

A Sensitivity Analysis Formulation for Steady-State Conjugate Heat Transfer Problems of Industrial Size

Christian Rauch

VIRTUAL VEHICLE Research and Test Center

ABSTRACT

In recent years, developments in virtual prototyping have gained significant interest. The promise of cost savings and the need for rapid simulations in order to be able to offer various platforms has led to automated coupling of software for various physical models. But relying heavily on simulation necessitates confidence in the results obtained. By incorporating various physical models the number of parameters to be set increases. The uncertainties associated with these input parameters are propagated to the simulation results. Uncertainty analysis aims at quantifying those influences. In this paper local first order sensitivity analysis for conjugate heat transfer problems including cavity radiation is introduced. The issue of memory requirement associated with this approach for industrial sized cases is discussed. A reformulation of sensitivity coefficients derived by the author previously for view factor, emissivity, heat transfer coefficient, and conductance is presented using the principle of reciprocity. The resulting uncertainty coefficients are compared to the standard approach for those parameters. A simple numerical stability analysis is conducted and the danger of neglecting interdependencies of the heat transfer modes is shown along with a study on the range of applicability of those factors.

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INTRODUCTION

As the desired development time for automobile innovations becomes increasingly smaller, engineers rely more and more on simulations, as wind tunnel tests are expensive and often limited by capacity. The introduction of more platforms and alternative propulsion systems, such as hybrid and electric systems, have put increased pressure on CFD and thermal engineers. Especially with the advent of new systems expert knowledge and experience is often missing. In order to better understand the quality of conclusions from a simulation, uncertainty analysis has evolved. Roy and Oberkampf [1] give an overview of sources of uncertainties that influence the outcome of a simulation. They identified uncertainties that stem from model inputs, such as model parameters, geometry, boundary and initial conditions, and external influences such as forces. They do occur in the manufacturing process, in natural resources, and might change during the life-cycle. Another source of uncertainties is of a numerical nature. To this category belong discretizations in space and time, computational round-off errors, iterative convergence errors, or programming errors. Finally, the model itself might be based on uncertain footings

such as assumptions on model simplifications. This paper actually will deal with one such model assumption.

In this paper first-order local sensitivity analysis is being performed on a conjugate heat transfer case. This method derives the first-order derivatives of the model equations and calculates those derivatives with the results of a simulation. According to Saltelli and Annoni [2], this method behaves similarly to one-at-a-time (OAT) screening in a high-dimensional parameter space. They stated that this method is more convenient to the engineer as it deals with results of the actual simulation without having to perform many simulations with parameters that are not of immediate interest. Starting from the simulation results small perturbations are excited in order to discover the surroundings of the solutions for all parameters of interest. In this way, the outcome of a perturbation can be ascribed to a particular parameter. Contrary to stochastic methods, non-influential parameters cannot falsely be identified as important. Furthermore, a model failure can be ascribed to one parameter. Finally, it helps the engineer to understand the behavior of the system in an easy to follow way. The drawback of this method is its limited validity for extrapolation of non-linear systems.

Whereas local sensitivity and uncertainty analysis for conduction or convection has received some attention, to the author's knowledge, publications on thermal radiation are rather scarce. Blackwell et al. [3] investigated a control volume finite element system with conjugate heat transfer, but they included only a simplified radiation equation for the interaction with the environment without the use of view factors and reflections within the geometry. To the best knowledge of the author, the first paper on thermal radiation analysis by uncertainty investigation with reflections between surfaces taken into account was published by Taylor et al. [4]. There, they investigated the sensitivities of view factors and emissivity when temperatures are given as boundary conditions. In later works [5,6] they expanded their research for changes in area and boundary conditions and for the case when a heat flux is specified instead of a temperature. One of their conclusions was that raising the quality of view factors by enforcing reciprocity and closure greatly improved accuracy. In [7] the authors elaborated more on various methods of closure and reciprocity correction methods.

