

Assessment of Similarity of a Set of Impact Response Time Histories

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ABSTRACT

Two methods of assessing the similarity of a set of impact test signals have been proposed and used in the literature, which are cumulative variance-based and cross correlation-based. In this study, a normalized formulation unites these two approaches by establishing a relationship between the normalized cumulative variance metric (v), an overall similarity metric, and the normalized magnitude similarity metric (m) and shape similarity metric (s): $v=1 - m \cdot s$. Each of these ranges between 0 and 1 (for the practical case of signals acquired with the same polarity), and they are independent of the physical unit of measurement. Under generally satisfied conditions, the magnitude similarity m is independent of the relative time shifts among the signals in the set; while the shape similarity s is a function of these. An optimal alignment is defined as the relative shifts corresponding to the minimum of the cumulative variance metric, or equivalently, to the maximum of the shape similarity metric. This system therefore quantifies the similarity of a given set of signals with an “as given” relative time alignment with the following: $v = 1 - m(p + s_a)$, i.e., its overall similarity is partitioned into the shift-invariant normalized magnitude metric m , and the inherent (optimally aligned) normalized shape metric s_a , and a normalized phase metric p which reflects the alignment.

An algorithm is provided for automatic search of the optimal shifts for a given set of signals using the cumulative variance as the objective function. The shifts are treated as real numbers instead of integers so that standard continuous variable optimization functions can be used to treat the discrete signals. This is facilitated by an interpolation procedure for shifting a signal with a real-valued time shift. Numerical examples are presented to demonstrate the general effectiveness of the algorithm, and also specific limitations on consistent convergence to the global optimal. Practical measures that would mitigate the limitations and alternative numerical methods that would fundamentally remove these are suggested.

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INTRODUCTION

A set of similar but nonidentical response time histories typically result from impact tests with anthropomorphic test devices (ATD), or a set of postmortem human subjects (PMHS), under nominally repeat test conditions. Summary metrics that quantify the degree of similarity of such a set are needed for assessing the ATD biofidelity and repeatability/reproducibility, including repeatability of test results from different laboratories.

A summary descriptor of the agreement between a set of two time histories was introduced by Morgan, Marcus and Eppinger [1]. It is based on the “cumulative variance” (CV) as shown in Eq. (1):

$$CV_{a_1 \text{ vs } a_2} = \sum_{i=1}^N (a_1[i] - a_2[i])^2 . \quad \text{Eq. (1)}$$

For assessing the similarity of a set of more than two tests, the following CV definition was also introduced which covers Eq. (1) as a special case:

$$CV_a = \frac{\sum_{j=1}^n \sum_{i=1}^N (a_j[i] - \bar{a}[i])^2}{n-1}, \quad \text{Eq. (2)}$$

where $\bar{a}$ denotes the average of the set a which includes n signals.

Furthermore, for the scenario of assessing the similarity of one test to a reference set of tests, as in the case of comparing an ATD test to its targeted PMHS reference which often contains multiple tests embodying the scatter in test subjects, the following metric based on the CV was proposed:

$$R = \frac{CV_{ATD \text{ vs } PMHS_average}}{CV_{PMHS_set}} . \quad \text{Eq. (3)}$$

One aspect intertwined with the CV-based similarity assessment is ‘alignment’, which is the relative shift in time of the pair in the case of Eq. (1), or relative shifts among the time histories in the set in the case of Eq. (2); In the case of Eq. (3), alignment underlies both the CVs. Because a set of responses (two or more) each having the same underlying impact physics may be shifted in time, either due to instrumentation triggering or inconsequential differences in test subjects (such as in subcutaneous fat layer), alignment is often necessary before the final CV-based similarity assessment as the CV value generally changes when the signals are shifted relative to each other. Morgan *et al.* [1] proposed that optimal alignment is achieved when the CV is minimized. However, a general algorithm for aligning a set of more than two signals was not proposed or numerically demonstrated.

Xu *et al.* [2], in assessing the repeatability of test results from ATD under development, defined and used pair-wise “magnitude” and “shape” metrics. These metrics are based on the correlation function between the pair of signals (zero-shift autocorrelation for the “magnitudes” and maximum crosscorrelation for the “shape”). Time alignment of the two time histories is achieved through maximizing the crosscorrelation between the two signals, as opposed to minimizing the CV. Finding the aligning time shift is straightforward with two signals, as the entire crosscorrelation function can be calculated efficiently numerically via Fast Fourier Transform for digital signals.

The problem of aligning more than two signals is addressed by Maltese *et al.* [3]. A numerical scheme was proposed which designates one signal that “appeared to have the most typical shape of all signals” as a standard, and then aligns each of the rest of the set to this standard. The designation of the ‘standard’ is subjective. Furthermore, unlike the crosscorrelation maximization scheme in [2], for each alignment, the CV is calculated in [3] as a function of all possible time step shifts for the digital signals to identify the optimal shift, i.e., via brute force.

