

Uncertainty Analysis of Static Plane Problems by Intervals

Naijia Xiao, Rafi L. Muhanna, and Francesco Fedele
Georgia Institute of Technology

Robert L. Mullen
Univ. of South Carolina

ABSTRACT

We present a new interval-based formulation for the static analysis of plane stress/strain problems with uncertain parameters in load, material and geometry. We exploit the Interval Finite Element Method (IFEM) to model uncertainties in the system. Overestimation due to dependency among interval variables is reduced using a new decomposition strategy for the structural stiffness matrix and the nodal equivalent load vector. Primary and derived quantities follow from minimization of the total energy and they are solved simultaneously and with the same accuracy by means of Lagrangian multipliers. Two different element assembly strategies are introduced in the formulation: one is Element-by-Element, and the other resembles conventional assembly. In addition, we implement a new variant of the interval iterative enclosure method to obtain outer and inner solutions. Numerical examples show that the proposed interval approach guarantees to enclose the exact system response.

KEYWORDS: Interval, Finite element method, Plane problems

CITATION: Xiao, N., Muhanna, R., Fedele, F., and Mullen, R., "Uncertainty Analysis of Static Plane Problems by Intervals," *SAE Int. J. Mater. Manf.* 8(2):2015, doi:10.4271/2015-01-0482.

INTRODUCTION

In recent years, there has been an increasing interest in the modeling and analysis of structural systems with uncertain parameters, which originate from incomplete information available about the system. Conventional probability theory models uncertainties as random variables [4, 10], and it is suitable when there is enough data to reliably estimate the ranges and distributions of the uncertain variables. However, in reality, we usually know some partial information about uncertainties. In such cases, one can apply non-probabilistic approaches such as evidence theory [7, 23, 24], fuzzy sets [13, 25, 26] and interval methods [3, 6, 11, 15].

In this work, we study plane stress/strain problems by using Interval Finite Element Method (IFEM, see [14, 19, 27, 28]). In particular, we use intervals to model uncertainties in load, material and geometry. Intervals are extension of real numbers. They describe an uncertain variable by its endpoints, with no assumption regarding the probability value distribution between the endpoints [12]. The associated interval arithmetic operations provide a guaranteed interval enclosure of the exact response of the system.

The main obstacle encountered in IFEM, and in interval analysis in general, is that naïve implementation of interval arithmetic suffers from the curse of overestimation caused by dependency among interval variables [12]. In order to reduce overestimation, we adopt the $M-\delta$ method [16] to eliminate load dependencies completely.

Further, we propose new decomposition strategies for the stiffness matrix to reduce and limit dependency. In addition, we use Lagrangian multipliers to solve for both derived and primary variables simultaneously and with the same accuracy via energy minimization, avoiding interval intermediate calculations [20]. The governing interval linear equations are then solved by means of a new variant of the iterative enclosure method [18].

The paper is organized as follows. First, we present the new matrix decomposition strategies to reduce overestimation. Then, we introduce direct and indirect approaches to compute derived quantities. This is followed by the description of the new variant of the iterative enclosure method to solve for the outer and inner solutions of the system response. Several numerical examples are finally discussed.

INTERVAL FINITE ELEMENT METHOD

Interval Finite Element Method (IFEM) uses intervals to model uncertain variables. An interval $\mathbf{x} = \{x \mid \underline{x} \leq x \leq \bar{x}, x \in \mathfrak{R}\}$ represents the set of real numbers within the specified range, where $\underline{x}$ and $\bar{x}$ denote the lower and upper bounds of the interval, respectively. Hereafter, *non-italic bold letters denote interval variables*. Detailed discussion on interval arithmetic and extensions to interval matrices and functions can be found in [1, 9, 12, 17].

Naïve implementation of IFEM yields overestimation of the interval solution, which is caused by the dependencies among the interval variables. For instance, to compute $\mathbf{f}(\mathbf{x}) = \mathbf{x} - \mathbf{x}^2$ with $\mathbf{x} = [0, 1]$, a naïve implementation yields $\mathbf{f}(\mathbf{x}) = [0, 1] - [0, 1]^2 = [0, 1] - [0, 1] = [-1, 1]$, which is much wider than the exact solution $f(\mathbf{x}) = [0, 1/4]$.

