

Fundamental Analysis of Spring-Variied, Free Piston, Otto Engine Device

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ABSTRACT

Conventional crank-based engines are limited by mechanical, thermal, and combustion inefficiencies. The free piston of a linear engine generator reduces frictional losses by avoiding the rotational motion and crankshaft linkages. Instead, electrical power is generated by the oscillation of a translator through a linear stator. Because the free piston is not geometrically constrained, dead center positions are not specifically known. This results in a struggle against adverse events like misfire, stall, over-fueling, or rapid load changes. It is the belief that incorporating springs will have the dual benefit of increasing frequency and providing a restoring force to aid in greater cycle to cycle stability. For dual free piston linear engines the addition of springs has not been fully explored, despite growing interest and literature. This investigation reviews the current modeling literature and advances the fundamental understanding of the free piston linear engine with springs by developing an idealized, nondimensional model. The model combines the dynamics of a damped, spring mass system with in-cylinder thermodynamic expressions. Simplifying assumptions are made to represent perfect springs, ideal gases, instantaneous heat addition and rejection, and an average friction force for work output dependent on stroke length. The model is executed for selected cases to represent the fundamental nature of the system. The model is explored both in and out of the context of the Otto cycle to demonstrate natural and forced stability over multiple operation cycles. Then, it is shown that system frequency and relative indicated mean effective pressure can be raised by increasing the baseline cylinder pressure, the bore diameter, or the amount of heat added through combustion relative to the stiffness of the spring. Lastly, the benefits of lower temperature operation are reinforced through parameterization of the specific heat ratio.

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INTRODUCTION

Free piston linear engines (FPLE) offer efficiency, cost, and power density advantages over conventional crank-based engines while also providing a potential platform for the employment of Homogeneous Charge Compression Ignition (HCCI) combustion [1, 2]. If successful, FPLEs married to linear alternators could be practically applied as generators in many settings including hybrid electric vehicles and portable power devices. Because of this potential, a variety of free piston linear engine configurations have been examined over the past years. Investigations at West Virginia University (WVU) [1, 3, 4, 5] and Sandia National Laboratories (SNL) [6, 7, 8] have involved a dual piston arrangement in which two opposite facing, rigidly connected pistons oscillate as a single translator between two engine cylinders. This differs from arrangements in which a single piston oscillates between an engine cylinder and an oppositely situated bounce device or mechanical spring. In both cases, power is drawn from the

motion of the translator by a stator mounted to the midsection of the engine. Using the dual piston approach and employing a two-stroke HCCI combustion cycle holds the potential for high power density and low NO_x emissions. Researchers at WVU believe that prior shortcomings can in part be attributed to low frequency of operation due to alternator mass. Overcoming adverse occurrences such as misfire, and rapid load changes requires the device to maintain adequate system frequency. This investigation is a continuation of their efforts to realize the dependable operation of a linear engine generator system.

LITERATURE REVIEW

Over the past couple of decades the potential of a linear engine generator has captured the focus of an increasing number of investigations. A majority of these have been simulation based - both numerical and analytical. While a few

have succeeded in working prototype models. Results from these various studies have greatly advanced the art, but difficult hurdles remain.

Early Work

In early days, the complexity of modeling an engine led to a reliance on available charts and tables. Farmer [9] spoke of this difficulty in 1947. The large amount of internal combustion engine research of the time laid the groundwork for a number of analytical efforts throughout the next decades. These works were often limited, however, and not able to incorporate the complex nature of engine thermodynamics with the difficult dynamics of the free piston.

In 1958 one of the earliest combined approaches was seen when Olson joined idealized thermodynamic processes to a detailed force balance into a differential equation and solved it as an initial value problem through an iterative approach [10]. Yet, the difficulties of combining the thermodynamics, combustion, and piston dynamics have persisted, and modern approaches, though more sophisticated, still rely on simplifications and iteration. For a more inclusive view of the history and more expansive discussion of the four main applications of free piston technology over the years, Aichlmayr [11] provides a comprehensive timeline and details some of the early free piston modeling work.

Recent Work

Over the past fifteen years, the power of modern computers and numeric techniques, combined with the need for better engine technologies, has led to a surge in free piston engine investigations. One of the earliest of these studies, a 1995 US Army funded project out of The University of Texas, aimed to evaluate the feasibility of a free piston linear engine device as an auxiliary power device for a hybrid electric vehicle [12]. The research covered a number of different engine and alternator configurations and took advantage of computer simulations as well as scaling parameters. The research showed that tailoring the generator device to the velocity profile of the piston assembly improved the efficiency of the device. The study set the modern stage for free piston linear engine modeling efforts by citing the highly complex interdependency of the engine device and the need for further modeling and development.

