

# An Analytical Model of a Two-Phase Jet with Application to Fuel Sprays in Internal Combustion Engines

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## ABSTRACT

The paper presents an analytical two-dimensional model of two-phase turbulent jets with focus on fuel sprays in internal combustion engines. The developed model allows prediction of the fuel spray parameters including local fuel concentration and mixture velocity.

The model proposed in this paper is based on the single-phase steady-state laminar axisymmetric jet flow field solution by Schlichting. This solution is amended to include transport of the discontinuous fuel phase in a stagnant air in the limit of a dilute fuel concentration. This two-phase jet flow model admits a closed form analytical solution for the fuel concentration distribution. This solution is then applied to turbulent jet flow as per the approach described by Schlichting and in other studies, and used to predict point-wise properties of fuel sprays in internal combustion engines.

The results of model simulations are compared with the available experimental data. It was found that the analytical model predicts satisfactorily spray properties without additional assumptions or fitting coefficient.

The advantage of the developed model is in its simplicity and rigor in comparison with empirical and numerical models.

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## INTRODUCTION

Increasingly stringent demands and regulations imposed on engine emissions and efficiency pose challenges for diesel engine manufacturers. To meet these demands engine developers perform computer-aided optimizations to improve the mixture formation and combustion processes, which usually rely on predictions of various physical processes of mixture formation and combustion in the engine [1]. Knowledge of the temporal and spatial behavior of fuel sprays is critical for prediction of engine performance and improvement of its design. Despite the great effort spent on fuel spray modeling, complex phenomena that take place at short time and length scales in diesel fuel sprays make spray modeling a difficult task.

Increasing computing power brings a growing use in computational fluid dynamics (CFD) models, which are a helpful tool in analyzing engine performance. Sprays in such CFD codes are widely modeled by Lagrangian-drop and Eulerian-Fluid method (LDEF) with some models adopting the Eulerian-liquid-Eulerian-gas (two-fluid) method [2]. In the LDEF method spray is divided into parcels containing a predefined mass of fuel that move within the gas-phase grid. The properties of the parcels are coupled with those of the gaseous

phase at every time step via mass, momentum and energy transfer between the phases. In the two-fluid method, no additional cells are used and the fuel mass is added to the air contained in the grid [1]. Both methods allow calculation of local and general spray properties in combustion chambers of internal combustion engines.

The accuracy of CFD models strongly depends on mesh selection. A CFD model poorly predicts even general (integral) properties of spray structures when a coarse CFD mesh is used [2, 3, 4]. Normally in these models the mesh sizes is chosen to decrease near the nozzle down to nozzle size resolution. Notwithstanding the results are sensitive with respect to the mesh size choice and still pose problems for engine designers, particularly if the engine size is either too small or too large for using conventional grid sizes [2].

To reduce numerical grid dependency in CFD models, different spray models have recently been developed [4, 5, 6, 7] which do not rely on CFD calculations but rather on analytical or empirical correlations. Some of these models combine CFD calculation of several spray properties with determination of the remaining properties from correlations. Such is the LDEF model of Abani et al. [2], where the drag force between gas

and fuel is calculated from the axial component of the ambient gas velocity derived from the gas jet theory rather than calculated in every time step. This approach is reported to significantly reduce numerical grid dependency in CFD models.

In other LDEF models properties of spray parcels are implied mostly from predictions of analytical or empirical studies of macroscopic spray properties [4], [6]. These models offer an advantage of reduced grid size sensitivity. Another advantage is usage of smaller computing resources (power or time) compared to the CFD models. Even with the available computing power nowadays, the above advantage is important in optimizations of injection parameters, where the spray flow models are used for calculations at various operating conditions. Therefore, analytical and empirical studies of spray properties, and the related models of spray behavior may contribute to the capability of CFD models to predict engine performance and perform optimizations. In this respect analytical fuel spray models, being a versatile tool requiring negligible computational resources, are always preferable.

In the empirical LDEF models, the local dynamic properties of sprays, namely parcel velocity components and fuel concentration within the parcel, are calculated using two macroscopic spray characteristics: spray tip penetration and spray dispersion angle [4], [6]. Using these two properties, an array of parcels traveling within the gas-phase grid can be defined. Axial distributions of velocity and fuel concentration are determined either by analytical or empirical correlations, or by using 1D equations of conservation of mass and momentum [4], [6]. The radial distributions of velocity and fuel concentration are assumed to be characterized by a constant spray dispersion angle [4], [6] or having Gaussian shape [5]. The radial distribution determines the ratio of velocity and fuel concentration between adjacent cells. This approach offers a method of simulation of a transient spray based on empirical correlations with little computing effort and grid size insensitivity. However it is important to notice that radial distributions of mixture velocity and fuel concentration are calculated using two approximations. Namely (i) the spray dispersion angle is calculated using empirical constants, and (ii) the distributions are then assumed to be certain functions of this angle.

