

An Efficient Method to Calculate the Failure Rate of Dynamic Systems with Random Parameters Using the Total Probability Theorem

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ABSTRACT

Using the total probability theorem, we propose a method to calculate the failure rate of a linear vibratory system with random parameters excited by stationary Gaussian processes. The response of such a system is non-stationary because of the randomness of the input parameters. A space-filling design, such as optimal symmetric Latin hypercube sampling or maximin, is first used to sample the input parameter space. For each design point, the output process is stationary and Gaussian. We present two approaches to calculate the corresponding conditional probability of failure. A Kriging metamodel is then created between the input parameters and the output conditional probabilities allowing us to estimate the conditional probabilities for any set of input parameters. The total probability theorem is finally applied to calculate the time-dependent probability of failure and the failure rate of the dynamic system. The proposed method is demonstrated using a vibratory system. Our approach can be easily extended to non-stationary Gaussian input processes.

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INTRODUCTION

The response of a vibratory system with random parameters excited by stationary Gaussian processes is a non-stationary random process. A time-dependent reliability analysis is thus, needed to calculate the probability that the system will perform its intended function successfully for a specified time.

Reliability is an important engineering requirement for consistently delivering acceptable product performance through time. As time progresses, the product may fail due to time-dependent operating conditions and material properties, component degradation, etc. The reliability degradation with time may increase the lifecycle cost due to potential warranty costs, repairs and loss of market share. In this article, we use time-dependent reliability concepts associated with the first-passage of non-repairable systems. Among its many applications, the time-dependent reliability concept can be used to reduce the lifecycle cost [1] or to set a schedule for preventive condition-based maintenance [2].

The time-dependent probability of failure, or cumulative probability of failure [1, 3], is defined as

$$P_f(0, T) = P\{\exists t \in [0, T]: g(\mathbf{X}, \mathbf{F}(t), t) \leq 0\} \quad (1)$$

where the limit state $g(\mathbf{X}, \mathbf{Z}(t), t) = 0$ depends on the vector $\mathbf{X} = [X_1 X_2 \dots X_m]$ of m input random variables, the vector $\mathbf{F}(t) = [F_1(t) F_2(t) \dots F_n(t)]$ of n input random processes. Failure occurs if $g(\cdot) \leq 0$ at any time $t \in [0, T]$ where T is the time of interest.

The time-dependent probability of failure of Eq. (1) can be calculated exactly as

$$P_f(0, t) = 1 - \left(1 - P_f^0\right) \exp\left\{-\int_0^t \lambda(\xi) d\xi\right\} \quad (2)$$

where P_f^0 is the instantaneous probability of failure at the initial time and $\lambda(t)$ is the failure rate. Eq. (2) indicates that $P_f(0, T)$ can be calculated if $\lambda(t)$ is known and vice versa.

In the commonly used up-crossing rate approach, the failure rate is approximated by the up-crossing rate

$$v^+(t) = \lim_{\Delta t \rightarrow 0} \frac{P[g(\bullet, t) > 0 \cap g(\bullet, t + \Delta t) \leq 0]}{\Delta t} \quad (3)$$

under the assumptions that the probability of having two or more out-crossings in $[t, t + \Delta t]$ is negligible, and the out-crossings in $[t, t + \Delta t]$ are statistically independent of the previous out-crossings in $[0, t]$. Eq. (2) is then used to estimate $P_f(0, T)$.

Monte Carlo simulation (MCS) can accurately estimate the probability of failure of Eq. (1) but it is computationally prohibitive for low failure of probability problems. To address the computational issue of MCS, analytical methods have been developed based on the out-crossing rate approach which was first introduced by Rice [4] followed by extensive studies [3, 5, 6, 7]. The PHI2 method [3] uses two successive time-invariant analyses based on FORM, and the binomial cumulative distribution to calculate the probability of the joint event in Eq. (3). A Monte-Carlo based set theory approach has been also proposed [8] using a similar approach with the PHI2 method. Analytical studies such as in [9, 10] have shown that the PHI2-based approach lacks sufficient accuracy for vibratory systems. Other analytical approaches have been however, proposed to estimate the time-dependent probability of failure with sufficient accuracy [11, 12].

The limited accuracy of the out-crossing rate approach has been improved by solving an integral equation involving $v^+(t)$ and $v^{++}(t, t_1)$, the joint up-crossing rate between times t and t_1 [13]. The up-crossing rate $v^+(t)$ is defined in Eq. (3) and the joint up-crossing rate $v^{++}(t, t_1)$ is defined as

$$v^{++}(t, t_1) = \lim_{\Delta t \rightarrow 0} \frac{P[g(\bullet, t) > 0 \cap g(\bullet, t + \Delta t) \leq 0 \cap g(\bullet, t_1) > 0 \cap g(\bullet, t_1 + \Delta t) \leq 0]}{(\Delta t)^2} \quad (4)$$

This approach has been adopted in [10].