Shortly after, work began at West Virginia University. In 1998, two works were produced by WVU regarding linear engine work. The first [3] presents the fundamental analysis of a linear engine system. In this, a translator - two rigidly connected, opposite facing, free pistons - oscillated between two combustion chambers. A two stroke cycle was modeled as an Otto cycle and work output was correlated to friction over the length of each stroke. Idealized thermodynamic processes were implemented to represent polytropic expansion and compression and instantaneous heat rejection and addition. Piston dynamics were modeled through the application of Newton's second law. The second order differential was solved

in nondimensional terms for a closed form look at the piston motion and velocity. The efforts revealed that the model was highly sensitive to the amount of heat added or subtracted from the system as well as giving insight into the nature of the piston dynamics. The investigation reported here stems from this work.

The second [1] of these detailed a prototype model built for experimental comparison. The bench model was built from inexpensive, off the shelf components and operated on a spark ignited, port injected, two-stroke cycle. A simple friction brake provided resistance to the oscillating shaft, and the geometry allowed for a maximum compression ratio of 8:1. Based on the drag applied to the piston shaft, a net positive power output of about 780 W was produced through sustained operation. Through the efforts, it was seen that stable operation is highly dependent on ignition timing.

In 1999 the work at WVU was followed by a numerical simulation [4]. The model was built and validated against the operation of the prototype. The numeric model also provided the basis for a parametric study of an engine similar to the constructed prototype. Through these efforts, it was seen that heat addition and combustion duration influenced peak pressure, engine frequency, and compression ratio similarly. Further, the translator mass is seen to affect the peak pressure and stroke length proportionally while affecting the frequency inversely.

Concurrently, a group at Sandia National Laboratories (SNL) began to study the use of Homogeneous Charge Compression Ignition (HCCI) in a free piston engine generator device. Towards this, a free piston rapid compression/expansion machine was used to experiment with a variety of fuels, fuel/air mixtures, and initial temperatures [6]. Measured from these experiments were the combustion emissions, thermal efficiencies, cylinder pressures, and changes in cylinder volume. Given the conditions of the experiment, the similarity between HCCI combustion and constant volume heat addition as seen in an Otto cycle was evident. It was seen that performance - regarding both thermal efficiency and NO_x emissions - was significantly improved with the use of HCCI compared to traditional spark ignition.

Following these experiments, SNL continued the idea of applying HCCI combustion to a free piston device - a dual free piston, oscillating between port scavenged, two stroke combustion events within a double ended cylinder [2]. A linear alternator was considered to be an integrated component of the cylinder. Hydrogen fuel combustion was modeled according to HCCI concepts using single zone kinetics dependent on piston dynamics, fuel mixture, port geometry, heat rejection, and the resulting cylinder pressure. An iterative process was used as part of the initialization process to configure the engine system so that, for a steady state cycle of the engine, combustion occurred at top dead center. This study showed that the compression ratio of the engine does vary and is highly interdependent on the operating parameters of the engine. Very rapid HCCI combustion was seen, pointing towards the potential of high thermal efficiency and low NO_x emissions.

relative to conventional crank based engine operation. Finally, it was seen that the scavenging process plays an important role in the reliable operation of the engine.

In 2000, FEV Engine Technology advanced the science of free piston engine generator technology through numerical simulation [13]. The simulation sought to describe the performance of a dual free piston oscillating between alternating, two stroke, compression ignition, combustion events. Port scavenging, direct injection, electromechanical exhaust valves, and a permanent linear alternator were idealized within the engine model. Novel to the work was the construction of a graphical user interface and the parametric study that followed. Piston mass, magnetic strength of alternator, exhaust valve opening/closing timing, and onset of combustion were all studied with respect to their effect on engine frequency, compression ratio, and overall efficiency.

The year 2001 saw a continuation of linear engine work by both WVU and SNL. The work at WVU evaluated the concept of a four-stroke linear engine generator [14]. The design and evaluation were based on simulations of a numerical model capable of employing either compression ignition or HCCI combustion. Four-stroke operation was achieved by coupling the translators from two dual, free piston linear engine systems. Synchronizing their motions properly allowed for the expansion stroke of one translator to drive both the exhaust stroke of its own opposite piston as well as the compression stroke of the other translator. The study provided insight into the possible operation range for a compression ignition system and the potential for improvements in efficiency. It also gave an indication of the difficulty of sustainable operation when combining free piston and HCCI technologies.

