

Probability of Failure of Dynamic Systems by Importance Sampling

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ABSTRACT

Estimation of the probability of failure of mechanical systems under random loads is computationally expensive, especially for very reliable systems with low probabilities of failure. Importance Sampling can be an efficient tool for static problems if a proper sampling distribution is selected. This paper presents a methodology to apply Importance Sampling to dynamic systems in which both the load and response are stochastic processes. The method is applicable to problems for which the input loads are stationary and Gaussian and are represented by power spectral density functions. Shinozuka's method is used to generate random time histories of excitation. The method is demonstrated on a linear quarter car model. This approach is more efficient than standard Monte Carlo simulation by several orders of magnitude.

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INTRODUCTION

The reliability of a mechanical system under stochastic loads can be measured by estimating the probability of first-exursion of the ultimate load. Reliability analysis usually requires the evaluation of an integral over the domain of input variables. Monte Carlo simulation (MCs) is a general technique to evaluate such integrals numerically. Metropolis & Ulam [1] published the first paper on MCs and named it after a famous casino in Monaco. Although the idea of MCs existed before Metropolis & Ulam published their paper (e.g. [2]), they pioneered the application of MCs in treating a complex problem using the Electronic Numerical Integrator And Computer (ENIAC), which is known as the first general-purpose computer.

Monte Carlo simulation is an accurate and robust tool for estimating the reliability of both static and dynamic systems. In static problems, the input variables and the system response are random variables. However, in dynamic systems, both the input loads and response attributes are time dependent. The additional variability due to the variation in time makes MCs for dynamic problems more expensive than for static problems. In spite of access to powerful computers, the computational cost of MCs is high, making it impractical for most real-life complex problems with high dimensions. Consequently, MCs is mostly used for validating new reliability assessment techniques, rather than for reliability analysis of alternative designs.

Many researchers try to improve the efficiency of MCs by reducing the variance of its estimates. There are several variance reduction techniques in the literature [3]. Among them, Importance Sampling (IS) and Conditional Monte Carlo methods are the most effective. Importance Sampling frequently reduces the variance dramatically provided that a proper sampling distribution is selected.

The first application of IS was in the estimation of probabilities of nuclear particles penetrating shields [4]. Other applications of IS include reliability assessment of digital communications [5, 6], simulation of flows in porous media [7] and analysis of queuing systems [8]. Hesterberg [9] reviewed other early applications of IS.

Engelund & Rackwitz [10] reviewed and compared several popular IS methods for the estimation of failure probabilities in several benchmark examples. They categorized IS methods into four major groups: Direct methods, Updating methods, Adaptive schemes and Spherical schemes. They concluded there is no technique that is consistently superior to others in all applications.

The challenge in application of IS to dynamic problems is the calculation of the likelihood ratio as will be described in the following section. In this paper, we propose an approach to extend the application of IS to linear or nonlinear dynamic problems. This new approach utilizes Shinozuka's method [11] to calculate the realizations of the excitation load. The performance of the proposed method is studied on a linear quarter car model.

IMPORTANCE SAMPLING

The idea of IS comes from the fact that in a MCs certain values of the random variables affect the results more than others. If these important values appear more frequently in a sample, then the estimator of the variance could be reduced, and consequently, the accuracy could be improved. Importance Sampling often reduces the variance dramatically, especially for low failure probabilities.

In order to estimate the probability of failure due to first excursion of a threshold, the following integral should be evaluated,

$$P_f^T = \int I(G(\mathbf{x}) \leq 0) f_{\mathbf{x}}^T(\mathbf{x}) d\mathbf{x}, \quad (1)$$

where $f_{\mathbf{x}}^T(\mathbf{x})$ is the true PDF. The indicator function, $I()$ becomes 1 when the response passes the threshold value, and zero otherwise.

Two major steps are involved in IS. The first is the choice of an appropriate sampling distribution. The use of a sampling PDF, $f_{\mathbf{x}}^S(\mathbf{x})$ would result in a biased estimation of the probability of failure. Therefore, in the second step, the results are weighted by the likelihood ratio of the true and sampling PDFs to correct the bias (Eq. 2) [12]. A good sampling distribution results in many failures with high likelihood.