Correlations for fuel spray penetration  $S$  [6, 7, 8, 9, 10, 11, 12, 13, 14] were obtained either analytically or empirically, being heavily based on the results of experiments. Several analytical studies considered spray as a two-phased jet with predetermined velocity distribution [11], [14], such as the work of Wakuri et al., with the resulting correlation for spray penetration:

$$S = \left( \frac{2C_c \Delta p}{\rho_a} \right)^{0.25} \left( \frac{2tr_i}{\tan\left(\frac{\theta}{2}\right)} \right)^{0.5} \quad (1)$$

where  $t$  is the time variable,  $\Delta p$ ,  $\rho_a$ , are the inlet pressure drop and the gas density respectively,  $r_i$  is the inlet jet radius, and  $\theta$  is the spray dispersion angle,  $C_c$  is the coefficient of contraction, defined below.

Hiroyasu and Arai [9] obtained  $S$  by solving the equations of conservation of momentum in the spray under the assumption of Levich's break-up length of high velocity liquid jets in the spray, which resulted in the correlation:

$$S = 2.95 \left( \frac{\Delta p}{\rho_a} \right)^{0.25} (2tr_i)^{0.5} \quad (2)$$

Other correlations either analytical or empirical [10] also obey the relation:

$$S \propto \left( \frac{\Delta p}{\rho_a} \right)^{0.25} (tr_i)^{0.5} \quad (3)$$

Correlations for spray dispersion angle obtained in several studies [15], [16], [9], [11] vary greatly in their functional forms. Some were obtained from experiments [16], whereas other hinge on fuel mass conservation equations appearing in one-dimensional spray models, such as Hiroyasu and Arai [9] and Wakuri et al. [11]. Naber & Siebers [10] proposed a correlation for  $\theta$  by examining experimental data collected for various operating conditions of spray penetration.

Correlations for cross-sectionally averaged fuel concentration  $\overline{C_f}$  were obtained on either conservation of fuel mass in one-dimensional spray models, Wakuri et al. [11], resulting in the formula:

$$\overline{C_f} = \frac{\sqrt{C_c \rho_f \rho_a r_i}}{x \tan\left(\frac{\theta}{2}\right)} \quad (4)$$

where  $\rho_f$  is the fuel drop material density. In contrast, other correlations [6], [4] in which the spray is regarded as an array of parcels were obtained by conservation of momentum in sprays, with incorporating correlations of spray penetration to determine the local mixture velocity in the spray. The correlation of Wakuri et al. was compared with the experimental data of Kamimoto et al. [17], which will also serve as a basis for validation of the present model in this paper.

Experimental correlations also exist for the radial distributions of mixture velocity and fuel concentration. Commonly [14], [18] a Gaussian profile is assumed for both the velocity and fuel concentration:

$$u_x(x, r) = u_x(x, 0) \exp \left( -\lambda \left( \frac{r}{x \tan(\theta/2)} \right)^2 \right)$$

$$C(x, r) = C(x, 0) \exp \left( -\lambda Sc \left( \frac{r}{x \tan(\theta/2)} \right)^2 \right) \quad (5)$$

Where  $x$ ,  $r$  are the axial and radial coordinates within the jet,  $Sc$  is the droplets' Schmidt number (see definition below) and a constant  $\lambda$  was determined experimentally.

It can be seen that in order to calculate the radial distribution of velocity and fuel concentration from equation (5) the spray dispersion angle must be measured or correlated. These correlations of radial distributions were validated with experiments in the above mentioned works

It may therefore be seen that existing empirical correlations enable calculation of integral and averaged spray properties. However, no existing correlation allows calculation of local spray properties, in particular local fuel concentration and local mixture velocity within the spray. In other words, in order to calculate local spray properties with existing analytical or empirical correlations, one must rely on either measurements or correlations of the spray dispersion angle, as well as choosing values for constants and assuming functional dependence of several spray quantities.

The goal of this study is to develop an analytical model of two-phase turbulent jets capable of predicting local spray properties without the need for additional correlations of general spray properties. We develop an analytical model for spatial distribution of mixture velocity and drop concentration in fuel sprays and compare the results with the available experimental data.