Sandia National Laboratories continued to pursue the use of hydrogen to fuel its free piston linear engine operating within the two-stroke HCCI combustion regime [7]. Their report details work towards both scavenging and linear alternator design and optimization. The scavenging investigation employed KIVA modeling software to develop a loop scavenged system and was aimed at maximizing thermal efficiency. The alternator investigation sought to build a predictive model of a stator previously designed in Sandia Labs.

In 2002, following the four-stroke concept development work, researchers at West Virginia University applied computer modeling techniques to simulate the operation of a free piston linear engine operating on two-stroke compression ignition [15]. The detailed model was employed to study the significance of fuel to air ratio, translator mass, friction, injection timing, and geometry with regards to system frequency and overall efficiency. Modeled parameters such as combustion duration, heat rejection, in-cylinder pressures, and compression ratio were also studied to provide a broader understanding of linear engine operation. It was seen that compression ratio is very dependent on a variety of parameters

and that the translator mass can have significant effects on engine operation - both in terms of compression ratio and engine speed.

Based on a 2002 benchmark review of prior investigations, Stanford University researchers decided to pursue a single free piston cylinder arrangement operating on a four-stroke working cycle [16]. The translator oscillates between combustion and a rebound device. Modeling was employed to aid in the design of a linear engine prototype. For this, idealizing assumptions were made to treat combustion gases within a single zone as ideal and to approximate mass exchange and heat release with well-known, empirically based models. Practical application of electrical and mechanical elements was then investigated and the research helped to establish the feasibility of continuing the development of a linear engine device.

Another thesis presented in 2002 came out of the University of Minnesota [11]. The DARPA-supported research investigated the feasibility of HCCI free piston engines on a micro-engine scale. Towards this, experimentation and numeric modeling were employed to show that micro-HCCI combustion can be realized for a wide range of operation. The combustion was modeled using detailed kinetics and then coupled to the dynamic piston model of a free piston engine compressor. In the end, the author concludes that a minimum size for such an engine does exist. This lower constraint, though difficult to determine, likely stems from limitations on fabrication and the chemical kinetics leading to combustion.

In 2003, researchers at SNL continued their work in scavenging system optimization for a dual free piston linear engine alternator device. It was seen that scavenging performance is not greatly affected by variations in compression ratio and that best overall performance might be had from a large tank supplying a steady, low temperature charge [8].

In the same year, Australian group Pempek Systems presented the FP3 device [17]. Their work sought to improve linear engine generator designs by eliminating the use of ports while maintaining the two stroke cycle, minimizing the mass of the translator, and maximizing frequency and power density. Their combination of three individual engine devices also decreased overall package size and aided in reducing system vibrations.

In 2004, researchers at Chalmers University of Technology investigated the multi-fuel capabilities of a dual free piston linear engine generator [18]. The work was based on a simulation consisting of piston and thermodynamics, working gas exchanges, and chemical kinetics processes. The potential for increased efficiency and reduced emissions was reinforced. Meanwhile, a 2004 computational study out of the IFP School in France demonstrated that engine indicated efficiencies above %50 and low soot and NO_x emissions can be achieved through high compression ratio, raised EGR rates and the employment of an advanced combustion regime such as HCCI [19].

In 2007, researchers out of the Sir Joseph Swan Institute for Energy Research at Newcastle University in the UK published a broad review of the history of the free piston engine including many of its various applications [20]. The investigation pointed to the potential of the free piston engine as an electrical power generator, and gave heed to the most difficult challenge that has yet to be successfully dealt with: appropriate control of the piston.

In 2008, the same group from Newcastle University, noting that operation of a typical dual free piston linear engine generator is constrained to two-stroke operation, showed through simulation that employing spark ignition in the engine does not lead to noticeable performance advantages over conventional spark ignited crank based engines [21]. The same year, in their examination of compression ignition engines [22], it was seen that, regardless of the general configuration of the free piston engine device, the motion profile of the translator only changes slightly. This motion is generally characterized by its asymmetry and a significantly shorter dead center residence time compared to conventional engines.

This characteristic translator motion profile for the dual free piston linear engine generator is observed in many investigations, including a 2008 investigation out of Shanghai, China [23]. In this, a single zone chemical kinetic and piston dynamic model are combined with a finite element model for the linear alternator to describe the nature of the system using various compression ratios and fuel to air ratios. It was observed that higher compression ratio and lower equivalence ratio led to increased efficiency and lower temperature operation.

