

Interval Finite Element Analysis of Structural Dynamic Problems

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ABSTRACT

We analyze the frequency response of structural dynamic systems with uncertainties in load and material properties. We introduce uncertainties in the system as interval numbers, and use Interval Finite Element Method (IFEM). Overestimation due to dependency is reduced using a new decomposition for the stiffness and mass matrices, as well as for the nodal equivalent load. In addition, primary and derived quantities are simultaneously obtained by means of Lagrangian multipliers that are introduced in the total energy of the system. The obtained interval equations are solved by means of a new variant of the iterative enclosure method resulting in guaranteed enclosures of relevant quantities. Several numerical examples show the accuracy and efficiency of the new formulation.

KEYWORDS: Interval, Finite element method, Frequency response

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INTRODUCTION

We analyze structural dynamic systems with uncertainties in load and material properties in the frequency domain, which has been the subject of extensive research in recent years [6, 21, 28, 29]. The conventional approach for the treatment of uncertainties evolves probability theory, which models system uncertain parameters as random variables [7, 12]. In the probabilistic approach, complete information about the uncertain parameters is required, such as the range of the random variable and its probabilistic distribution within that range. However, such complete information is usually not available in practice. Usually one is exposed to only partial information about the uncertain parameters. Then the probabilistic approach relies on a prior assumption on the nature of the uncertain parameters, the validity of which is under debate [13, 32]. In such case, one can alternatively turn to non-conventional approaches, such as fuzzy sets [1, 10, 16], evidence theory [3, 33, 34], and intervals [5, 9, 14, 19].

In this work, we assume that only range information is known about the system uncertain parameters, and we use the intervals to model these uncertain parameters. Intervals [1, 11, 15, 22] are extension of real (or complex, see [8]) numbers. One interval denotes a set of real (or complex) numbers instead of one single real (or complex) number. The corresponding interval operations provide a guaranteed enclosure of the exact solution. *Hereafter we will denote interval variable in bold non-italic font.* We use Interval Finite Element Method (IFEM, see [17, 18, 24, 27, 31]) as the cornerstone for the proposed method.

To reduce overestimation due to dependency of interval variables, which is the curse of interval arithmetic [15], we adopt: *i)* a new decomposition for interval vectors and matrices in the formulation, such as nodal equivalent load vector $\mathbf{f}$ (see [20]), stiffness matrix $\mathbf{K}$, and mass matrix $\mathbf{M}$; *ii)* Lagrangian multipliers (see [25]) for the calculation of derived quantities, to avoid interval intermediate results; and *iii)* a new variant of iterative enclosure method (see [23]), which is applicable to the interval linear system as in the frequency response analysis.

The paper is organized as follows. First we present a new decomposition strategy for the interval vectors and matrices encountered in the formulation. Then we propose to enclose the frequency response of the structure via a new variant of iterative enclosure method. The formulation encloses the real and imaginary parts of the response. Numerical simulations show that the proposed method provides a guaranteed enclosure of the exact response.

INTERVAL FINITE ELEMENT METHOD

We base our proposed formulation on Interval Finite Element Method (IFEM). To reduce overestimation, the nodal equivalent load $\mathbf{f}$ and stiffness matrix $\mathbf{K}$ are decomposed into [20, 30]

$$\mathbf{f} = F\delta, \quad \mathbf{K} = A \text{diag}(\Lambda \alpha) A^T,$$

(1.)

where F, A, Λ are scalar matrices, δ is the interval load uncertainty vector, and α is the interval stiffness parameter vector. For mass matrix $\mathbf{M}$, the decomposition is similar to that of $\mathbf{K}$,

$$\mathbf{M} = A_m \text{diag}(\Lambda_m \alpha_m) A_m^T, \quad (2.)$$

where A_m, Λ_m are scalar matrices, and α_m is the interval mass parameter vector. Details on the decompositions of $\mathbf{f}$, $\mathbf{K}$, and $\mathbf{M}$ [see Eqs. (1) and (2)] are discussed in the following subsections.