We propose to reduce overestimation due to dependency as follows. From the equilibrium equation $\mathbf{Ku} = \mathbf{f}$, the stiffness matrix $\mathbf{K}$ and the nodal equivalent load $\mathbf{f}$ are decomposed as

$$\mathbf{K} = A \text{diag}(\Lambda \mathbf{a}) A^T, \quad \mathbf{f} = M \delta, \quad (1)$$

where A , Λ , M are scalar matrices, $\mathbf{a}$, δ are interval vectors, and $\text{diag}(\mathbf{v})$ maps a vector $\mathbf{v}$ into a diagonal matrix, whose diagonal is $\mathbf{v}$. By doing so, we separate scalar and interval parts and avoid multiple occurrences of the same variables. The dependency in the load variables is eliminated completely by decomposing $\mathbf{f}$ in accord with the M - δ method [16]. Note that the introduction of the scalar matrix Λ makes the proposed method different from that by Impollonia [5] and Neumaier and Pownuk [18]. The decomposition in Eq. (1) is carried out in two steps. First, the element stiffness matrix $\mathbf{K}_e$ and the element nodal equivalent load $\mathbf{f}_e$ are decomposed into A_e , Λ_e , M_e , $\mathbf{a}_e$ and δ_e . Then, A_e , Λ_e , M_e , $\mathbf{a}_e$ and δ_e are assembled into A , Λ , M , $\mathbf{a}$ and δ in the global system. Two different assembly strategies are hereafter adopted: *i*) an Element-By-Element assembly and *ii*) a modified conventional FEM assembly. This formulation is the bases for a bounded uncertain field of variation of material and geometric properties, viz. $\mathbf{E}$ and $\mathbf{t}$, where a specific transition function can be used between elements.

Decomposition of $\mathbf{K}_e$ and $\mathbf{f}_e$

Hereafter, we use the standard 8-node rectangular isoparametric element. Assume the element thickness $\mathbf{t}$ and Young's modulus $\mathbf{E}$ are modeled by intervals, and all other quantities in the stiffness matrix $\mathbf{K}_e$ are scalars. Then,

$$\mathbf{K}_e = \iint_{\Omega} B_e^T \mathbf{E}_e B_e \mathbf{t} d\Omega, \quad (2)$$

where Ω is the integration domain, i.e. the entire element, B_e is the strain-displacement matrix. For plane stress, the constitutive matrix $\mathbf{E}_e$ is given by

$$\mathbf{E}_e = \frac{\mathbf{E}}{1-\nu^2} \begin{Bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{Bmatrix}, \quad (3)$$

and for plane strain

$$\mathbf{E}_e = \frac{\mathbf{E}}{(1+\nu)(1-2\nu)} \begin{Bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & (1-2\nu)/2 \end{Bmatrix}, \quad (4)$$

where ν is Poisson's ratio. The double integral in Eq. (2) is computed numerically using a 3×3 Gaussian integration rule, that is

$$\mathbf{K}_e = \sum_{i=1}^9 B_e^T(\xi_i) \mathbf{E}_e(\xi_i) B_e(\xi_i) w_i J(\xi_i) \mathbf{t}(\xi_i), \quad (5)$$

where the coordinates ξ_i and weights w_i for all the integration points in the standard domain $[-1, 1] \times [-1, 1]$ are given and J is the determinant of the Jacobian of the coordinate transformation between the local and global reference system.

In Eq. (5), the only interval term $\mathbf{E} \mathbf{t}$ can be factored out, viz. $\mathbf{K}_e = K_e \mathbf{E} \mathbf{t}$, where K_e is a scalar matrix. This can be decomposed as $K_e = A_e \text{diag}(\Lambda_e) A_e^T$, where Λ_e is the eigenvalue matrix and the columns of A_e are the corresponding eigenvectors. Note that Λ_e includes three zero eigenvalues, which correspond to rigid body motion of translation and rotation. They are dropped and Eq. (5) can be rewritten as

$$\mathbf{K}_e = A_e \text{diag}(\Lambda_e) A_e^T \mathbf{a}_e = A_e \text{diag}(\Lambda_e \mathbf{a}_e) A_e^T, \quad (6)$$

where $\mathbf{a}_e = \{\mathbf{E} \mathbf{t}\}$ is the only interval quantity. One can also use an LDL decomposition of K_e . As a result, A_e is a lower unitriangular matrix, and Λ_e is also different. In both cases, K_e is explicitly computed in order to obtain A_e and Λ_e .