Operation of the engine at low temperatures offers many benefits including reduced NO_x emissions. In 2009, the Newcastle University group employed a computational fluid dynamics model to predict emissions from a free piston diesel engine [24]. The results of the study were compared to a similarly scaled conventional diesel engine, and, for the free piston case, NO_x emissions were potentially reduced.

In 2010, the Newcastle University research team reported on their efforts to control the motion of the free piston engine [25]. Their technique simulated the use of feed forward prediction of the piston top dead center position to control fuel flow. This and the accompanying works [26, 27] were put forth to provide a foundation for controlling the free piston motion. It is stated that much work still remains if FPLE generator success is to be realized.

Current Work

Despite the amount of effort that has been applied towards successful FPLE operation, difficult challenges remain. The ability of the engine to self-adjust its compression ratio during operation means that advanced combustion regimes such as HCCI remain a possibility. In this, the translator is free to continue cylinder gas compression until auto-ignition occurs providing a stable feedback between the cylinder gas chemistry and the device itself. At the same time, a single

misfire or sudden increase in load propagates through the system with each stroke and can quickly result in a stall unless the energy of the translator can be maintained or increased until the next combustion event and re-stabilization of the system. In a conventional engine, the flywheel stores substantial energy; in the linear engine, energy storage is minimal when springs are absent. It is thought that higher engine speeds have a positive impact on both stable operation and power density. This research at WVU investigates the use of mechanical springs to directly augment the piston dynamics by increasing system frequency and bolstering the translator through varied cycles of operation.

FUNDAMENTAL SYSTEM MODELING

For this work, the engine geometry is greatly simplified by the lack of valves or ports in the cylinders. Based on the idea that springs can be used to bolster the performance of the linear engine, simple compression springs are added behind each of the piston heads. In reality, other positions and types of springs can be explored. During operation, each spring is pre-compressed between its respective piston head and a dividing wall located centrally with respect to the engine cylinders. The system in question is illustrated in Figure 1.

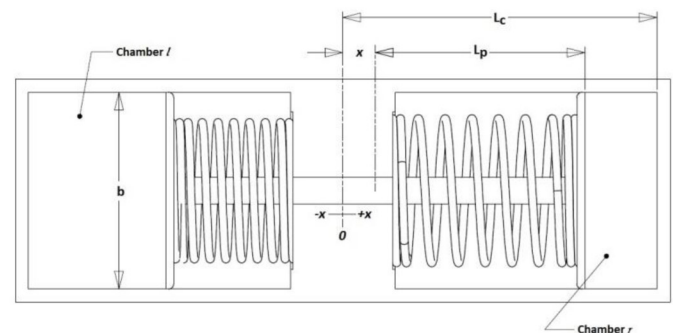


Figure 1. Illustration of simple spring augmented FPLE system

In the figure, L_c represents the length of either cylinder, L_p represents the length of either piston, b represents piston bore diameter, and x describes the variable position of the translator with the system origin found at the center of the engine. The left side of the system, to include the chamber, piston head, and spring, is denoted as side l while the right side is denoted as side r .

The Basic System

The following assumptions apply to the system during this analysis.

- Frictionless translation of piston assembly
- Perfect, linearly variable compression springs
- Adiabatic cylinder walls and piston heads
- No heat transfer through piston heads
- Polytropic compression and expansion of chamber gases (equal coefficients for compression and expansion)

- Constant properties, ideal air for chamber gases
- No leakage of gases from cylinder or around piston heads
- Symmetric system geometry disregarding instantaneous piston position and spring compression

From these assumptions and given the dimensions described above, the following basic variables are used to define the system.

- Translator mass, m
- Translator bore, b
- Translator instantaneous position, x
- Translator instantaneous velocity, v
- Translator instantaneous acceleration, a
- Piston length (translator half-length), L_p
- Cylinder half-length, L_c
- Spring constant, k
- Midpoint pressure, P_{mp}
- Specific heat of chamber gas, C_p
- Specific volume of chamber gas, C_v
- Time, t

From these basic variables, the specific heat and specific volume are constants which define the polytropic expansion and compression of the chamber gases. The midpoint pressure describes the pressure of the chamber gases when the translator is in the center of the system ($x = 0$). If the translator is at rest ($v = 0$) at this position with equal midpoint pressures in both chambers and symmetric springs, the translator will be held static. At this position, the gas and spring forces are equal so that the total summation of the forces acting on the translator is equal to zero. If the translator is moved to the left or right of the origin, a change in the position x , energy is stored in the system in the form of spring potential energy and compressed gas potential energy. If the translator is then released, it will be forced to a new position with a velocity and acceleration dependent upon the parameters listed above. For this scenario, peak translator velocity will be reached when $x = 0$, and the translator will travel to the exact opposite position within the system. In other words, if the translator is moved to a starting position x_l to the left of the origin, it will travel to position x_r to the right of the origin where $x_r = -x_l$ and will adopt a stable oscillation between these positions. The following analysis describes this motion and lays the foundation for the parameterization of the system as an engine with heat addition, heat rejection, and work output.

