

# Estimation of Surface Heat Flux in IC Engines Using Temperature Measurements: Processing Code Effects

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## ABSTRACT

Heat transfer in internal combustion engines is taking on greater importance as manufacturers strive to increase efficiency while keeping pollutant emissions low and maintaining adequate performance. Wall heat transfer is experimentally evaluated using temperature measurements both on and below the surface using a physical model of conduction in the wall. Three classes of model inversion are used to recover heat flux from surface temperature measurements: analytical methods, numerical methods and inverse heat conduction methods; the latter method has not been previously applied to engine data. This paper details the inherent assumptions behind, required steps for implementation of, and merits and weaknesses of these heat flux calculation methods. The analytical methods, which have been most commonly employed for engine data, are shown to suffer from sensitivity to measurement noise that requires *a priori* signal filtering. The numerical methods are computationally demanding and do not offer any advantage over the analytical methods unless the wall is a composite structure. The inverse methods are found to be relatively insensitive to noise because of the use of future-time information, the inclusion of a regularization term that relaxes the exact-matching constraint, and the fact that finite derivatives are not required to estimate the heat flux.

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## INTRODUCTION

Heat flux measurements in IC engines are commonly performed by measuring the transient surface temperature with a specialized coaxial surface thermocouple. Heat flux is then inferred from the surface temperature measurement. A coaxial surface thermocouple measurement is, in principle, a point measurement, and this configuration readily lends itself to a semi-infinite one-dimensional (1-D) solution despite the fact that the measurement system is composed of many materials with widely varying thermophysical properties and is influenced by multi-dimensional effects [10,41]. The common methods used to calculate the 1-D heat flux are usually variations of analytical techniques and finite difference methods. An in-depth study of these methods will be presented which examines their strengths and weaknesses. An inverse solver utilizing Beck's sequential function specification method (SFS) with a form of [Tikhonov] regularization will also be detailed. This powerful solver not only mitigates electrical noise issues but also uses measurement statistics to influence the estimated surface heat flux to provide the surface temperature estimate within the variance of the temperature data. Inverse solvers have been utilized to predict surface heat transfer rates with subsurface

measurements for years in the aerospace industry, but to the authors' knowledge this is the first time they have been implemented to process IC engine data. The periodicity of the engine cycle and ensemble averaging of the engine data increases the value of the inverse solver compared to other processing techniques and these unique benefits will be detailed and discussed as well.

## BACKGROUND

### *Heat Flux Measurements in IC Engines*

Heat flux measurements have been conducted by numerous investigators since a satisfactory method of constructing a fast-response surface temperature measurement was established. The invention and introduction of the Bendersky-type [1] thermocouple enabled accurate, microsecond response [2], crank-angle resolved temperature measurements in IC engines. This led to the exploitation of the coaxial thermocouples as the surface temperature measurement device of choice for use in heat transfer measurements within the IC engine community, and this is still the case today [10,11,12], even though other niche technologies exist that can make direct heat flux measurements without measuring surface temperature

[14,15]. The bulk of studies conducted [3,4,5,6,7] have focused on using the point temperature measurements to infer the global heat transfer rate in gasoline [8,9] engines.

### Analytical Methods

Analytical methods are characterized as exact temperature matching techniques; conservation of energy dictates that the solution produced exactly matches the data recorded, *i.e.* errors in the temperature data will be reflected in the estimated surface heat flux. Secondly, closed-form analytical solutions are linear (temperature-dependent properties are not accounted for), and only one material can be represented in the model. This precludes composite bodies from being examined.