In structural mechanics there exist some formulations for conjugate heat transfer with cavity radiation. In [8] a Finite Element Method (FEM) formulation that includes participating media and does not treat view factors as black box entries is presented along with an adjoint method formulation. In [9,10] sensitivity factors for FEM are formulated with the radiation exchange formulation being coupled weakly between two meshes of different resolutions. These factors were employed to estimate changing form factors due to mechanical and thermal deformations [9]. In a follow-up paper [10] a way of investigating temperature dependent parameters were presented in a transient setting by reformulating the equations by placing those parameters on the main diagonal.

The current author derived local steady-state first order sensitivity equations for conjugate heat transfer within the finite difference method that incorporates cavity radiation [11,12,13]. The aim of the following chapter is to address a memory issue with those equations. In the theory section the principle of reciprocity for view factors is applied to the sensitivity equations. The special case for sensitivities for view factors is treated by a throughout derivation. Then, two test cases are introduced. In the analysis section the equations are subjected to numerical stability analysis. The effect of the heat transfer modes on uncertainty factors is demonstrated along with a comparison of the results with and without reciprocity. The range of applicability for uncertainty factors is demonstrated.

MEMORY ISSUE

For understanding the issue of memory one equation of the net heat flux method for cavity radiation is shown [14],

which serves as the starting point for the equations presented in this paper.

$$q_{o,k} - \frac{1}{A_k} \sum_{s=1}^N A_s q_{o,s} F_{sk} = b_k \quad (1)$$

Equation (1) states, that the thermal node k emits energy and receives energy from all other nodes N of the system. Here, q_o is the radiosity that includes emittance and reflection, F_{sk} the view factor from node s to k , A the area, and b_k the boundary heat flux. The summation term indicates a dense left-hand side (LHS) matrix. The view factor F runs from s to k and this value needs to be supplied in that order. There are several ways to deal with this issue.

The first option would be to load all of the view factors into memory. The advantage is the immediate access to all factors. Equation (1) has to be written for all k nodes and solved simultaneously for the unknown radiosity. This necessitates $8 \cdot N^2$ Bytes of memory when using 64bit double precision variables. As radiation has to deal with dense matrices, matrix compression techniques are not effective, especially in this paper when an LU solver is applied. Consequently, reading in all view factors for industrial cases is not viable.

The second option would be to transform the view factor file by transposition as indicated in Fig. 1.

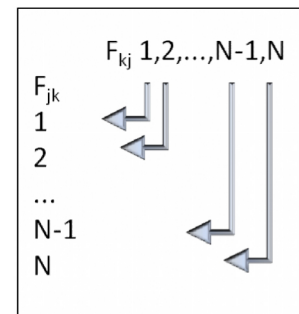


Figure 1. Transpose of a view factor file.

This is done most simply by reading all view factors and save them in a new file as a preprocessor step before the solution matrix is built. In this way factors that are saved for a node k to all N nodes as shown in the first line of Fig. 1 would be swapped into their respective columns.

The third option is filling the solution matrix for all N rows simultaneously. This is easily done for Eq. (1) and manageable with some extra storage in the equations presented in the theory section. But it is cumbersome for a follow-up equation of the next section.

The fourth possibility is making use of the reciprocity relationship:

$$A_i F_{ij} = A_j F_{ji} \quad (2)$$

It states, that a view factor can be transformed by taking into account the size of the nodes. This paper will apply this principle in reformulating the original equations. This has the advantage of reading view factors line by line. As an additional benefit the changes of uncertainty coefficients by changing the model assumption of reciprocity can be observed.

THEORY

The author has previously derived sensitivity equations for the parameters view factor [13], emissivity, fluid temperature, and conductance [11,12] without presuming that Eq. (2) holds. The reformulations will be discussed for each of those parameters using Eq. (2). To help understand the following treatment the geometry of two thermal nodes is shown in Fig. 2.

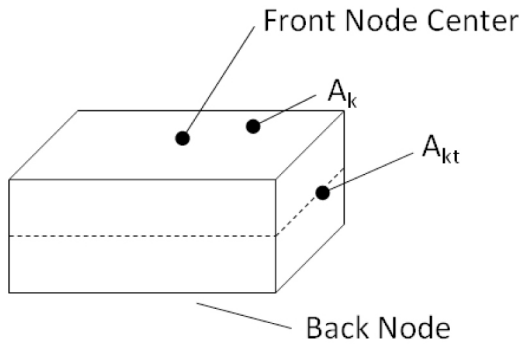


Figure 2. Thermal node.