The midpoint volume of either chamber is given by the following,

$$L = L_c - L_p$$

$$V_{mp} = L \frac{\pi b^2}{4}$$

where L is the static equilibrium length of either chamber.

The instantaneous chamber volume can also be calculated using L .

$$V_l = (L + x) \frac{\pi b^2}{4}$$

$$V_r = (L - x) \frac{\pi b^2}{4}$$

The general form for the polytropic compression and expansion of an ideal gas is given as the following.

$$P_1 V_1^{\frac{C_p}{C_v}} = P_2 V_2^{\frac{C_p}{C_v}}$$

Applying this to the case at hand, the instantaneous chamber pressure can be calculated.

$$P_l = P_{mp} \left(\frac{V_{mp}}{V_l} \right)^{\frac{C_p}{C_v}} = P_{mp} \left(\frac{L \frac{\pi b^2}{4}}{(L + x) \frac{\pi b^2}{4}} \right)^{\frac{C_p}{C_v}} = P_{mp} \left(\frac{L}{L + x} \right)^{\frac{C_p}{C_v}}$$

$$P_r = P_{mp} \left(\frac{L}{L - x} \right)^{\frac{C_p}{C_v}}$$

These chamber pressures can be converted to forces acting on the translator through the area of the piston head.

$$F_{gas,l} = \frac{\pi b^2}{4} P_{mp} \left(\frac{L}{L + x} \right)^{\frac{C_p}{C_v}}$$

$$F_{gas,r} = \frac{\pi b^2}{4} P_{mp} \left(\frac{L}{L - x} \right)^{\frac{C_p}{C_v}}$$

The force acting on the translator due to the springs being compressed is linearly related to the reduction in length of the springs. The change in compressed spring length can be expressed using the free length of the springs, L_{fk} , and system geometry.

$$F_{spring,l} = k(L_{fk} - |x - L_p|)$$

$$F_{spring,r} = k(L_{fk} - |x + L_p|)$$

Making the reasonable assumption that the left piston will never move to the right of the origin and that the right piston will never move to the left of the origin, the absolute values can be eliminated from the equations as follows.

$$F_{spring,r} = k(L_{fk} - (x + L_p))$$

These forces can now be applied to a free body diagram of the translator. This is shown in Figure 2.

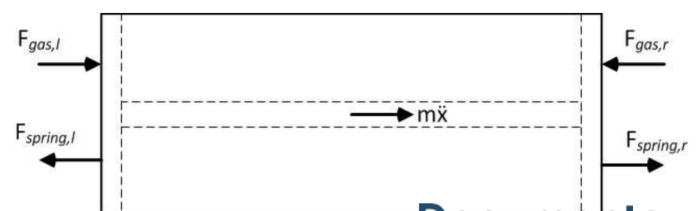


Figure 2. Simple system free body diagram of translator

From this, forces can be summed according to Newton's second law and rearranged to the following form.

$$\frac{\pi b^2}{4} P_{mp} \left[\left(\frac{L}{L+x} \right)^{\frac{C_p}{C_v}} - \left(\frac{L}{L-x} \right)^{\frac{C_p}{C_v}} \right] - 2kx = m\ddot{x}$$

$$x = x(t) \quad (1)$$

Forming nondimensional terms aids in gaining a fundamental understanding of the system parameters and resulting behaviors.

Examining the form of the analytical solution, apparent nondimensional terms are available.

