895
	$n=4$		$n=5$	
	Ref [1]	Present Method	Ref [1]	Present Method
0	69.666	69.665	90.739	90.739
1	140.108	140.107	171.803	171.809
2	229.519	229.606	271.428	271.496
3	338.411	340.349	390.390	397.882
4	–	487.921	–	550.003

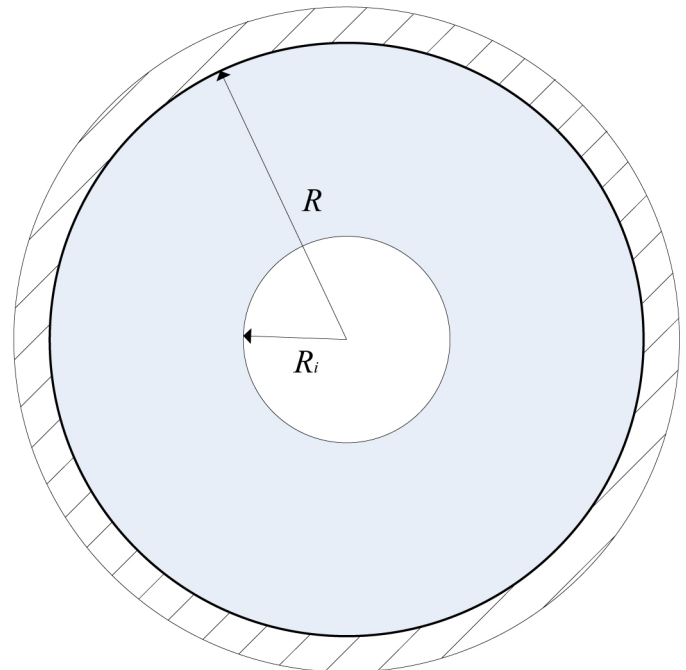


Figure 3. Annular plate with clamped free boundary condition

Table 2. Dimensionless natural frequencies Ω of an annular plate ($R_i/R=0.1$) clamped on the outside and free on the inside by using $M=10$ orthogonal polynomials

number of radial nodes	$n=0$		$n=1$	
	Ref. [1]	Present method	Ref. [1]	Present Method
0	10.2	10.137	21.1	21.190
1	39.5	39.393	60.0	60.006
2	90.4	90.192	–	116.862
3	–	163.952	–	192.118
4	–	261.666	–	288.295
	$n=2$		$n=3$	
	Ref. [1]	Present method	Ref. [1]	Present method
0	34.5	34.552	51.0	50.987
1	83.4	83.525	–	110.780
2	–	151.405	–	189.515
3	–	238.118	–	287.324
4	–	343.401	–	405.152

CASE 2: ANNULAR PLATE WITH OUTER EDGE CLAMPED AND INNER EDGE FREE

A plate with outer radius R and with a concentric hole of radius R_i has been considered as shown in Figure 3. Table 2 shows the natural frequencies of an annular plate with $R_i / R=0.1$. The results of the Rayleigh Ritz Method have been compared with those obtained by using the Bessel's function formulation in Ref. [1]. Ten orthogonal polynomials are used in the Rayleigh Ritz Method ($M=10$) in order to find the natural frequencies. Only the first five frequencies of each group have been presented in this research. Since 10 functions have been used, the first five natural frequencies of each group are in good agreement. Table 3 shows the dimensionless eigen-values of an annular plate with $R_i / R=0.25$.