A thermal node k consists of a center and a virtual extension. The node has an area A_k where it can exchange energy with the surrounding by convection and radiation. The dashed line in Fig. 2 marks the border to the thermal back node, if applicable. Through that border exchange from the surface node happens by conduction, only. The area A_{kt} marks the area to the neighboring node t , where heat is transferred from the surface node k to t by conduction. The distance between two node centers is named l_{kt} .

VIEW FACTOR

The change of radiosity with respect to the view factor between i and j has been published as follows [13]:

$$\begin{aligned} & \sum_{s=1}^N \left[\delta_{sk} (1 - F_{sk}) - (1 - \delta_{sk}) \frac{A_s}{A_k} F_{sk} \right] \frac{\partial q_{o,s}}{\partial F_{ij}} \\ & - \frac{1}{4\epsilon_k \sigma T_k^3} \left(-h_k - \frac{1}{A_k} \sum_{t=1}^{N_{\text{Cond}}} \frac{k_{kt}}{l_{kt}} A_{kt} \right) \\ & * \sum_{s=1}^N \left[\delta_{sk} - \delta_{sk} (1 - \epsilon_k) \frac{A_s}{A_k} F_{sk} - (1 - \delta_{sk}) (1 - \epsilon_k) \frac{A_s}{A_k} F_{sk} \right] \frac{\partial q_{o,s}}{\partial F_{ij}} \\ & - \frac{1}{A_k} \sum_{t=1}^{N_{\text{Cond}}} \frac{k_{kt}}{l_{kt}} A_{kt} \frac{1}{4\epsilon_t \sigma T_t^3} \\ & * \sum_{s=1}^N \left[\delta_{st} - \delta_{st} (1 - \epsilon_t) \frac{A_s}{A_t} F_{st} - (1 - \delta_{st}) (1 - \epsilon_t) \frac{A_s}{A_t} F_{st} \right] \frac{\partial q_{o,s}}{\partial F_{ij}} \\ & = \delta_{si} \delta_{kj} \frac{A_i}{A_j} q_{o,i} \\ & - \delta_{si} \delta_{kj} (1 - \epsilon_j) \frac{A_i}{A_j} q_{o,i} \left(-h_k - \frac{1}{A_k} \sum_{t=1}^{N_{\text{Cond}}} \frac{k_{kt}}{l_{kt}} A_{kt} \right) \frac{1}{4\epsilon_k \sigma T_k^3} \\ & - \delta_{si} \delta_{ij} \frac{1}{A_k} \frac{k_{kj}}{l_{kj}} A_{kj} \frac{1}{4\epsilon_j \sigma T_j^3} (1 - \epsilon_j) \frac{A_i}{A_j} q_{o,i} \end{aligned} \quad (3)$$

Here, δ is the Kronecker delta, ϵ the emissivity, k the thermal conductivity, h the convective heat transfer coefficient, T the node temperature, and σ the Stefan-Boltzmann constant. The first term on the right-hand side (RHS) and LHS treats the radiative mode, the second term treats both convection and conduction, and the final terms on both sides of Eq. (3) are attributed to conduction. As the change in radiosity usually is not as much of interest as the change in temperature, Eq. (4) needs to be used.

$$\begin{aligned} & \frac{\partial T_k}{\partial F_{ij}} = \frac{1}{4\epsilon_k \sigma T_k^3} \\ & * \left[\sum_{s=1}^N \left[\delta_{sk} - \delta_{sk} (1 - \epsilon_k) \frac{A_s}{A_k} F_{sk} - (1 - \delta_{sk}) (1 - \epsilon_k) \frac{A_s}{A_k} F_{sk} \right] \frac{\partial q_{o,s}}{\partial F_{ij}} \right. \\ & \left. - \delta_{si} \delta_{kj} (1 - \epsilon_j) \frac{A_i}{A_j} q_{o,i} \right] \end{aligned} \quad (4)$$

Equation (4) shows the clumsiness in using the third option in section 'memory issue', namely filling the whole solution matrix with the correct view factor. For once Eq. (3) is solved one cannot simply substitute the radiosity derivative into Eq. (4) without reading in all the view factors again.