Decomposition of $\mathbf{f}_e$

We start the decomposition on the element level. According to conventional Finite Element Method (FEM, see [4]), the element nodal equivalent load

$$\mathbf{f}_e = \sum_{i=1}^n N^T(\xi_i) \mathbf{f}_c(\xi_i) + \int_V N^T(\xi) \mathbf{f}_d(\xi) dV, \quad (3.)$$

where n is the number of concentrated load acting on the element, N is the displacement interpolation matrix, ξ_i are coordinates where the concentrated loads act, $\mathbf{f}_c, \mathbf{f}_d$ are respectively the concentrated and distributed loads, and V is the integration domain where distributed load $\mathbf{f}_d$ is non-zero. We assume there are uncertainties in load, and model $\mathbf{f}_c, \mathbf{f}_d$ by intervals. Then we further assume $\mathbf{f}_c, \mathbf{f}_d$ are parameterized by an interval vector δ_e , viz. $\mathbf{f}_c(\xi_i) = L_c(\xi_i) \delta_e$, $\mathbf{f}_d(\xi) = L_d(\xi) \delta_e$. As a result,

$$\mathbf{f}_e = F_e \delta_e, \quad (4.)$$

where

$$F_e = \sum_{i=1}^n N^T(\xi_i) L_c(\xi_i) + \int_V N^T(\xi) L_d(\xi) dV. \quad (5.)$$

Decomposition of $\mathbf{K}_e$

For the element stiffness matrix $\mathbf{K}_e$, we assume that there are uncertainties in the Young's modulus, viz.

$$\mathbf{K}_e = \int_V B_e^T \mathbf{E}_e B_e dV, \quad (6.)$$

where B_e is the element strain-displacement matrix, $\mathbf{E}_e$ is the element constitutive matrix modeled by intervals, and V is the integration domain for the element. Suppose numerical integrations are adopted to evaluate the integral in Eq. (6), viz.

$$\mathbf{K}_e = \sum_{k=1}^m w_k J(\xi_k) B_e^T(\xi_k) \mathbf{E}_e(\xi_k) B_e(\xi_k), \quad (7.)$$

where m is the number of numerical integration points, w_k, ξ_k are respectively their weights and coordinates, J is the determinant of the Jacobian for the transformation between the local and global coordinate systems. Note that Eq. (7) can be brought into

$$\mathbf{K}_e = \left\{ B_e^T(\xi_1) \Phi_e \cdots B_e^T(\xi_m) \Phi_e \right\} \begin{Bmatrix} w_1 \varphi_e J(\xi_1) \mathbf{E}(\xi_1) \\ \vdots \\ w_m \varphi_e J(\xi_m) \mathbf{E}(\xi_m) \end{Bmatrix} \begin{Bmatrix} \Phi_e^T B_e(\xi_1) \\ \vdots \\ \Phi_e^T B_e(\xi_m) \end{Bmatrix}. \quad (8.)$$

Scalar matrix Φ_e and scalar vector φ_e in Eq. (8) are obtained from the decomposition of $\mathbf{E}_e$ at the integration points, viz.

$$\mathbf{E}_e(\xi_k) = \Phi_e \text{diag}\{\varphi_e \mathbf{E}(\xi_k)\} \Phi_e^T, \quad (9.)$$

where $\mathbf{E}(\xi_k)$ denote Young's modulus at the k -th integration point ξ_k . Eq. (8) can be recast into the following compact form

$$\mathbf{K}_e = A_e \text{diag}(\Lambda_e \alpha_e) A_e^T, \quad (10.)$$

where A_e, Λ_e are scalar matrices, and α_e contains all interval variables in $\mathbf{K}_e$,

$$A_e = \left\{ B_e^T(\xi_1) \Phi_e \cdots B_e^T(\xi_m) \Phi_e \right\}, \quad \Lambda_e = \begin{Bmatrix} w_1 J(\xi_1) \varphi_e & & \\ & \ddots & \\ & & w_m J(\xi_m) \varphi_e \end{Bmatrix}, \quad \alpha_e = \begin{Bmatrix} \mathbf{E}(\xi_1) \\ \vdots \\ \mathbf{E}(\xi_m) \end{Bmatrix}. \quad (11.)$$