## DEVELOPMENT OF THE MODEL

### Physical Model and Assumptions

We consider fuel injected in the combustion chamber as a two-phase axisymmetric circular jet, flowing into an effectively semi-infinite gas space, as schematically depicted in Figure 1.

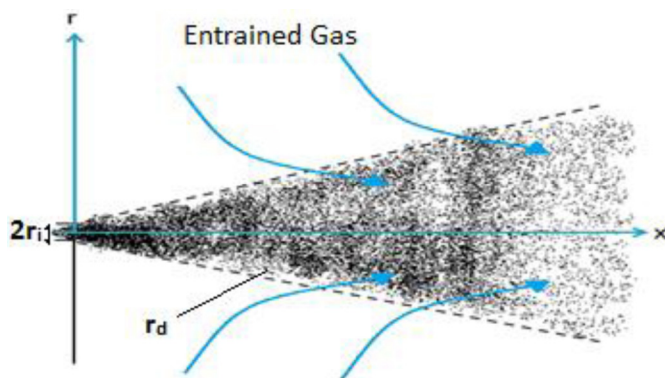


Figure 1. A schematic of the physical model of the fuel jet flow

The jet inlet is assumed to have point size. This implies that the model will be valid for axial distances far exceeding the nozzle inlet diameter. It is assumed also that the spray consists only of fuel droplets suspended in a gas, as the conditions of fuel injection in internal combustion engines correspond to the atomization break-up regime [19]. As such, the jet disintegrates into droplets at a very short distance from the nozzle. The droplets constantly collide and break-up into smaller ones. Yet, the droplet size distribution is rather narrow in comparison to the average droplet radius, as was observed in the internal combustion engines [20].

The fuel ( $f$ ) and gas ( $a$ ) concentrations  $C_f$  and  $C_a$  are related to the respective fuel and gas densities  $\rho_f$  and  $\rho_a$  via the relations:

$$C_f = \alpha \rho_f, \quad C_a = (1 - \alpha) \rho_a, \quad (6)$$

where  $\alpha$  and  $(1 - \alpha)$  are the corresponding volumetric fractions of fuel and ambient gas, respectively. We consider the region of the spray where  $\alpha$  is sufficiently small so that the effect of droplets on the gas momentum transfer is negligible. In other words, the droplets are considered as passive tracers in the moving gas.

Droplets are assumed to experience only the drag force, and therefore their deceleration is expressed as:

$$\frac{d\mathbf{u}_d}{dt} = \frac{\mathbf{F}_d}{m_d} = -\frac{3}{8r_d} C_D \frac{\rho_a}{\rho_f} |\mathbf{u}_d - \mathbf{u}_a| (\mathbf{u}_d - \mathbf{u}_a) \quad (7)$$

where  $r_d$  is the droplet radius;  $\mathbf{u}_d$  and  $\mathbf{u}_a$  are the droplet and gas velocities, respectively;  $C_D$  is the drag coefficient, which dependence on the droplet Reynolds number is established empirically [21]:

$$Re = \frac{2|\mathbf{u}_d - \mathbf{u}_a| \rho_a r_d}{\mu_a} \quad (8)$$

For Reynolds numbers smaller than 1 the drag coefficient is described by  $C_D = 24/Re$ . For Reynolds numbers above 1000  $C_D = 0.44$

During this process the droplets bring the surrounding gas into motion [22], [12]. After a short period of time (of the order of the droplets' relaxation time) the ambient gas' velocity and the fuel velocity equalize and the injection reaches a steady state. The time during which a steady state flow regime is achieved is related to the droplets characteristic relaxation time which for 30  $\mu\text{m}$  drops is of the order of several hundreds of microseconds. Practically it is significantly shorter since the droplets break up during their motion, resulting in smaller droplets, having smaller relaxation time. Therefore, the relaxation time during which a typical droplet will accommodate

to the moving gas is small compared to the injection duration. Since fuel injection is an unsteady process, each droplet in the spray penetrates to a certain distance and accelerates the air around it. If we consider axial distances less than the spray penetration distance  $S$ , it may be safely assumed that the jet is at a steady state. This assumption also implies that in the laminar flow regime the gas-fuel relative velocity vanishes and therefore a single flow field may be considered:

$$\mathbf{u}_f = \mathbf{u}_a = \mathbf{u} \quad (9)$$

In internal combustion engines, movement of the piston and gas flow through valves in the combustion chamber cause movements in the ambient gas surrounding the jet. Also, in some combustion chambers, particularly in smaller engines, droplets may impinge the inner chamber walls. The present model is limited to the circumstances where the effect of this gas motion on the fuel injection is weak and the fuel jet effectively develops in a quiescent gas. Also, it is assumed that the combustion chamber to which fuel is injected is large enough or injection process is designed to prevent fuel droplets from reaching any walls.