Applying reciprocity principle of Eq. (2) to Eq. (3) and (4) yields:



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$$\begin{aligned}
& \sum_{s=1}^N [\delta_{sk}(1-F_{ks}) - (1-\delta_{sk})F_{ks}] \frac{\partial q_{o,s}}{\partial F_{ij}} \\
& - \frac{1}{4\varepsilon_k \sigma T_k^3} \left(-h_k - \frac{1}{A_k} \sum_{t=1}^{N_{\text{Cand}}} \frac{k_{kt}}{l_{kt}} A_{kt} \right) \\
& * \sum_{s=1}^N [\delta_{sk} - \delta_{sk}(1-\varepsilon_k)F_{ks} - (1-\delta_{sk})(1-\varepsilon_k)F_{ks}] \frac{\partial q_{o,s}}{\partial F_{ij}} \\
& - \frac{1}{A_k} \sum_{t=1}^{N_{\text{Cand}}} \frac{k_{kt}}{l_{kt}} A_{kt} \frac{1}{4\varepsilon_t \sigma T_t^3} \\
& * \sum_{s=1}^N [\delta_{st} - \delta_{st}(1-\varepsilon_t)F_{ts} - (1-\delta_{st})(1-\varepsilon_t)F_{ts}] \frac{\partial q_{o,s}}{\partial F_{ij}} \\
& = \delta_{si} \delta_{kj} \frac{A_i}{A_j} q_{o,i} \\
& - \delta_{si} \delta_{kj} (1-\varepsilon_j) \frac{A_i}{A_j} q_{o,i} \left(-h_k - \frac{1}{A_k} \sum_{t=1}^{N_{\text{Cand}}} \frac{k_{kt}}{l_{kt}} A_{kt} \right) \frac{1}{4\varepsilon_k \sigma T_k^3} \\
& - \delta_{si} \delta_{kj} \frac{1}{A_k} \frac{k_{kj}}{l_{kj}} A_{kj} \frac{1}{4\varepsilon_j \sigma T_j^3} (1-\varepsilon_j) \frac{A_i}{A_j} q_{o,i}
\end{aligned} \tag{5}$$

$$\begin{aligned}
& \frac{\partial T_k}{\partial F_{ij}} = \frac{1}{4\varepsilon_k \sigma T_k^3} \\
& * \left[\sum_{s=1}^N [\delta_{sk} - \delta_{sk}(1-\varepsilon_k)F_{ks} - (1-\delta_{sk})(1-\varepsilon_k)F_{ks}] \frac{\partial q_{o,s}}{\partial F_{ij}} \right. \\
& \left. - \delta_{si} \delta_{kj} (1-\varepsilon_j) \frac{A_i}{A_j} q_{o,i} \right]
\end{aligned} \tag{6}$$

Equations (5) and (6) shall be called ‘partial’, meaning that the model itself is not including the reciprocity relation but only in form of the view factors as parameters that are presumed to obey that principle. In order to incorporate reciprocity into the model itself the derivations with respect to view factors are done in [Appendix A](#), in order to make this paper self-contained, as all the sensitivity equations for the other parameters follow the same reasoning for their derivations. The resulting relation is:

$$\begin{aligned}
& \sum_{s=1}^N [\delta_{sk}(1-F_{ks}) - (1-\delta_{sk})F_{ks}] \frac{\partial q_{o,s}}{\partial F_{ij}} \\
& - \left(-h_k - \frac{1}{A_k} \sum_{t=1}^{N_{\text{Cand}}} \frac{k_{kt}}{l_{kt}} A_{kt} \right) \frac{1}{4\varepsilon_k \sigma T_k^3} \\
& * \sum_{s=1}^N [\delta_{sk} - \delta_{sk}(1-\varepsilon_k)F_{ks} - (1-\delta_{sk})(1-\varepsilon_k)F_{ks}] \frac{\partial q_{o,s}}{\partial F_{ij}} \\
& - \frac{1}{A_k} \sum_{t=1}^{N_{\text{Cand}}} \frac{k_{kt}}{l_{kt}} A_{kt} \frac{1}{4\varepsilon_t \sigma T_t^3} \\
& * \sum_{s=1}^N [\delta_{st} - \delta_{st}(1-\varepsilon_t)F_{ts} - (1-\delta_{st})(1-\varepsilon_t)F_{ts}] \frac{\partial q_{o,s}}{\partial F_{ij}} \\
& = \delta_{ki} \delta_{sj} q_{o,j} \\
& - \delta_{ki} \delta_{sj} (1-\varepsilon_i) q_{o,j} \left(-h_k - \frac{1}{A_k} \sum_{t=1}^{N_{\text{Cand}}} \frac{k_{kt}}{l_{kt}} A_{kt} \right) \frac{1}{4\varepsilon_k \sigma T_k^3} \\
& - \delta_{ki} \delta_{sj} \frac{1}{A_k} \frac{k_{ki}}{l_{ki}} A_{ki} \frac{1}{4\varepsilon_i \sigma T_i^3} (1-\varepsilon_i) q_{o,j}
\end{aligned} \tag{7}$$

When compared to [Eq. \(5\)](#) the RHS of [Eq. \(7\)](#) has changed indices as expected, thus incorporating the reciprocity principle fully.

EMISSIONITY

The change of radiosity with respect to emissivity has been published as follows [[11](#),[12](#)] observing that the solution parameters of the LHS of [Eq. \(3\)](#) without reciprocity and the solution parameters of the LHS of [Eq. \(5\)](#) with reciprocity is always the same for all the parameter derivations under investigation. Just now, the unknown is the radiosity derivation with respect to emissivity on the LHS:

$$\begin{aligned}
RHS &= \left(-h_k - \frac{1}{A_k} \sum_{t=1}^{N_{\text{Cand}}} \frac{k_{kt}}{l_{kt}} A_{kt} \right) \frac{1}{4\varepsilon_k \sigma T_k^3} \\
& * \left(\frac{\delta_{ki}}{A_i} \sum_{s=1}^N A_s q_{o,s} F_{si} - \delta_{ik} \sigma T_i^4 \right) \\
& + \frac{1}{A_k} \sum_{t=1}^{N_{\text{Cand}}} \frac{k_{kt}}{l_{kt}} A_{kt} \frac{1}{4\varepsilon_t \sigma T_t^3} \left(\frac{\delta_{ti}}{A_i} \sum_{s=1}^N A_s q_{o,s} F_{si} - \delta_{it} \sigma T_i^4 \right)
\end{aligned} \tag{8}$$

Since [Eq. \(A6\)](#) is not applicable, there is no distinction between partial and full reciprocity as is the case for view factors.

$$\frac{\partial T_k}{\partial \varepsilon_i} = \left[\sum_{s=1}^N \left[\delta_{ks} (1 - (1 - \varepsilon_k) F_{kk}) - (1 - \delta_{ks}) (1 - \varepsilon_k) \frac{A_s}{A_k} F_{sk} \right] \frac{\partial q_{o,s}}{\partial \varepsilon_i} + \right. \\ \left. + \frac{\delta_{ki}}{A_i} \sum_{s=1}^N A_s q_{o,s} F_{si} - \delta_{ik} \sigma T_i^4 \right] \frac{1}{4 \varepsilon_k \sigma T_k^3} \quad (9)$$

Simply substituting Eq. (2) into Eq. (8) and (9) yields:

$$RHS = \delta_{ki} \delta_{sj} q_{o,j} \\ - \delta_{ki} \delta_{sj} (1 - \varepsilon_i) q_{o,j} \left(-h_k - \frac{1}{A_k} \sum_{t=1}^{N_{Cond}} \frac{k_{kt}}{l_{kt}} A_{kt} \right) \frac{1}{4 \varepsilon_k \sigma T_k^3} \\ - \delta_{ii} \delta_{sj} \frac{1}{A_k} \frac{k_{ki}}{l_{ki}} A_{ki} \frac{1}{4 \varepsilon_i \sigma T_i^3} (1 - \varepsilon_i) q_{o,j} \quad (10)$$

$$\frac{\partial T_k}{\partial \varepsilon_i} = \left[\sum_{s=1}^N \left[\delta_{ks} (1 - (1 - \varepsilon_k) F_{kk}) - (1 - \delta_{ks}) (1 - \varepsilon_k) F_{ks} \right] \frac{\partial q_{o,s}}{\partial \varepsilon_i} + \right. \\ \left. + \frac{\delta_{ki}}{A_i} \sum_{s=1}^N A_s q_{o,s} F_{is} - \delta_{ik} \sigma T_i^4 \right] \frac{1}{4 \varepsilon_k \sigma T_k^3} \quad (11)$$