Decomposition of $\mathbf{M}_e$

For the element mass matrix $\mathbf{M}_e$, we assume that there are uncertainties in mass density, viz.

$$\mathbf{M}_e = \int_V \rho N^T N dV, \quad (12.)$$

which is also evaluated via numerical integration, viz.

$$\mathbf{M}_e = \sum_{k=1}^m w_k J(\xi_k) \rho(\xi_k) N^T(\xi_k) N(\xi_k), \quad (13.)$$

where $\rho(\xi_k)$ denote mass density at the k -th integration point ξ_k . Following the same procedure as in the derivation of $\mathbf{K}_e$, we decompose $\mathbf{M}_e$ into

$$\mathbf{M}_e = A_{me} \text{diag}(\Lambda_{me} \alpha_{me}) A_{me}^T, \quad (14.)$$

where scalar matrices A_{me}, Λ_{me} and interval vector α_{me} are given by

$$A_{me} = \begin{Bmatrix} N^T(\xi_1) & \cdots & N^T(\xi_m) \end{Bmatrix},$$

$$\Lambda_{me} = \begin{Bmatrix} w_1 J(\xi_1) \mathbf{e}_e \\ \vdots \\ w_m J(\xi_m) \mathbf{e}_e \end{Bmatrix}, \quad \mathbf{a}_{me} = \begin{Bmatrix} \mathbf{p}(\xi_1) \\ \vdots \\ \mathbf{p}(\xi_m) \end{Bmatrix}. \quad (15.)$$

Note that $\mathbf{e}_e$ is an $n \times 1$ vector with all entries set to 1, where n is the number of translational Degrees of Freedom (DOF) at each node. For instance, a planar beam element has 2 translational DOF at each node, $n = 2$; a spatial truss element has 3 translational DOF, $n = 3$.

Element Assembly Strategy

We assembly the element variables $\mathbf{f}_e$, $\mathbf{K}_e$, and $\mathbf{M}_e$ into their global counterparts $\mathbf{f}$, $\mathbf{K}$, and $\mathbf{M}$. Suppose the element nodal displacement vector $\mathbf{u}_e$ is linked to its global counterpart $\mathbf{u}$ via $\mathbf{u}_e = T_e \mathbf{u}$, where T_e is the element transformation matrix. According to conventional assembly strategy [4, 35]

$$\mathbf{f} = \sum_e T_e^T \mathbf{f}_e, \quad \mathbf{K} = \sum_e T_e^T \mathbf{K}_e T_e, \quad \mathbf{M} = \sum_e T_e^T \mathbf{M}_e T_e. \quad (16.)$$

By inserting the decompositions of $\mathbf{f}_e$ [see Eq. (4)], we obtain

$$\mathbf{F} = \left\{ T_e^T F_e \quad \cdots \quad T_e^T F_e \right\} \begin{Bmatrix} L_\delta \\ \vdots \\ L_\delta \end{Bmatrix} = \sum_e T_e^T F_e L_\delta. \quad (17.)$$

where L_δ relates element load uncertainty vector δ_e to its global counterpart δ , viz. $\delta_e = L_\delta \delta$. Similarly, noting the decomposition of $\mathbf{K}_e$ [see Eq. (10)], the assembly rules for A and Λ are given by

$$A = \left\{ T_e^T A_e \quad \cdots \quad T_e^T A_e \right\}, \quad \Lambda = \begin{Bmatrix} \Lambda_e L_\alpha \\ \vdots \\ \Lambda_e L_\alpha \end{Bmatrix}. \quad (18.)$$