Drops with sizes below one micron are involved in molecular Brownian motion, resulting from collisions with fast-moving gas molecules [21]. Such impingements instantaneously accelerate the droplet relative to the surrounding gas, thereby causing diffusive droplet flux, governed by the droplets Brownian Diffusivity  $D$ . Brownian motion and diffusive flux of drops with sizes exceeding one micron is rather weak and, in the absence of external forces, droplets closely follow the gas flow trajectory. On the other hand, in circumstances where the characteristic jet Reynolds number is rather high (about  $10^4$  [23]) the jet flow becomes turbulent short distance after exiting the injection nozzle [19]. In such a situation turbulent gas motion combined with droplet inertia creates droplet-gas relative motion. The concomitant droplet turbulent flux is characterized by the corresponding turbulent diffusivity [24].

We first consider a laminar two-phase jet propagating in a gas having molecular viscosity  $\nu_a$ . Then we use this solution in situation where the jet is turbulent upon replacement of  $\nu_a$  with the appropriate turbulent viscosity  $\nu_t$ , in accordance with the solutions of Schlichting [24] and other studies [25].

### Inlet Spray Properties

Fuel injection is a process which includes several physical phenomena, including fuel jet motion, its break-up, droplet formation and their propagation within the chamber. When the fuel jet leaves the nozzle hole, it is assumed to have a circular cross-section with a diameter proportional to the nozzle-hole radius  $r_p$  and a uniform initial velocity  $u_i$ . The ratio between the cross-sectional areas of the jet and the nozzle is the coefficient of contraction  $C_c$  [10]. The initial velocity of the fuel jet is the product of the ideal velocity determined by the Bernoulli equation [26] and the coefficient of velocity of the nozzle  $C_v$ :

$$u_i = C_v \sqrt{2\Delta p / \rho_f} \quad (10)$$

By determining the inlet velocity of the fuel jet as in [equation \(10\)](#), the typical expressions for the inlet fuel mass flux and momentum flux are obtained:

$$\begin{aligned} \dot{m}_{fi} &= C_c \pi r_i^2 \rho_f u_i = \pi C_v C_c r_i^2 \sqrt{2\rho_f \Delta p} \\ \dot{J}_i &= C_c \pi r_i^2 \rho_f u_i^2 = 2\pi C_v^2 C_c r_i^2 \Delta p \end{aligned} \quad (11)$$

### Governing Equations

The flow of each phase (fuel and ambient gas) can be generally described by the laws of conservation of mass and momentum, each governing spatial and temporal evolution of fuel phase mass density and flow velocity.

The steady-state general equations of conservation is written in the approximation of a small volumetric fuel density  $\alpha \ll 1$  and the ensuing condition (8), justifiable sufficiently far from the jet inlet. Namely we will use the following equations [27]:

1. Conservation of mass for the two phases:

$$\begin{aligned} \nabla \cdot (C_f \mathbf{u}) &= D_f \nabla^2 C_f \\ \nabla \cdot (C_a \mathbf{u}) &= D_a \nabla^2 C_a \end{aligned} \quad (12)$$

where  $D_f$  and  $D_a = D_f = D$  are coefficients of diffusions of the binary fuel-air mixture.

2. The steady-state conservation of momentum for the two phases:

$$\begin{aligned} \nabla \cdot (C_a \mathbf{u} \mathbf{u}) &= -\nabla p_a + \nabla \cdot \boldsymbol{\tau}_a + \mathbf{F} \\ \nabla \cdot (C_f \mathbf{u} \mathbf{u}) &= -\mathbf{F} \end{aligned} \quad (13)$$

In [equation \(13\)](#)  $p_a$  is the gas pressure, and  $\mathbf{F}$  is the volumetric force density vector acting on the gas phase and stemming from the drag force acting on the droplets. The hydrodynamic Newtonian stress tensor  $\boldsymbol{\tau}_a$  is characterized by an effective mixture viscosity  $\mu$ , dependent on the fuel volume fraction [28]. In the dilute fuel volume fraction limit, the pressure and stress tensor induced by the fuel phase is neglected. Furthermore, by the same reason we set in [equation \(13\)](#)  $1 - \alpha \approx 1$ ,  $\mathbf{F} = 0$ , and neglect the dependency on  $\alpha$  of all terms therein. Thus, the effective mixture viscosity  $\mu$  is very close to the constant gas viscosity  $\mu_a$ .