### Fourier Method

A commonly used analytical model is based on the semi-infinite solid solution modified for periodic boundary conditions. According to Carslaw and Jaeger [13], the surface temperature can be represented as a truncated harmonic function of time (Fourier series) having the following form:

$$T_w(t) = \bar{T}_w(0) + \sum_{n=1}^N [A_n \cos n\omega t + B_n \sin n\omega t] \quad (1)$$

where  $\bar{T}_w(0)$  is the time-averaged surface temperature,  $A_n$  and  $B_n$  for  $n = 1$  to  $N$  are the Fourier coefficients used to represent the surface temperature history,  $t$  is time, and  $\omega$  is one half the angular velocity of the engine

The measured surface temperature history,  $T(0, t) = T_w(t)$  can be used in conjunction with a boundary condition at a distance  $\delta$  below the surface  $T(\delta, t) = T(\delta)$  where the temperature is constant. The semi-infinite idealization is necessary to solve the one-dimensional unsteady heat conduction equation

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha} \frac{dT}{dt} \quad (2)$$

where  $\alpha = \frac{k}{\rho c}$  is the thermal diffusivity. The instantaneous surface heat flux as a function of time,  $q_w''(t)$ , can be found as

$$q_w''(t) = \frac{k}{\delta} (\bar{T}_w - T_\delta) +$$

$$k \sum_{n=1}^N \sqrt{\frac{n\omega}{2\alpha}} [A_n (\cos n\omega t - \sin n\omega t) + B_n (\sin n\omega t + \cos n\omega t)] \quad (3)$$

where the first term represents the steady-state heat transfer, and the second term is the time-varying heat transfer component. Hereafter, [equation \(3\)](#) is identified as the Fourier method (FFT) for its use of the fast Fourier transform in generating the solutions. If only a surface temperature measurement is available, then the transient component of

[equation 3](#) can be used to calculate the dynamic heat flux component as implemented by Hoag [16]. The dynamic component captures all relevant combustion information (beside the steady state heat transfer rate) and this will be explored further in a later section. The periodic boundary conditions used to solve [equation 2](#) eliminate the startup transient. The FFT method is the most commonly used method for processing surface temperature data, and the results generated by this method will be used as a baseline reference for comparison with the results calculated with the other methods outlined in later sections.

The startup transient is the initial response of the system to an input at the beginning of the system disturbance; waiting for an engine to warm before taking data is the mathematical equivalent of waiting for the startup transient to die off. Most of the classic heat transfer solutions for a 1-D semi-infinite body, as found in engineering texts such as [17], require that the domain is initially at a constant, uniform temperature. Most applications, including engines, that have a net heat transfer do not have a uniform wall temperature, thus these solutions will contain a startup transient wherein the steady-state wall profile is established. This restrictive requirement can be handled with the use of numerical methods. One can iterate a solution until the startup transient is eliminated and the sustained response is achieved, *i.e.* the mathematical model is cycled until the "engine" is warm. Eliminating the requirement of the uniform initial condition opens up a field of numerically implemented, analytically based solutions such as the Cook - Felderman technique

A closed-form solution of surface heat flux for a step change in surface temperature from  $T_0$  to  $T_1$  at  $t = 0$  can be found from [equation 2](#) as [18],

$$q_w''(t) = \frac{\sqrt{k\rho c}}{\sqrt{\pi}} \frac{T_1 - T_0}{\sqrt{t - t_0}} \quad (4a)$$

where the subscript 0 refers to the initial condition. Since the problem is linear, the principle of superposition can be used to extend the solution for a generic number of step changes as

$$q_w''(t) = \frac{\sqrt{k\rho c}}{\sqrt{\pi}} \sum_{j=1}^n \frac{T_j - T_{j-1}}{\sqrt{t_n - t_{j-1}}} \quad (4b)$$

where  $n$  is the total number of time steps up to the time of interest. Using Duhamel's Theorem and assuming that the estimated flux is constant between discrete measured surface temperatures allows calculation of the transient surface heat flux. The analytical closed form solution of [equation \(4b\)](#) was derived by Schultz and Jones [19]. The numerical algorithm used to implement this solution is known as the Cook-Felderman technique [20], given by [equation 4c](#).

$$q_w''(t) = \frac{2\sqrt{k\rho c}}{\sqrt{\pi}} \sum_{n=1}^N \frac{T_j - T_{j-1}}{\sqrt{t_n - t_j} + \sqrt{t_n - t_{j-1}}} \quad (4c)$$

The Cook-Felderman (C-F) technique uses a truncated summation series to numerically calculate the surface heat flux. Gatowski [22] uses the C-F technique to analyze calibration data for coaxial thermocouple probes but did not use the procedure to calculate engine heat fluxes.