FLUID TEMPERATURE

The same procedure as for emissivity applies to the fluid temperature of a node [11,12]:

$$RHS = \delta_{ik} h_i \quad (12)$$

$$\frac{\partial T_k}{\partial T_{f,i}} = \frac{1}{4 \varepsilon_k \sigma T_k^3} \\ * \sum_{s=1}^N \left[\delta_{ks} (1 - (1 - \varepsilon_k) F_{sk}) - (1 - \delta_{ks}) (1 - \varepsilon_k) \frac{A_s}{A_k} F_{sk} \right] \frac{\partial q_{o,s}}{\partial T_{f,i}} \quad (13)$$

With reciprocity the LHS of Eq. (5) is applicable and the RHS is identical with Eq. (12). For the temperature gradient one gets:

$$\frac{\partial T_k}{\partial T_{f,i}} = \frac{1}{4 \varepsilon_k \sigma T_k^3} \\ * \left(\sum_{s=1}^N \left[\delta_{ks} (1 - (1 - \varepsilon_k) F_{ks}) - (1 - \delta_{ks}) (1 - \varepsilon_k) F_{ks} \right] \frac{\partial q_{o,s}}{\partial T_{f,i}} \right) \quad (14)$$

CONDUCTANCE

The conductance is a lumped parameter defined in Eq. (15) [11,12]:

$$C_{kt} = \frac{k_{kt}}{l_{kt}} A_{kt} \quad (15)$$

$$RHS = -\frac{1}{A_k} \sum_{t=1}^{N_{Cond}} \left[(T_k - T_t) (\delta_{ik} \delta_{jt} + \delta_{it} \delta_{jk}) \right] \quad (16)$$

With reciprocity Eq. (16) remains the same and again the LHS of Eq. (5) has to be used with the unknown gradients appropriately renamed. For the temperature gradient one has:

$$\frac{\partial T_k}{\partial C_{ij}} = \frac{1}{4 \varepsilon_k \sigma T_k^3} \\ * \sum_{s=1}^N \left[\delta_{sk} - \delta_{sk} (1 - \varepsilon_k) F_{ks} - (1 - \delta_{sk}) (1 - \varepsilon_k) F_{ks} \right] \frac{\partial q_{o,s}}{\partial C_{ij}} \quad (17)$$

UNCERTAINTY ANALYSIS

Once the sensitivity coefficients are calculated they can be summed-up in the following way to form the combined standard uncertainty U_c with u as standard uncertainties for all the parameters:

$$\frac{U_c}{T_k} = \sqrt{\sum_{s=1}^N \left(\frac{1}{T_k} \frac{\partial T_k}{\partial \varepsilon_s} \right)^2} u_\varepsilon + \sqrt{\sum_{s=1}^N \left(\frac{T_{f,s}}{T_k} \frac{\partial T_k}{\partial T_{f,s}} \right)^2} u_{T_f} + \sqrt{\sum_{s=1}^N \sum_{t=1}^{N_{Cond}} \left(\frac{C_{st}}{T_k} \frac{\partial T_k}{\partial C_{st}} \right)^2} u_{C_k} + \\ + \sqrt{\sum_{s=1}^N \sum_{t=1}^N \left(\frac{1}{T_k} \frac{\partial T_k}{\partial F_{st}} \right)^2} u_{FV} \quad (18)$$

The terms with the square root are the dimensionless uncertainty factors. The standard uncertainties can be supplied as difference for each parameter. Provided the standard uncertainties for all parameters are the same, the uncertainty factor can be directly compared. Ideally the standard uncertainty should be drawn into the square root but often a detailed knowledge of standard deviations, biases, or errors for each thermal node is not available. In this paper no distinction between errors, bias, or deviations is being made which would be beyond the scope of this work. Alternatively, dimensional sensitivity coefficients could be summed, but then the standard uncertainties need to be given in value differences and the combined standard uncertainty would have the dimension K. In that case the units for the uncertainty factors would be dimensionless of the fluid

temperature, K for emissivity and view factor, and K^2/W for the conductance.