Equation 1 can be rewritten as follows,

$$P_f^T = \int I(G(\mathbf{x}) \leq 0) \frac{f_{\mathbf{x}}^T(\mathbf{x})}{f_{\mathbf{x}}^S(\mathbf{x})} f_{\mathbf{x}}^S(\mathbf{x}) d\mathbf{x}. \quad (2)$$

The above integral could be approximated by the following unbiased equation,

$$\hat{P}_f^T = \frac{1}{K} \sum_{k=1}^K I(G(\mathbf{x}^S) \leq 0) \frac{f_{\mathbf{x}}^T(\mathbf{x}_k^S)}{f_{\mathbf{x}}^S(\mathbf{x}_k^S)}. \quad (3)$$

where K is the sample size. By defining likelihood ratio as in Eq. 4, we can rewrite Eq. 3 as in Eq. 5.

$$w_k = \frac{f_{\mathbf{x}}^T(\mathbf{x}_k^S)}{f_{\mathbf{x}}^S(\mathbf{x}_k^S)}, \quad (4)$$

$$\hat{P}_f^T = \frac{1}{K} \sum_{k=1}^K I(G(\mathbf{x}^S) \leq 0) w_k. \quad (5)$$

In the above equation, the observations of random variables are drawn from the sampling distribution instead of the true one. Then the standard deviation could be estimated by,

$$\sigma_{\hat{P}_f^T} = \frac{1}{\sqrt{K(K-1)}} \sqrt{\sum_{k=1}^K [I(G(\mathbf{x}^S) \leq 0) w_k]^2 - K(\hat{P}_f^T)^2}. \quad (6)$$

The confidence interval (CI) is estimated as follows,

$$CI_{(1-\alpha)\%} = \left[\hat{P}_f^T - t_{K-1, 1-\alpha/2} \sigma_{\hat{P}_f^T}, \hat{P}_f^T + t_{K-1, 1-\alpha/2} \sigma_{\hat{P}_f^T} \right], \quad (7)$$

where in the above equation, $t_{K-1, 1-\alpha/2}$ is the $1-\alpha/2$ percentile of the t -distribution with $K-1$ degrees of freedom.

Fonseca et al. [13], and Ridgeway & Madigan [14] suggest normalizing the estimator in Eq. 5 by the average likelihood ratio. This can improve the accuracy if the support of the sampling distribution does not completely cover that of the true spectrum, but it has higher density around the failure region. However, normalizing the result from Eq. 5 yields a biased estimator of the probability of failure.

IMPORTANCE SAMPLING FOR DYNAMIC SYSTEMS

The challenge in applying IS to a dynamic problem involving a stochastic process is to calculate the likelihood ratio in Eq. 4. This is because a stochastic process is an ordered set of an infinite number of variables, which means that the likelihood ratio is the product of infinity terms.

The following presentation is confined to problems where the input loads are represented by stationary, Gaussian random processes. In this case, these loads are completely specified by PSD functions. We can circumvent the above difficulty by generating random time histories by Shinozuka's method [11]. In this method, the time histories of a stationary, Gaussian process with zero mean, can be approximated as follows,

$$L(t) = \sum_{i=1}^N \sqrt{\frac{2}{N}} \sigma \cos(\omega_i t - \varphi_i), \quad (8)$$

where φ is a random phase angle that is uniformly drawn from $[0, 2\pi]$. Frequencies ω_i are independent random variables whose PDF is the one-sided PSD function normalized by the area under it.

Use of Eq. 8 enables calculation of the likelihood ratio as follows,

$$w_k = \prod_{k=1}^N \frac{f_{\mathbf{x}}^T(\omega_k^S)}{f_{\mathbf{x}}^S(\omega_k^S)} \quad (9)$$

The phase angles are still drawn from the true PDF; therefore, these PDFs do not appear in Eq. 9.

The selection of a sampling spectrum is critical in performing IS. A good choice can improve the accuracy by several orders of magnitude. For the structures under