Alternatively, we can use a different strategy, hereafter referred to as the B-matrix approach. Here, we can decompose $\mathbf{K}_e$ without explicitly computing K_e . To do so, we write $E_e = \Phi \text{diag}(\varphi \mathbf{E}) \Phi^T$, where Φ is a 3×3 scalar matrix and φ is a 3×1 scalar vector, both depending on the Poisson's ratio ν . Then, from Eq. (5)

$$\begin{aligned} \mathbf{K}_e &= \sum_{i=1}^9 B_e^T(\xi_i) \Phi \text{diag}(\varphi) \Phi^T B_e(\xi_i) w_i J(\xi_i) (\mathbf{E} \mathbf{t}) \\ &= \left\{ B_e^T(\xi_1) \Phi \quad \dots \quad B_e^T(\xi_9) \Phi \right\} \\ &\quad \begin{Bmatrix} \text{diag}(\varphi) w_1 J(\xi_1) \\ \vdots \\ \text{diag}(\varphi) w_9 J(\xi_9) \end{Bmatrix} \begin{Bmatrix} \Phi^T B_e(\xi_1) \\ \vdots \\ \Phi^T B_e(\xi_9) \end{Bmatrix} (\mathbf{E} \mathbf{t}), \end{aligned} \quad (7)$$

where $\mathbf{E}$ and $\mathbf{t}$ are factored out and collected into $\mathbf{a}_e = \{\mathbf{E} \mathbf{t}\}$, and A_e and Λ_e are given by

$$A_e = \left\{ B_e^T(\xi_1) \Phi \quad \dots \quad B_e^T(\xi_9) \Phi \right\}, \quad \Lambda_e = \begin{Bmatrix} w_1 J(\xi_1) \varphi \\ \vdots \\ w_9 J(\xi_9) \varphi \end{Bmatrix}. \quad (8)$$

And Eq. (7) simplifies to

$$\mathbf{K}_e = \underset{16 \times 16}{A_e} \underset{16 \times 27}{\text{diag}(\Lambda_e)} \underset{27 \times 1}{A_e^T} \underset{16 \times 27}{\mathbf{a}_e} = \underset{16 \times 27}{A_e} \underset{27 \times 1}{\text{diag}(\Lambda_e \mathbf{a}_e)} \underset{27 \times 16}{A_e^T}. \quad (9)$$

The B-matrix approach yields A_e and Λ_e directly from B_e and $\mathbf{E}_e$, but matrices are larger than the those from the eigen or LDL decomposition method. Indeed, in the eigen or LDL decomposition we need to decompose a 16×16 matrix K_e and A_e is 16×13 and Λ_e is 13×1 . In the B-matrix approach, a 3×3 matrix $\mathbf{E}_e$ is decomposed and A_e is 16×27 and Λ_e is 27×1 . Clearly, the B-matrix approach saves time in the decomposition process, but it is more time-consuming in the final computation. Comparisons between the two decomposition strategies are presented below in several numerical examples.

The nodal equivalent load vector $\mathbf{f}$ is decomposed by using the M - δ method [16]. For a standard 8-node rectangular isoparametric element,

$$\mathbf{f}_e = \sum_{i=1}^n N^T(\xi_i) \underset{16 \times 2}{\mathbf{f}_{c,i}} + \iint_{\Omega} N^T \underset{16 \times 2}{\mathbf{f}_b} d\Omega + \sum_{i=1}^4 \int_0^L N^T \underset{16 \times 2}{\mathbf{f}_{l,i}} dl, \quad (10)$$

where N is the displacement interpolation matrix, $\mathbf{f}_{c,i}$ are concentrated loads, n is the number of concentrated loads in the element, $\mathbf{f}_b$ is the distributed body load, and $\mathbf{f}_{l,i}$ are the distributed line loads on the edges.

We can further simplify Eq. (10) as follows. First, rewrite $\mathbf{f}_{c,i} = L_{c,i} \delta_e$, $\mathbf{f}_b = L_b \delta_e$ and $\mathbf{f}_{l,i} = L_{l,i} \delta_e$ as function of the interval vector δ_e , where $L_{c,i}$, L_b , $L_{l,i}$ are scalar matrices. Then,

$$\begin{aligned} \mathbf{f}_e &= \sum_{i=1}^n N^T(\xi_i) L_{c,i} \delta_e + \iint_{\Omega} N^T L_b \delta_e d\Omega + \sum_{i=1}^4 \int_0^L N^T L_{l,i} \delta_e dl \\ &= \left\{ \sum_{i=1}^n N^T(\xi_i) L_{c,i} + \iint_{\Omega} N^T L_b d\Omega + \sum_{i=1}^4 \int_0^L N^T L_{l,i} dl \right\} \delta_e = M_e \delta_e, \end{aligned} \quad (11)$$

where m is the size of δ_e , and M_e is defined as the matrix in braces. Each component of δ_e is assumed to vary independently.