We also note that in view of (6) the sum of [equations \(12\)](#) yields the usual mass continuity equation for the whole mixture.

$$\nabla \cdot \mathbf{u} = 0 \quad (14)$$

The above relation will be used in place of the second equation (12).

For circular nozzle geometry the jet is axially symmetric and the above equations (12), (13), (14) may be rewritten in the forms:

$$u_x \frac{\partial \alpha}{\partial x} + u_r \frac{\partial \alpha}{\partial r} = D \left[ \frac{\partial^2 \alpha}{\partial x^2} + \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \alpha}{\partial r} \right) \right] \quad (15)$$

$$\frac{\partial}{\partial x} (\alpha u_x^2) + \frac{1}{r} \frac{\partial}{\partial r} (r \alpha u_x u_r) = 0 \quad (16)$$

$$\frac{\partial}{\partial x} (u_x^2) + \frac{1}{r} \frac{\partial}{\partial r} (r u_x u_r) = - \frac{1}{\rho_a} \frac{\partial p}{\partial x} + \nu_a \left[ \frac{\partial^2 u_x}{\partial x^2} + \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial u_r}{\partial r} \right) \right] \quad (17)$$

$$\frac{\partial u_x}{\partial x} + \frac{1}{r} \frac{\partial (r u_r)}{\partial r} = 0 \quad (18)$$

This system should generally be amended by the corresponding equation for the radial momentum components. However, as will be shown below it is unnecessary in view of the boundary layer nature of the jet assumed below.

For sufficiently long distances from the inlet, as expressed by the requirement that the characteristic jet Reynolds number greatly exceeds one, the radial jet width is significantly smaller than the corresponding axial distance from the inlet and also  $u_x \gg u_r$  [24]. Therefore the radial momentum and mass conservation equations contain terms which are significantly smaller than those appearing in the axial transport equations. Furthermore, performing usual order of magnitude analysis in the axial transport equations, one obtains the equations of the two-phase jet in the following form:

$$u_x \frac{\partial \alpha}{\partial x} + u_r \frac{\partial \alpha}{\partial r} = \frac{D}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \alpha}{\partial r} \right), \quad (19)$$

$$\frac{\partial}{\partial x} (r u_x^2) + \frac{\partial}{\partial r} (r u_x u_r) = \nu_a \frac{\partial}{\partial r} \left( r \frac{\partial u_x}{\partial r} \right), \quad (20)$$

to be considered jointly with (18). The boundary conditions for the above systems are:

$$\begin{aligned} r = 0: \quad u_r = 0; \quad \frac{\partial u_x}{\partial r} = 0; \quad \frac{\partial \alpha}{\partial r} = 0 \\ r \rightarrow \infty: \quad u_x \rightarrow 0; \quad \alpha \rightarrow 0 \end{aligned} \quad (21)$$

### Solution for Steady State Two-Phase Laminar Jet

The governing equations (18), (20) were solved by Schlichting [24] for the single phase axisymmetric jet. A solution for the fuel volumetric fraction can then be obtained by using the equation of conservation of mass for the fuel phase (equation (19)).

Integrating the equation of conservation of mass for fuel (equation (19)) over the whole jet width and using conditions (21), we get:

$$2\pi \int_0^\infty \rho_f \alpha u_x r dr = \text{Const} = \dot{m}_{fi} \quad (22)$$

This means that the fuel mass flow in every cross-section of the spray is constant and equal to the fuel mass flux at the nozzle outlet,  $\dot{m}_{fi}$ .

Integrating the sum of the equations of conservation of momentum for the two phases jointly with (21) yields the condition of conservation of momentum flux in the axial direction, which is equal to the momentum flux at the nozzle outlet  $\dot{J}_i$ :

$$\begin{aligned} 2\pi \rho_a \int_0^\infty u_x^2 r dr + \\ 2\pi (\rho_f - \rho_a) \int_0^\infty \alpha u_x^2 r dr = \text{Const} = \dot{J}_i \end{aligned} \quad (23)$$

In view of the smallness of the fuel volume fraction, the second integral term in equation (23) may be neglected. The requirement for a small volume fraction can thus be quantified as follows:

$$\alpha \ll \frac{\rho_a}{\rho_f - \rho_a} \quad (24)$$

Equations (18), (20) can be solved by defining a stream function  $\Psi$  similarly to the case of the single-phase laminar jet [24], where:

$$r u_x = \frac{\partial \Psi}{\partial r}, \quad r u_r = - \frac{\partial \Psi}{\partial x} \quad (25)$$