Among the simulation-based methods, a MCS approach was proposed in [14] to estimate the time-dependent failure rate over the product lifecycle and its efficiency was improved using an importance sampling method considering a decorrelation length [15] in order to reduce the high dimensionality of the problem. Subset simulation [16] has been also developed as an efficient simulation method for computing small failure probabilities for general reliability problems. Its efficiency comes from introducing appropriate intermediate failure sub-domains to express the low probability of failure as a product of larger conditional failure probabilities which are estimated with much less computational effort. An extreme value method has also been proposed [17] using the distribution of the extreme value of the response. Recently, a time-dependent reliability analysis was proposed [18] using the total probability theorem and the concept of composite limit state.

In this paper, we present a time-dependent reliability analysis for dynamic systems with random parameters excited by a stationary Gaussian process using the total probability theorem. Metamodels are used to estimate conditional probabilities needed by the total probability theorem. An advantage of our approach is that we can easily handle non-normal and correlated random variables without additional computational effort.

The paper is arranged as follows. Section 2 describes the proposed methodology with all necessary details. Section 3 uses a beam example to demonstrate all developments and Section 4 summarizes, concludes and highlights future research.

PROPOSED APPROACH

We consider the following n degree-of-freedom (DOF) linear vibratory system with random parameters

$$[m(\mathbf{X})]\ddot{\mathbf{Y}}(t) + [c(\mathbf{X})]\dot{\mathbf{Y}}(t) + [k(\mathbf{X})]\mathbf{Y}(t) = \mathbf{F}(t) \quad (5)$$

where $\mathbf{Y}(t) = [Y_1(t) Y_2(t) \dots Y_n(t)]$ is the vector of output (response) random processes, $\mathbf{F}(t) = [F_1(t) F_2(t) \dots F_n(t)]$ is the vector of input (force) random processes, and $\mathbf{X} = [X_1 X_2 \dots X_m]$ includes m input random variables (see Figure 1). The mass $[m]$, damping $[c]$ and stiffness $[k]$ matrices depend on $\mathbf{X}$. Although our approach can handle all n random processes in $\mathbf{Y}(t)$ and $\mathbf{F}(t)$, we will consider the case with only one input $F_i(t)$, $1 \leq i \leq n$ and one output $Y_j(t)$, $1 \leq j \leq n$ for simplicity. Figure 1 provides a schematic with the input-output notation.

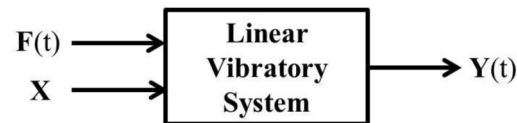


Figure 1. Schematic of linear vibratory system.

Total Probability Theorem Approach

The calculation of the time-dependent probability of failure (i.e., the probability of the response exceeding a threshold) is very challenging because of the random parameters $\mathbf{X}$. For each realization of $\mathbf{X}$ however, the linear vibratory system of Eq. (5) has constant coefficients (matrices $[m]$, $[c]$ and $[k]$). The problem then becomes much easier if the system is excited by wide-sense stationary processes $\mathbf{F}(t)$. The total probability theorem allows us to calculate the $P_f(0, t)$ of the dynamic system with random parameters in terms of conditional probabilities of dynamic systems with constant parameters.

According to the total probability theorem, the time-dependent probability of failure of Eq. (1) can be expressed as [18]

$$P_f(0, T) = P(F) = \int_{\Omega} P\left(\frac{F}{\mathbf{X}}\right) f_{\mathbf{X}}(\mathbf{x}) d\mathbf{x} \quad (6)$$

where $P(F/\mathbf{x})$ is a time-dependent conditional probability of failure, $f_{\mathbf{X}}(\mathbf{x})$ is the joint PDF of the input random variables $\mathbf{x}$ and Ω is the support of $f_{\mathbf{X}}(\mathbf{x})$. The integral of Eq. (6) can be calculated using numerical integration schemes if the number n of random variables is small (e.g. less than 5). Otherwise, Monte Carlo simulation or importance sampling methods can be used. In all cases, $P(F/\mathbf{x})$ is calculated directly or using a pre-built metamodel as is the case in

this paper. An advantage of the total probability theorem is that non-normal and correlated random variables are handled without additional computational effort or loss of accuracy.

Calculation of Conditional Probability $P^{(F/X)}$

We assume that the input random process $F(t)$ is a wide-sense stationary and Gaussian random process with zero mean and a constant standard deviation, characterized fully by the power spectral density (PSD) $S_{FF}(\omega)$. In this case, for each realization of the input random variables $\mathbf{X}$, the output process $Y(t)$ is also wide-sense stationary and Gaussian with zero mean, a constant standard deviation and a PSD of

$$S_{YY}(\omega) = |H(\omega)|^2 S_{FF}(\omega) \quad (7)$$

where $H(\omega)$ is the frequency response function of the system between the i^{th} (input) and j^{th} (output) degrees of freedom (DOF) [19].