The C-F technique benefits from ease of use and adequate accuracy given its assumptions (1-D and linear). Since this method is based on the assumption an initial uniform wall temperature (equation 4a), the steady state temperature gradient in the cylinder wall must be accounted for by eliminating the startup transient. This is achieved in engine applications by successively applying the measured surface temperature data. If the predicted heat flux in the  $n+1^{\text{st}}$  cycle does not match (within some tolerance) that of the  $n^{\text{th}}$  cycle, then another cycle is calculated. If the C-F technique is iterated to find the sustained response, the calculated C-F heat flux will match the transient heat flux term from the Fourier solution (the second term of Eqn. 3). Similar to the dynamic portion of the FFT, the calculated heat flux is scaled by a constant that is a function of the thermal diffusivity, in this

case  $\frac{2\sqrt{k\rho c}}{\sqrt{\pi}}$ . Additionally, it can be seen that the finite temperature difference (wrt time) predicts the response of the entire domain (this is also the case for the other analytical methods as well). Finally, the square root terms in the denominator amplify whatever common mode noise is present in the temperature measurements  $T_j$  and  $T_{j-1}$ . The amplification of noise through the use of finite differences / derivatives and the method by which they are calculated is the major weakness of any analytical method and generally manifests itself as a form of numerical oscillation.

### Other Analytical Solutions

Based on the general form of the C-F technique, many other possible variations of this solution are made possible by altering the calculation of the temperature-time summation term ever so slightly. The Kendall-Dixon method [21] is one of these variations and is shown in equation 5.

$$q_w(t) = \frac{\sqrt{k\rho c}}{\sqrt{\pi}} \sum_{n=1}^N \frac{Y_j + Y_{j-1}}{\sqrt{t_n - t_j} + \sqrt{t_n - t_{j-1}}} \Delta t \quad (5a)$$

The Kendall-Dixon procedure actually calculates an integrated energy flow,  $q_w(t)$  so a scheme must be devised to recover the heat flux. The differencing scheme suggested by Hollis [24] converts the energy flow calculated by equation 5a into a heat flux averaged across a large number of steps. This provides a large amount of smoothing and is given by equation 5b.

$$q_w''(t) = \frac{-2q_{n-8} - q_{n-4} + q_{n+4} + 2q_{n+8}}{40\Delta t} \quad (5b)$$

It is also apparent that the Kendall Dixon method will exhibit a greater degree of stability at the cost of significantly filtering the calculated solution.

Finally, an interesting approach to mitigating noise amplification in the finite derivative term was taken by Diller [18] in another modified form of the C-F. Instead of averaging over a number of time steps, Diller alters the method by which the temperature changes in the algorithm. With the C-F technique, the temperature is assumed to change in a piecewise linear manner between two steps, Diller proposed a step change between timesteps to enhance stability. Diller's equation is shown in 6.

$$q_w'' = \frac{\sqrt{k\rho c}}{\sqrt{\pi\Delta t}} \sum_{j=1}^n \frac{T_j - T_{j-1}}{\sqrt{n+1-j}} \quad (6)$$

While similar, equations 3, 4, 5, and 6 all produce slightly different results. The analytical solutions presented will be implemented and compared to other methods in a later section of this paper.