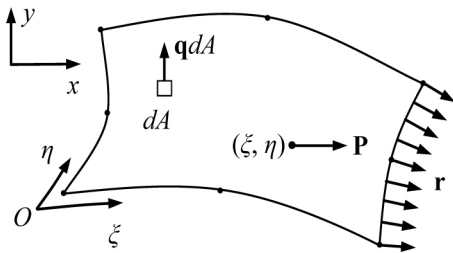


Figure 1. Loads acting on an 8-node rectangular isoparametric element.

As an example, consider the element shown in Figure 1. This is subject to a concentrated load $\mathbf{P}$ in the x -direction at (ξ, η) , a uniformly distributed body load $\mathbf{q}$ in the y -direction, and a uniformly distributed line load $\mathbf{r}$ acting on the right edge. Then, $\delta_e = \{\mathbf{P}, \mathbf{q}, \mathbf{r}\}^T$ and M_e is given by

$$M_e = \underset{16 \times 3}{\begin{Bmatrix} N_1(\xi, \eta) & 0 & \int_{x=0}^L N_1 J_{11} dl \\ 0 & \iint_{\Omega} N_1 d\Omega & \int_{x=0}^L N_1 J_{21} dl \\ \vdots & \vdots & \vdots \\ N_8(\xi, \eta) & 0 & \int_{x=0}^L N_8 J_{11} dl \\ 0 & \iint_{\Omega} N_8 d\Omega & \int_{x=0}^L N_8 J_{21} dl \end{Bmatrix}}, \quad (12)$$

where N_i are displacement interpolation matrix with respect to the i -th node, L is the length of the right edge, and $\{J_{11}, J_{21}\}^T$ is the first column of the Jacobian matrix, accounting for the distributed load $\mathbf{r}$ acting normal to the right edge.

Note that our proposed method is general and it can be applied to other plane stress/strain elements. The only difference is the size of the matrices A_e , Λ_e , and M_e . For instance, if one uses a 6-node triangular isoparametric element, $\mathbf{K}_e$ is 12×12 . From the eigen-decomposition, the resulting A_e is 12×9 and Λ_e is 9×1 ; instead, if the B-matrix approach and a 7-point numerical integration rule are adopted, A_e is 12×21 and Λ_e is 21×1 .

Element Assembly Strategies

We consider the Element-by-Element approach [20] and the conventional FEM assembly strategy [2, 29].

In the Element-by-Element approach, the structure is modeled by free elements and connecting common nodes [20]. Structural nodal displacement vector $\mathbf{u}$ is composed of the element nodal displacement vector $\mathbf{u}_e$ and displacement vector $\mathbf{u}_n$ of the common nodes. Then

$$\mathbf{u} = \begin{Bmatrix} \mathbf{u}_e \\ \vdots \\ \mathbf{u}_e \\ \mathbf{u}_n \end{Bmatrix}, \quad \mathbf{K} = \begin{Bmatrix} \mathbf{K}_e & & \\ & \ddots & \\ & & \mathbf{K}_e \\ & & & \mathbf{0} \end{Bmatrix}, \quad \mathbf{f} = \begin{Bmatrix} \mathbf{f}_e \\ \vdots \\ \mathbf{f}_e \\ \mathbf{f}_n \end{Bmatrix}, \quad (13)$$

where $\mathbf{f}_n$ denotes concentrated loads applied on the common nodes. Note $\mathbf{K}$ is a singular matrix.

Using the decomposition rules for $\mathbf{K}$ and $\mathbf{f}$ in Eq. (1), the corresponding assembly rules for the scalar matrices A , Λ , and M and interval vectors $\mathbf{a}$, δ are given by

$$A = \begin{Bmatrix} A_e & & \\ & \ddots & \\ & & A_e \\ 0 & \dots & 0 \end{Bmatrix}, \quad \Lambda = \begin{Bmatrix} \Lambda_e & & \\ & \ddots & \\ & & \Lambda_e \end{Bmatrix}, \quad \mathbf{a} = \begin{Bmatrix} \mathbf{a}_e \\ \vdots \\ \mathbf{a}_e \end{Bmatrix};$$

$$M = \begin{Bmatrix} M_e & & \\ & \ddots & \\ & & M_e \\ & & & M_n \end{Bmatrix}, \quad \delta = \begin{Bmatrix} \delta_e \\ \vdots \\ \delta_e \\ \delta_n \end{Bmatrix},$$