WORK-FLOW

When performing an uncertainty analysis the following work-flow can be applied as described in Fig. 3.

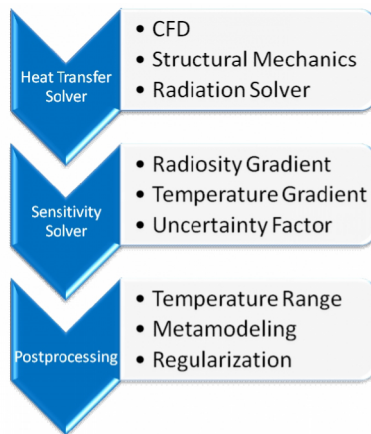


Figure 3. Work-flow for uncertainty analysis.

The sensitivity and uncertainty analysis used here is performed a posteriori. First a simulation has to be run, be it a structural mechanics, a CFD, or a specialized heat transfer code. Then the radiosity gradients of Eq. (5), (7), (10), (12) and (16) have to be calculated. This is followed by solving Eq. (6), (A22), (11), (14), and (17) for the temperature gradients, respectively. When uncertainty factors are desired, these temperature gradients need to be summed. This requires a lot of radiosity gradients. Wherefore, it is recommended to employ LU factorization for the LHS of the first set of equations, followed by matrix vector multiplication of the RHS of those relations. Sensitivity coefficients can be used for inverse calculations, where measurements are not possible, or for model reduction. When using Eq. (18) a temperature range can be obtained.

EXAMPLES

MUFFLER

For demonstration purposes a simple test case of a muffler and heat shield is being calculated. The geometry is shown in Fig. 4. The red circles indicate node 28 on the heat shield and node 96 on the muffler. The geometry consists of 260 elements or 460 thermal nodes. The side of the heat shield facing away from the muffler is insulated.

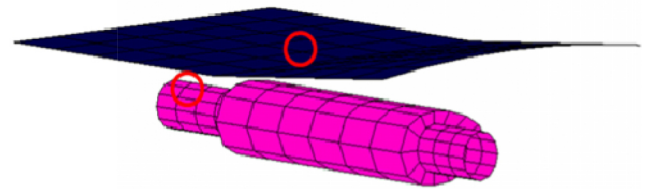


Figure 4. Geometry of Muffler.

The case is set up in the commercial radiation solver RadThermIR 9.1. A built-in heat transfer model was used to provide convective heat transfer coefficients. The exterior boundary conditions were chosen in a way that would be similar to a stop-and-go traffic with rusty and soiled metal. They are shown in Table 1. The exhaust gas throughput is assumed to be 0.04 kg/s.

Table 1. Boundary conditions for muffler.

Entity	Unit	Heat Shield	Muffler	
			Outside	Inside
Emissivity	[-]	0.7	0.8	0.9
Thickness	[m]	0.002	0.001	0.001
Conductivity	[W/m/K]	17.3056	52.019	52.019
Air Velocity	[m/s]	0.2	0.2	
Air Temperature	[K]	293	293	380

The form factors were calculated with 512 rays per face. The exhaust gas is modeled as air without soot formation and nonparticipating for radiation. Though severe simplifications, the exhaust gas serves merely as a heat source. The exhaust gas itself is held at a constant temperature.

The resulting heat fluxes for two thermal nodes are shown in Table 2.

Table 2. Heat fluxes for nodes 28 and 96.

Node	28	96
	[W/m ²]	
Radiation	44.13	-137.95
Convection	-25.33	-212.29
Conduction	-18.72	358.30

STEP

As a second example, a simple step has been created as shown in Fig. 5. It consists of three squares as thermal nodes with a side length of 0.1 m. The backside has no heat exchange. On node '1' a heat of 100 W is imposed. The step is placed inside a box in order to simulate the environment by a constant temperature of 293.15 K and an emissivity of 1. Further properties and boundary conditions can be taken from Table 3.