### Finite Difference Methods

A one-dimensional finite difference code can be implemented to process surface temperature data to mitigate some of the issues associated with analytical methods. In general, numerical methods, such as finite differences or finite element methods allow the effects of multiple materials to be accounted for and also permits non-linear analysis. In addition, more boundary conditions can be accommodated than analytical methods because the calculated surface heat flux is based on a converged temperature distribution resulting from the solution of a generic conduction model. The surface heat flux is calculated by performing an energy balance on the first control volume that borders the surface. The energy balance includes the finite energy storage term of the surface node which has half the volume of an interior node [25].

$$q_w''(t) = \frac{\Delta x}{2} \rho c \frac{\partial T}{\partial t} + k \frac{\partial T}{\partial x} \quad (7)$$

The recorded surface temperature data are used as a boundary condition at the front of the domain and the backside temperature (if available) sets the boundary condition at the back of the domain, just as with the analytical models. Information about matrix methods used with direct conduction solvers can be found in [26]. In this study, an implicit Crank-Nicolson integration routine was used to integrate the surface temperature data and advance the solution timestep. Other routines frequently utilized for integration include explicit Euler [27, 28] and Runge-Kutta

integration schemes. It is beyond the scope of this study to fully detail integration routines but readers are directed to Press *et al* [29] for further information.

For the case of a single material, the solution generated by a finite difference solution should exactly match that of any of the analytical methods mentioned previously. There are, however, some issues unique to finite difference methods. Because the surface temperature exactly matches the measured value, the heat flux estimate includes the temperature measurement errors and the finite differences that approximate derivatives will amplify these errors. Also, to process temperature data from an IC engine, the sustained response of the wall must be fully captured by the numerical method, which requires an iteration scheme to ensure the thermal gradient within the wall is fully converged, *i.e.* the sustained response is recovered. Third, care must be taken to avoid stability errors from exceeding the critical time step; this is especially true with explicit solvers. Fourth, numerical methods in general require considerably more time to implement than the algebraic analytical methods discussed above. Therefore, unless the complexity of a non-linear model is required to account for temperature-dependent properties, for instance with a ceramic coated piston, or a multilayered device, analytical methods should produce exactly the same solution as a finite difference method at a fraction of the programming cost without increasing the possibility of errors due to numerical integration instabilities.

### Inverse Heat Conduction Solvers

Temperature measurements made below the surface, *i.e.* inside a given wall, can be utilized to estimate the surface heat flux. The resulting problem is designated as an ill-posed problem whose solution is characterized by a lack of existence, instability, and a lack of uniqueness as specified by Hadamard [30]. Ill-posed problems can be solved, but they require a much more sophisticated solution methodology than those of the analytical and numerical methods previously mentioned. Temperature data from surface thermocouples can also be used with an inverse solver since the solution technique is very flexible and allows using data from anywhere in the domain to estimate surface conditions (even the back of a given wall). In this case, since the thermocouple is on the surface the estimated surface heat flux is a unique solution, and the inverse solution methodology can be used to mitigate instability arising from noise and measurement errors in the system. The inverse solver is also used in this study to make use of statistical data provided by the ensemble averaging of the surface temperature data to improve the estimate of surface heat flux.

### Background on Inverse Heat Conduction Problem

Previous efforts made at reconstructing the surface heat flux given an internal temperature measurement, such as the Stolz [31] method and Burggraf [32] solution, relied on inversions of analytical solutions. Burggraf developed an exact analytical solution with a rapidly convergent series

using the lumped capacitance approximation as the leading term. The heat flux problem was subdivided into two separate entities as shown in Figure 1 for a slab configuration.

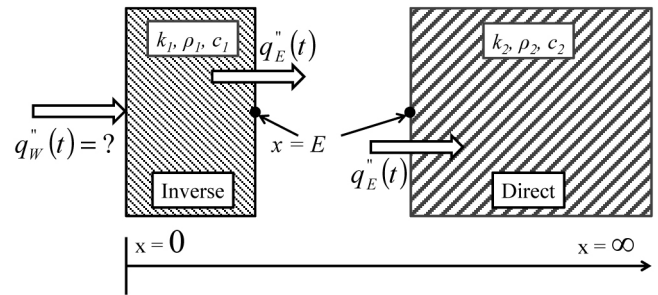


Figure 1. Subdivision of an inverse heat conduction problem into an inverse and direct problem.