- The specific heat ratio, a given constant for the ideal air assumption

$$\gamma = \frac{C_p}{C_v}$$

- The dimensionless translator position with theoretical bounds of $(-1,1)$

$$x^* = \frac{x}{L}$$

- The pressure ratio

$$R_p = \frac{\pi b^2 P_{mp}}{4Lk}$$

- The dimensionless time

$$t^* = \frac{t}{\sqrt{\frac{m}{k}}}$$

The analytical solution can be rewritten applying these terms.

$$\frac{R_p}{(1+x^*)^\gamma} - \frac{R_p}{(1-x^*)^\gamma} - 2x^* = \frac{d^2 x^*}{dt^{*2}} = a^* \quad (2)$$

Solving this equation analytically for the position as a function of time proves difficult. However, introducing the dimensionless velocity yields an analytical solution for the dimensionless velocity as a function of dimensionless position.

$$v^* = \frac{dx^*}{dt^*}$$

$$\frac{d^2 x^*}{dt^{*2}} = v^* \frac{dv^*}{dx^*}$$

$$\frac{R_p}{(1+x^*)^\gamma} - \frac{R_p}{(1-x^*)^\gamma} - 2x^* = v^* \frac{dv^*}{dx^*} \quad (3)$$

This can be integrated to yield the following.

$$\frac{1}{2} v^{*2} = \frac{-R_p}{\gamma-1} [(1+x^*)^{1-\gamma} + (1-x^*)^{1-\gamma}] - x^{*2} - C^*$$

$$C^* = \frac{-R_p}{\gamma-1} [(1+x_l^*)^{1-\gamma} + (1-x_l^*)^{1-\gamma}] - x_l^{*2} \quad (4)$$

The constant of integration, C^* , is found by applying the initial condition, $x^* = x_l^*$, $v^* = 0$. This condition indicates that the translator is moved to the left then released from its initial position with an initial velocity of zero.

To break down the nature of the motion further, this system is composed of two distinct cases. For each case, the governing equation can be arrived at in a similar fashion.

The first case is that which does not involve the influences of the mechanical springs. In essence, this is the simplified form of the free piston engine system previously investigated by WVU [1, 3, 5], SNL [2, 6, 7], and other research groups [16, 20]. The analysis yields the following governing equation.

$$\frac{1}{2} v^{*2} = \frac{-R_p}{\gamma-1} [(1+x^*)^{1-\gamma} + (1-x^*)^{1-\gamma}] - C^*$$

$$C^* = \frac{-R_p}{\gamma-1} [(1+x_l^*)^{1-\gamma} + (1-x_l^*)^{1-\gamma}] \quad (5)$$

The second case is that which does not involve the influences of the chamber pressures on the translator. Essentially this is a simple mass spring system and the governing equation is found to be:

$$\frac{1}{2} v^{*2} = -x^{*2} + x_l^{*2} \quad (6)$$

The total system is then just a linear combination of the two individual systems.

For each of these cases, the dimensionless velocity can be calculated as a function of the dimensionless position given values for the starting position and pressure ratio. Example plots of these can be seen in Appendix [Figure 5](#).

The idealizations in place result in symmetry of the velocity profile curves. This implies that the stroke length is twice the absolute value of the starting position. Second, adjectives such as "steep sides" and "flatness" can be used to describe the shapes of the curves and actually correspond to the nature of the acceleration. In the springs-only case, the starting position only affects the magnitude of the curve and not the general shape of the curve which is more rounded indicating a more linear acceleration of the translator. In the case where only cylinder pressures influence translator travel, the starting position and the pressure ratio affect both the magnitude of the curve and the flatness. As the stroke is increased, the sides of the curve steepen indicating a much more rapid acceleration of the translator due to the rapid increase of gas pressures in.

either of the chambers. Meanwhile, the acceleration during the middle of the stroke is less making the velocity profile curve flatter on the top. The pressure ratio is proportionally related to the magnitude of the curve, but as the theoretical limit of the stroke is approached, the effects of the midpoint pressure are significantly reduced in comparison to the maximum chamber pressures that are reached.

Adding Heat and Work

A model that includes work output, heat addition, and heat rejection is addressed next. A few simplifying assumptions are made to facilitate building this model.

- The engine chambers experience opposing, ideal Otto cycles including the following reversible processes:
 - Constant volume heat addition
 - Isentropic expansion with work output
 - Constant volume heat rejection
 - Isentropic compression
- Heat addition and rejection occurs instantaneously at top and bottom dead center piston positions
- Isentropic expansion and compression processes are polytropic with the polytropic constant being the ratio of specific heats for both expansion and compression
- Chamber gases behave according to cold air ideal gas assumption
- Work output is represented by an average friction force over the length of the translator stroke

Consider the case that the initial position of the translator is to the left of the origin ($-1 < x_i < 0$). Thus the initial pressure and volume in the left chamber are known.

$$P_{l,1} = P_{mp} \left(\frac{L}{L + x_i} \right)^\gamma \quad V_{l,i} = (L + x_i) \frac{\pi b^2}{4}$$