where L_α relates $\mathbf{a}_e$ to $\mathbf{a}$, viz. $\mathbf{a}_e = L_\alpha \mathbf{a}$. Noting the decomposition of $\mathbf{M}_e$ [see Eq. (14)], the assembly rules for A_m and Λ_m are given by

$$A_m = \left\{ T_e^T A_{me} \quad \cdots \quad T_e^T A_{me} \right\}, \quad \Lambda_m = \begin{Bmatrix} \Lambda_{me} L_\alpha \\ \vdots \\ \Lambda_{me} L_\alpha \end{Bmatrix}. \quad (19.)$$

where we assume the same scalar matrix L_α is used to relate $\mathbf{a}_{me}$ to $\mathbf{a}_m$, viz. $\mathbf{a}_{me} = L_\alpha \mathbf{a}_m$.

To apply essential boundary conditions, the nodal displacement vector $\mathbf{u}$ must satisfy additional constraint $\mathbf{C}\mathbf{u} = \mathbf{0}$, with C denoting a constraint matrix [25]. Each row of C denotes one constraint, and the corresponding entry is set equal to 1, leaving the rest of the row null. By introducing a Lagrangian multiplier λ to enforce the constraint

equation $\mathbf{C}\mathbf{u} = \mathbf{0}$, we apply essential boundary conditions to the system, where λ denote reaction forces associated to the essential boundary conditions. For instance, for a structural static system with equilibrium condition $\mathbf{K}\mathbf{u} = \mathbf{f}$, the governing equation is given by

$$\begin{Bmatrix} \mathbf{K} & C^T \\ C & 0 \end{Bmatrix} \begin{Bmatrix} \mathbf{u} \\ \lambda \end{Bmatrix} = \begin{Bmatrix} \mathbf{f} \\ \mathbf{0} \end{Bmatrix}. \quad (20.)$$

We note that $C^T \lambda$ is added to the equilibrium equation to account for the essential boundary conditions.

ENCLOSURE OF FREQUENCY RESPONSES

We propose to solve a structural dynamic system in the frequency domain, which is governed by

$$(\mathbf{K} + i\omega\mathbf{C} - \omega^2\mathbf{M})\mathbf{u} = \mathbf{f}, \quad (21.)$$

where $i = \sqrt{-1}$ is the imaginary unit, ω is the angular frequency under consideration, and $\mathbf{C}$ is the damping matrix. Assuming Rayleigh damping is used,

$$\mathbf{C} = a\mathbf{M} + b\mathbf{K}, \quad (22.)$$

where a and b are the damping coefficients. At given frequency ω , the equivalent damping ratio ζ is given in terms of a and b by

$$\zeta = a/(2\omega) + b\omega/2. \quad (23.)$$

Interval Governing Equations

According to Eq. (21), the effective stiffness matrix is given by

$$\mathbf{K}_{eff} = \mathbf{K} + i\omega\mathbf{C} - \omega^2\mathbf{M}. \quad (24.)$$

Noting that we use Rayleigh damping [see Eq. (22)], $\mathbf{K}_{eff}$ is given by

$$\mathbf{K}_{eff} = (1 + ib\omega)\mathbf{K} + (-\omega^2 + ia\omega)\mathbf{M}, \quad (25.)$$

which can be decomposed into

$$\mathbf{K}_{eff} = A_{eff} \text{diag}(\Lambda_{eff} \mathbf{a}_{eff}) B_{eff}, \quad (26.)$$

where

$$A_{eff} = \begin{Bmatrix} (1 + ib\omega)A & (-\omega^2 + ia\omega)A_m \end{Bmatrix},$$

$$\Lambda_{eff} = \begin{Bmatrix} \Lambda & 0 \\ 0 & \Lambda_m \end{Bmatrix}, \quad \mathbf{a}_{eff} = \begin{Bmatrix} \mathbf{a} \\ \mathbf{a}_m \end{Bmatrix}, \quad B_{eff} = \begin{Bmatrix} A^T \\ A_m^T \end{Bmatrix}.$$