If the temperature-time history and the heat flux,  $q_E''(t)$  are known at a depth of  $x=E$  then the surface heat flux,  $q_w''(t)$  can be calculated according to equation 8. Note:  $q_E''(t)$  can be calculated using an analytical method such as the C-F technique.

$$q_w''(t) = q_E''(t) + k_1 \sum_{n=1}^{\infty} \frac{E^{2n-1}}{(2n-1)!} \frac{1}{\alpha_1^n} \left[ \frac{d^n T}{dt^n} \right]_{x=E} + \sum_{n=1}^{\infty} \frac{E^{2n}}{(2n)!} \frac{1}{\alpha_1^n} \left[ \frac{d^n q_E}{dt^n} \right]_{x=E} \quad (8)$$

All time derivatives are explicit, and since the higher order terms are calculated from experimental data, the series must converge rapidly otherwise inaccuracies in the higher order derivatives may lead to numerical oscillations or large errors in calculated surface heat flux. The Burggraf method suffers from numerical instabilities due to the use of finite derivatives. Furthermore if higher order derivatives are required in a given solution, numerical instability will be amplified even further. As stated by Beck [36], the utility of the Burggraf solution is to provide beneficial insight into inverse heat conduction problems. If we are seeking the solution for the surface heat flux using temperature data on or below the surface, using finite derivatives will increase instability. As the frequency of the signal we are trying to reconstruct increases, higher order terms will need to be accounted for to maintain accuracy and this will increase instability as well. These shortcomings motivate the need for a different solution technique; inverse heat conduction problem (IHCP) solvers (*i.e.* inverse solvers) meet this need.

The generalized methodology of inverse solutions can be primarily attributed to Beck [34] and was developed to calculate the surface heat flux of atmospheric re-entry vehicles where surface ablation limits thermocouple mounting options to subsurface locations for removed from

the incident surface flux. The methodology developed allowed for nonlinearities in the solution [35], e.g. temperature-dependent thermal properties, and composite bodies. The solution is very flexible and allows a wide range of boundary conditions and geometries (including 2D and 3D domains). The solution uses temperature data from a limited number of locations to calculate the surface heat flux and minimizes the error between a calculated temperature distribution (described in detail below) and the recorded temperature data. In essence, inverse methods are numerical techniques that use temperature data from timesteps in the future, and (possibly) multiple locations within the solid to directly calculate the surface heat flux at the current timestep. The future temperature information enables the model to look ahead in time to predict the surface heat flux, and provide a degree of smoothing that is independent of a specified filtering frequency, allowing the solution to capture both high and low frequency components. Furthermore, inverse methods directly calculate the heat flux at the surface (as compared to other methods that deduce the heat flux from a calculated temperature distribution), and thus eliminate the need to calculate finite derivatives, which improves stability.

### Beck's Sequential Function Specification Method (SFS)

The major component of the inverse method discussed in this study follows the strategy outlined by Beck, Blackwell, and St. Clair [36]. A description of the solution process and the inverse algorithm can be summarized as follows:

1. An initial surface heat flux  $q_o^m$ , is assumed at the surface at the domain at the time step  $m$
2. This heat flux is used to solve for the temperature distribution,  $T_j^m$ , in the model using an appropriate direct conduction solver, i.e. a finite difference code. Note: As stated earlier, temperature-dependent thermal properties and multi-dimensional models are allowed.
3. An objective function  $S$  is created according to [equation 9a](#) that describes the error between the measured temperature,  $Y_j$ , at location  $j$ , for the current timestep  $m$  and  $i=1$  to  $r$  future times and the calculated temperature distribution from step 2,  $T_j$  at the same times and locations. Note: the objective function allows multiple sensors and the use of multiple future-time temperature information. The solver makes the temporary assumption that the heat flux from the current timestep remains constant ( $q_o^m = C$ ) when calculating the temperature distribution for  $r$  future timesteps.