Constant volume heat addition to the left chamber increases the temperature and pressure within that chamber. If the chamber is considered to be a closed system with no work or changes in potential and kinetic energy, the first law energy analysis can be written in the following form.

$$Q_{in} = \Delta U = m_{air} C_v (T_2 - T_1)$$

In this form, T_1 represents the temperature of the gas before heat addition and T_2 represents the gas temperature after heat addition. In both cases the ideal gas law applies.

$$Q_{in} = m_{air} C_v \left(\frac{P_{l,2} V_{l,i}}{m_{air} R} - \frac{P_{l,1} V_{l,i}}{m_{air} R} \right) = \frac{V_{l,i} C_v}{R} (P_{l,2} - P_{l,1})$$

$$R = C_p - C_v$$

$$Q_{in} = \frac{V_{l,i} C_v}{C_p - C_v} (P_{l,2} - P_{l,1}) = \frac{V_{l,i}}{\gamma - 1} (P_{l,2} - P_{l,1})$$

$$Q_{in} = \left(\frac{\pi b^2}{4} \right) \left(\frac{L + x_i}{\gamma - 1} \right) \left[P_{l,2} - P_{mp} \left(\frac{L}{L + x_i} \right)^\gamma \right]$$

This can be rearranged to represent the chamber pressure before expansion as a function of the heat addition.

$$P_{l,2} = Q_{in} \frac{4(\gamma - 1)}{\pi b^2 (L + x_i)} + P_{mp} \left(\frac{L}{L + x_i} \right)^\gamma$$

Now, because chamber gases expand polytropically starting at pressure $P_{l,2}$, the pressure in the left chamber can be calculated for any position of the translator via the following.

$$P_{l,2} V_{l,i}^\gamma = P_l V_l^\gamma$$

$$P_l = \left[Q_{in} \frac{4(\gamma - 1)}{\pi b^2 (L + x_i)} + P_{mp} \left(\frac{L}{L + x_i} \right)^\gamma \right] \left(\frac{L + x_i}{L + x} \right)^\gamma$$

From which the force of the gas in the left chamber can be calculated.

$$F_{gas,l} = \frac{Q_{in}(\gamma - 1)(L + x_i)^{\gamma-1}}{(L + x)^\gamma} + \frac{\pi b^2}{4} P_{mp} \left(\frac{L}{L + x} \right)^\gamma$$

The free body diagram of the translator can be reexamined to include the friction force (Figure 3).

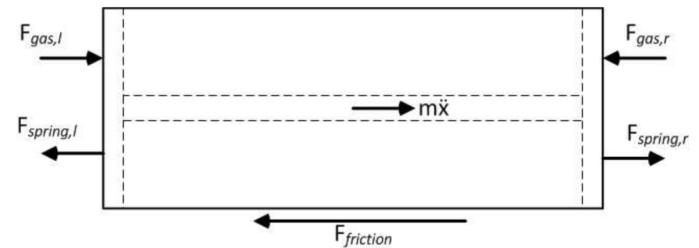


Figure 3. Free body diagram of left-to-right translator travel including friction

The force balance can be written as follows using equation (1) as a reference.

$$\left[\frac{Q_{in}(\gamma - 1)(L + x_i)^{\gamma-1}}{(L + x)^\gamma} + \frac{\pi b^2}{4} P_{mp} \left(\frac{L}{L + x} \right)^\gamma \right] - \frac{\pi b^2}{4} P_{mp} \left(\frac{L}{L - x} \right)^\gamma - 2kx - F_f = m\ddot{x}$$

This can be simplified to the following.

$$\frac{Q_{in}(\gamma - 1)(L + x_i)^{\gamma-1}}{(L + x)^\gamma} + \frac{\pi b^2}{4} P_{mp} \left[\left(\frac{L}{L + x} \right)^\gamma - \left(\frac{L}{L - x} \right)^\gamma \right] - 2kx - F_f = m \frac{d^2 x}{dt^2}$$

(7)



Previously, dimensionless terms were used to reduce the number of input parameters required in fundamental analysis of the engine system. Using the same dimensionless terms, equation (7) can be rewritten into dimensionless form in the following way.

$$\frac{Q_{in}}{kL^2} \left[\frac{(\gamma - 1)(1 + x_l^*)^{\gamma-1}}{(1 + x^*)^\gamma} \right] + R_p \left[\frac{1}{(1 + x^*)^\gamma} - \frac{1}{(1 - x^*)^\gamma} \right] - 2x^* - \frac{F_f}{kL} = \frac{d^2 x^*}{dt^{*2}} = a^* \quad (8)$$

From this, two dimensionless parameters are apparent.