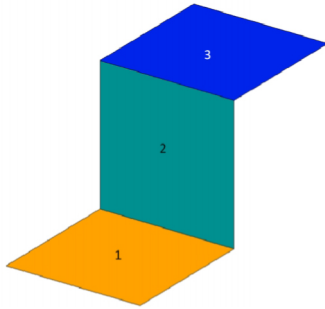


Figure 5. Geometry of Step.

Table 3. Boundary conditions for Step.

Entity	Unit	Step
Emissivity	[-]	0.9
Thickness	[m]	0.01
Conductivity	[W/m/K]	52.019
Heat Transfer Coefficient	[W/m²/K]	1
Air Temperature	[K]	293.15

ANALYSIS

First, it shall be investigated, whether numerical stability is influenced significantly with the new formulation.

NUMERICAL STABILITY

The Gerschgorin circle theorem estimates eigenvalues λ and thus stability of a solver because it gives the maximum possible value of λ by the Gerschgorin radius r . The eigenvalue λ assures stability if it is less than or equal to 1. This is guaranteed when the radius r is less than or equal to 0.5. In that case it can be expected, that the set of equations can be handled by an iterative solver without preconditioning.

$$r = \sum_{\substack{j=1 \\ j \neq i}}^N |a_{ij}| \quad (19)$$

Figure B1 shows the ratios of the diagonal entry to the Gerschgorin radius for nodes 28 and 96 of the muffler. Using reciprocity for node 28 actually decreases stability slightly as compared to the formulation without reciprocity (Conventional), whereas for node 96 the ratio remains virtually the same. This should show that the system behaves differently at different places. Diagonal dominance assures numerical stability regardless of the actual size of the radius. The value for the radius for node 96 is about 9 indicating an ill-conditioned behavior. The author has shown [12] with the same geometry but different boundary conditions that the

inside of the muffler is the main source of instability which happens to be the case for this example, too.

The infinity norm $\|M\|_{\infty}$ of the matrix M is another indicator for stability.

$$\|M\|_{\infty} = \max_{1 \leq i \leq N} \sum_{j=1}^N |a_{ij}| \quad (20)$$

Figure B2 shows the infinity norm of the solution matrix of the muffler for sensitivity coefficients for formulations with (Reciprocity) and without (Conventional) reciprocity of the LHS of Eq. (5) and (3), respectively. The set of bars denote whether only the radiation term (Rad), the radiation and convection terms (Conv), radiation and conduction terms (Cond), or all three terms (ConvCond) for the sensitivity coefficients had been used. It can be seen that for Rad the infinity norm is quite low. The introduction of the conduction term lets the norm rise significantly, also suggesting instability.

PARTIAL UNCERTAINTIES

As Fig. B2 suggests different numerical behavior when using various heat transfer modes, the same can be expected for uncertainty coefficients. Figure B3 shows, how incorporating various heat transfer modes alter the dimensionless uncertainty coefficients for node 28 of the muffler as they are used in Eq. (18). The uncertainty factor for fluid temperature is rather low as expected for natural convection. Emissivity and view factor show the same tendency. At this point two remarks have to be given. First, a comparison between uncertainty factors is only valid when all standard uncertainties for all parameters in the whole domain are the same. Second, local sensitivity analysis cannot predict the behavior of the system far away from the simulation. For example, the results of this analysis cannot predict the influence of the fluid temperature in the forced convection regime.

In Fig. B4 the same is shown for node 96. The difference lies in the order of magnitude difference of uncertainty factors as compared to node 28. This suggests that node 96 is more susceptible to change in view factors than node 28.

RECIPROCITY

What remains to be shown is how the uncertainty coefficients with and without reciprocity deviate from each other. In Fig. B5 the unsigned differences between the coefficients divided by the coefficient without reciprocity and multiplied by hundred are shown for node 28. The same is done for node 96 of the muffler in Fig. B6.

Even in the worst case for node 28 the difference is less than 5% with radiation as the sole heat transfer mode. With combined heat transfer modes it is even less. The differences