$$S = \sum_{i=1}^r \sum_{j=1}^J (Y_j^{m+i-1} - T_j^{m+i-1})^2 \quad (9a)$$

4.  $S$  is minimized by taking the partial derivative of the objective function with respect to the surface heat flux giving

$$\frac{\partial S}{\partial q_o^m} = 0 = \sum_{i=1}^r \sum_{j=1}^J (Y_j^{m+i-1} - T_j^{m+i-1}(q_o^m)) \left( \frac{\partial T_j^{m+i-1}}{\partial q_o^m} \right) \quad (9b)$$

$$X_j^{m+i-1} \equiv \left( \frac{\partial T_j^{m+i-1}}{\partial q_o^m} \right) \Big|_{q_o^{m-1}} \quad (9c)$$

5. The temperature is expressed as a two-term Taylor series expansion in heat flux, as given by [9d](#), which provides the relation that enables the temperature solution to advance by one timestep,

$$T_j^{m+i-1}(q_o^m) = T_j^{m+i-1}(q_o^{m-1}) + \frac{\partial T_j^{m+i-1}}{\partial q_o^{m-1}} \Delta q^m \quad (9d)$$

6. Substitution of [9d](#) into [9b](#) yields the heat flux correction term ( $\Delta q^m$ ) in algebraic ([9e-1](#)) and matrix form ([9e-2](#)) where  $\mathbf{X}$  is a matrix of sensitivity coefficients

$$\Delta q^m = \left[ \frac{\sum_{i=1}^r \sum_{j=1}^J (Y_j^{m+i-1} - T_j^{m+i-1}(q_o^{m-1})) (X_j^{m+i-1})}{\sum_{i=1}^r \sum_{j=1}^J (X_j^{m+i-1})^2} \right] \quad (9e-1)$$

$$\mathbf{X}^T \mathbf{X} = \Delta q^m \mathbf{X}^T (\mathbf{Y} - \mathbf{T}_0) \quad (9e-2)$$

$$q_o^{m'} = q_o^m + \Delta q^m \quad (9f)$$

7. The initially assumed heat flux ( $q_o^m$ ), is corrected by [equation 9e](#) and the heat flux estimate for the current timestep ( $q_o^{m'}$ ) is calculated in [equation 9f](#). The algorithm then advances the timestep by one and returns to step 1 until the simulation is finished. Note: [equation 9e-2](#) is a linear set of equations.

The solution methodology highlighted by equation group 9 summarizes the algorithm used to commonly solve the IHCP with the sequential function specification method. Looking closer at [Eqn. 9e](#) we realize that spatial information is not directly incorporated into the heat flux correction term which means that this algorithm can be utilized for one-dimensional or multi-dimensional problems. Secondly, we can also use the inverse algorithm with surface temperature data (even though this algorithm was originally designed for subsurface inverse problems). Third, since derivatives of the

temperature distribution are not used to calculate the estimated flux, errors in the temperature measurements are not amplified by the estimated heat flux solution as is the case with the Burggraf or Stolz methods. Finally, examining the heat flux correction term, we recognize that the only unknown in the equation group above is the variable  $X_j^{m+i-1}$ , which is termed the sensitivity step function coefficient. The derivation of the sensitivity step function coefficients are detailed next.

Step function sensitivity coefficients are calculated weighting factors sensitive to their spatial location. Their effect on the calculated surface heat flux varies based on their spatial location and the timestep (current or future) they are calculated for.