- The heat addition ratio

$$R_{Qin} = \frac{Q_{in}}{kL^2}$$

- The work output ratio

$$R_W = \frac{F_f}{kL}$$

The velocity transformation seen in equation (3) can be applied to equation (8) to allow for integration and a closed form solution for the dimensionless velocity as a function of dimensionless translator position.

$$a^* = v^* \frac{dv^*}{dx^*} = R_{Qin} \left[\frac{(\gamma - 1)(1 + x_l^*)^{\gamma-1}}{(1 + x^*)^\gamma} \right] + R_p \left[\frac{1}{(1 + x^*)^\gamma} - \frac{1}{(1 - x^*)^\gamma} \right] - 2x^* - R_W \quad (9)$$

Integrating this and solving for the constant of integration with the initial condition (when $x^* = x_l^*$, $v^* = 0$) yields equation (10).

$$\frac{1}{2} v^{*2} = -R_{Qin} \left[\frac{(1 + x_l^*)^{\gamma-1}}{(1 + x^*)^\gamma} \right] - \frac{R_p}{\gamma - 1} [(1 + x^*)^{1-\gamma} + (1 - x^*)^{1-\gamma}] - x^{*2} - R_W x^* - C^* \\ C^* = -R_{Qin} - \frac{R_p}{\gamma - 1} [(1 + x_l^*)^{1-\gamma} + (1 - x_l^*)^{1-\gamma}] - x_l^{*2} - R_W x_l^* \quad (10)$$

Equation (9) is important because it represents the analytical solution for the dimensionless acceleration as a function of the dimensionless position and the various input parameters. If no heat or work are added (i.e. $R_Q = R_W = 0$), equations (9) and (10) reduce to the simple cases in which only springs and symmetric gas pressures affect the motion of the translator. An example is shown in Figure 6 in the Appendix.

For Figure 6, Figure 7, Figure 8, Figure 9, Figure 10, Figure 11, Figure 12, Figure 13, Figure 14, Figure 15, it is of note that the circular shaped marker indicates the starting location of the

translator, the square shaped marker indicates the ending location, and the diamond shaped marker on the position/velocity plot indicates the peak velocity of the translator. The effects of the heat and work ratios will now be examined in some depth.

Given a starting position and pressure ratio, adding heat to the left chamber (increasing the heat ratio) will increase the initial pressure in the left chamber. This increased pressure causes asymmetry of the velocity profile, and if no work is removed, will increase both the magnitude of the velocity profile and the compression ratio of the left chamber. This is shown in Figure 7 in the Appendix.

Conversely, the heat ratio can be kept at zero while the work ratio is raised. This will have the opposite effect and will serve to skew the velocity profile towards the left side of the system, lowering the compression ratio and velocity profile magnitude. Figure 8 in the Appendix demonstrates this.

So, it is seen that heat input and work out of the system result in asymmetry of the velocity profile and cause changes in compression ratio of the system. It is more interesting to discover what happens to the translator over many operation cycles rather than just one stroke. To do this, an equation must be derived for the return stroke, from right to left. To do this, the position variable is denoted as follows.

- Dimensionless left side translator position when translator velocity equals zero ($v^* = 0$)
 x_l^*
- Dimensionless right side translator position when translator velocity equals zero ($v^* = 0$)
 x_r^*

These represent the two dead center positions. For a given stroke, the translator might start at dead center left, x_l^* then travel to dead center right, x_r^* the position of the translator at the end of the stroke. From this position, x_r^* becomes the starting position for the return stroke while x_l^* becomes the ending position.

For translator travel from right to left, the free body diagram changes to Figure 4.

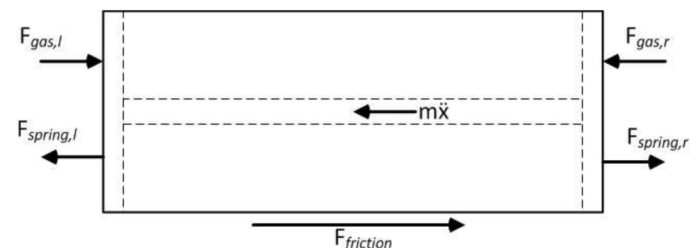


Figure 4. Free body diagram of translator traveling from right to left

Applying the same approach as before, the system equations become the following (equation (11) and (12)).