For a real-life, multi-degree of freedom system with millions of DOF, the calculation of $H(\omega)$ is computationally intensive. It is desired therefore to minimize the number of times $H(\omega)$ is calculated. Our approach using the total probability theorem reduces the number $H(\omega)$ is calculated considerably.

According to the Wiener-Khinchine theorem, the Fourier transform of the $S_{YY}(\omega)$ spectrum provides the autocorrelation function $R_{YY}(\tau)$ of the output process. Because both $S_{YY}(\omega)$ and $R_{YY}(\tau)$ are real and even functions, we have [19]

$$R_{YY}(\tau) = \frac{1}{\pi} \int_0^{+\infty} S_{YY}(\omega) \cos(\omega \tau) d\omega \quad (8)$$

Note that for a wide-sense stationary process,

$$R_{YY}(\tau) = E[Y(t) \cdot Y(t + \tau)] = \rho \sigma_Y^2 + \mu_Y^2 \quad (9)$$

where $-1 \leq \rho \leq +1$ is the correlation coefficient and $E[Y(t)] = \mu_Y$ and $\sigma_{Y(t)} = \sigma_Y$. The covariance function is

$$\begin{aligned} C_{YY}(t, t + \tau) &= R_{YY}(t, t + \tau) - E[Y(t)] \cdot E[Y(t + \tau)] = \\ &= C_{YY}(\tau) = R_{YY}(\tau) - \mu_Y^2 \end{aligned} \quad (10)$$

Knowing the covariance function of the output process, we can use a spectral decomposition method to express the output process as a linear combination of the eigenvectors of the covariance matrix where the coefficients are independent standard normal random variables. The time interval of interest $[0, T]$ is discretized using N discrete times $t_1 = 0, t_2, \dots, t_N = T$ and the $N \times N$ covariance matrix $\Sigma = [Cov(t_i, t_j)]$, $i = 1, 2, \dots, N; j = 1, 2, \dots, N$ is formed where $Cov(t_i, t_j + \tau) = C_{YY}(t_i, t_j) = R_{YY}(\tau)$ is the covariance between times t_i and t_j provided by Eq. (10).

Let $\Sigma = \Phi \cdot \Lambda \cdot \Phi^T$ be the spectral decomposition of the covariance matrix Σ where $\Phi = [\Phi_1 \Phi_2 \dots \Phi_N]$ is the orthonormal matrix of the eigenvectors Φ_i , $i = 1, \dots, N$ and $\Lambda = diag[\lambda_1 \lambda_2 \dots \lambda_N]$ is the diagonal matrix of the corresponding eigenvalues. Also, let $\mathbf{Z} = (Z_1 Z_2 \dots Z_N)$ be a vector of N independent standard normal variables. Because of the affine transformation property of the multi-normal distribution, the following spectral representation holds

$$Y(t) = \mu_Y(t) + \sum_{i=1}^r \sqrt{\lambda_i} \cdot \Phi_i^T(t) \cdot Z_i \quad (11)$$

where $t = t_1, t_2, \dots, t_N$ and $r \leq N$ is the number of dominant eigenvalues. Eq. (11) can be used to generate sample functions (trajectories) of $Y(t)$. The Expansion Optimal Linear Estimation method (EOLE) or the Orthogonal Series Expansion (OSE) [20] can also be used.

Since Eq. (11) provides a discretized version of the output process, we can use Monte Carlo simulation to calculate the time-dependent probability of failure. However, MCS will be computationally very demanding considering that the time of interest T can be long and the time step Δt very small. To reduce the computational effort, we use in this paper the total probability theorem.

We use two approaches to estimate $P^{(F/X)}$. The first one assumes that the up-crossings are statistically independent while the second approach does not.

Approach 1: Statistically Independent Up-Crossings

The $P^{(F/X)} = P_f[0, T]$ is estimated using Eq. (2) where the failure rate $\lambda(t)$ is approximated by the up-crossing rate $v^+(t)$ (i.e., $\lambda(t) = v^+(t)$). Because the excitation process is Gaussian and wide-sense stationary and the dynamic system is linear with constant parameters, the up-crossing rate of threshold α is constant given by [19]

$$v^+(t) = v^+ = v_0^+ e^{-0.5 \left(\frac{\alpha}{\sigma_Y} \right)^2} \quad (12)$$

where

$$v_0^+ = \frac{\sigma_{\dot{Y}}}{2\pi\sigma_Y} \quad (13)$$

is the up-crossing rate for $\alpha = 0$ and

$$\sigma_{\dot{Y}} = \sqrt{\int_{-\infty}^{\infty} \omega^2 S_{YY}(\omega) d\omega} \quad (14)$$

is the standard deviation of the derivative process $\dot{Y}(t)$.