From a physical standpoint, the sensitivity coefficients in the algorithm are based on the solution of the direct conduction problem. The sensitivity coefficients predict how the temperature distribution and thermal properties change within the wall due to thermal conduction. Since they are based on the heat conduction characteristics of the domain, they are more sophisticated than a simple filter that attenuates a signal based on a given cutoff frequency, or a weighted average that calculates filter coefficients based solely on timing. The sensitivity coefficient matrix places higher importance on sensors closest to the surface at times that are closest to the current timestep while accounting for different materials that may exist within the domain. This means that a sensor at  $x/L = 0.25$  will have a larger impact on the estimated surface heat flux than a sensor buried at  $x/L = 1.0$ , and the sensitivity coefficients will be larger at  $t_{m+1}$  than at  $t_{m+3}$  ( $r = 3$  future times) assuming a constant heat flux between times.

The step-function sensitivity coefficients can be found by differentiating the heat diffusion equation and its associated boundary conditions with respect to  $q''$ . The governing differential equation and its boundary conditions is shown in Eqn. 8 taken from [36].

$$-k \frac{\partial X}{\partial x} \Big|_{x=0} = 1 \quad t \geq t_m \quad (8a)$$

$$\rho c \frac{\partial X}{\partial t} = \frac{\partial}{\partial x} \left( k \frac{\partial X}{\partial x} \right) \quad (8b)$$

$$-k \frac{\partial X}{\partial x} \Big|_{x=L} = 0 \quad (8c)$$

$$X(x, t_{m-1}) = 0 \quad (8d)$$

At the active surface,  $x = 0$ , for all times greater than or equal to the current timestep  $t_m$ , the sensitivity coefficient rate of change with respect to  $x$  is  $1/k$ . At all times before the current timestep, the coefficients are equal to zero. This simplistic representation conveys the idea that the sensitivity coefficients act like weighting factors that gauge the domain response to a unit step function spanning from 0 to 1.

The same routines used to solve a direct conduction problem can be utilized to solve the governing differential equation for the sensitivity coefficients (Eqn. 8b) subject to the boundary conditions identified in Eqn. 8.

### Regularization Techniques

Tikhonov regularization, or more simply regularization, is an extension of the SFS technique that incorporates the noise present in discrete measurements into the model to reduce fluctuations in the unknown function (heat flux) being estimated [38]. Regularization adds stability by relaxing the requirement that the estimated temperature distribution exactly match the recorded data by intentionally adding a small of bias to the solution. Relaxation is allowed within the variance of the measured temperature data. For instance if the variance calculated at a specific crank angle is  $T \pm 0.3K$  then a regularization code will allow the solver to calculate a heat flux that produces a temperature in the range  $T \pm 0.3K$ . This type of solution procedure dampens oscillations and increases solution stability. The regularization technique incorporates the temperature measurement variance ( $\sigma^2$ ) into the inverse model by adding a term to the objective function  $S$ ,

$$S = \sum_{i=1}^r \sum_{j=1}^J (Y_j^{m+i-1} - T_j^{m+i-1})^2 + \sum_{i=1}^{r-1} \alpha_i (q_{i+1}'' - q_i'')^2 \quad (9)$$

$$\alpha_i \equiv \left( \frac{\sigma_i}{q_i} \right)^2 \quad (10)$$

where  $\alpha$  is the regularization parameter. If  $r = 2$ , the scheme depicted here is first order in nature, and this limits the change in flux gradients. If  $r = 3$ , a second order regularization scheme results. Based on the definition of the variance, which can be a constant or a function of time (or temperature), the influence of noisy data can be controlled. The objective function can be minimized and then rearranged to solve for the flux correction akin to Equation 9e given by [39] in matrix notation.

$$\mathbf{X}^T \Psi^{-1} \mathbf{X} + \alpha \mathbf{W}_i \mathbf{H}^T \mathbf{H} = \Delta q \left[ \mathbf{X}^T \Psi^{-1} (\mathbf{Y} - \mathbf{T}_o) - \alpha \mathbf{W}_i \mathbf{H}^T \mathbf{H} q_o \right] \quad (11)$$

$$\Psi \equiv \sigma^2 \mathbf{I} \quad (12)$$