Nusholtz *et al.* [4] proposed two schemes for aligning a set of signals. Recognizing that there are only $n-1$ independent time shifts to align a set of n signals, one of the two methods seeks to maximize the following:

$$C(\tau_2, \tau_3, \dots, \tau_n) = \frac{1}{n(n-1)} \left(\left(\sum_{i=1}^n \sum_{j=1}^n c_{ij}(\tau_i, \tau_j) \right) - n \right) = \frac{1}{n(n-1)} \sum_{i \neq j} c_{ij}(\tau_i, \tau_j), \quad \text{Eq. (4)}$$

which is the average of all the permutation of the crosscorrelation functions under a given set of shifts. Therefore, this approach solves the multiple time shifts simultaneously by formulating the problem into a multivariable optimization problem. The optimization search was numerically carried out using the function `fminunc.m` in Matlab Optimization Toolbox, with a modification to allow for integer shifts associated with the digital signals. This alignment algorithm as a result is a formal gradient-based iterative optimization search. Donnelly and Moorhouse [5] tackled the same multi-signals

alignment problem, and used essentially the same objective function as Nusholtz *et al.* [4]. A significant difference however is that instead of $n-1$ independent shifts for a set of n signals, relative shifts between each permutation pairs of signals is set as a variable, resulting in a total number of search variables to $n(n-1)$. Since these are in fact not independent of each other, their relationships have to be introduced in the optimization as constraints, and in [5] the Lagrangian multipliers method was used to solve the constrained optimization. The computational method presented for the optimization search is approximate because it used an assumption that the derivative of the crosscorrelation function is constant [5]. This method was applied in Kang *et al.* [6] to align PMHS test results.

Once a set of time histories are aligned, the similarity of the aligned set may be further analysed in terms of “magnitude” and “shape” qualities. In Xu *et al.* [2] these were defined to be normalized metrics, but they were not combined to form an overall metric. Moorhouse *et al.* [7] extended the CV based summary metric in Eq. (3) by breaking down its numerator into three components and arrived at the following partition:

$$R = R_{\text{phase}} + R_{\text{amp}} + R_{\text{shape}}, \quad \text{Eq. (5)}$$

where R_{phase} (given by Eq. C7 in [7]) is essentially the following:

$$R_{\text{phase}} = \frac{\sum (x_{\text{ATD}} - x_{\text{mean PMHS}})^2 - \sum (x_{\text{ATD, aligned}} - x_{\text{mean PMHS}})^2}{CV_{\text{PMHS}}}, \quad \text{Eq. (6)}$$

which means that it is a normalized reduction of the CV due to the aligning the ATD time history to the mean time history of the PMHS. Therefore, the quality of alignment of the ‘as collected’ ATD and PMHS mean responses is reflected by R_{phase} . The remaining two factors measure the similarity of the “aligned” ATD and PMHS mean responses. Their meanings are clearer, if they are rewritten into the following forms from those given in Eqs. C9 and C10 in [7]:

$$R_{\text{amp}} = \frac{\sum x_{\text{ATD}}^2}{CV_{\text{PMHS}}} \left(1 - \rho_{\text{max}} \frac{\sqrt{\sum x_{\text{ATD}}^2}}{\sqrt{\sum x_{\text{mean PMHS}}^2}} \right), \quad \text{Eq. (7)}$$

$$R_{\text{shape}} = \frac{\sum x_{\text{mean PMHS}}^2}{CV_{\text{PMHS}}} (1 - \rho_{\text{max}}^2), \quad \text{Eq. (8)}$$

$$\text{where, } \rho_{\text{max}} = \frac{C_{\text{ATC v mean PMHS}}(\tau_{\text{max}})}{\sqrt{\sum x_{\text{ATD}}^2} \sqrt{\sum x_{\text{mean PMHS}}^2}}. \quad \text{Eq. (9)}$$

ρ_{max} is the normalized crosscorrelation at the time shift τ_{max} corresponding to the maximum of the crosscorrelation function, as used in [2]. Since ρ_{max} appears in Eqs. (8), it means that the partition proposed in Moorhouse *et al.* [7] does not truly completely separate the shape information from the amplitude information (another way to view this is that based on Eqs. (7) and (8), R_{amp} and R_{shape} are not independent metrics).

The approach of using separate phase, amplitude, and shape metrics, and also combining these into one overall rating is also followed by Gehre *et al.* [8] and Gehre and Stahlschmidt [9]. Their three individual metrics are fundamentally based on the crosscorrelation function. The normalization of these metrics and their combination into the overall rating involve weighting factors whose values need to be chosen. The metrics and the final rating therefore generally change depending on the choice of these. Only the two-signal case is covered in [8, 9].

The above review has highlighted the following aspects of the topic of similarity assessment of a set of impact response signals: (1) What are the meaningful metrics to quantify the ‘similarity’? The CV has been proposed and used as a summary metric; and separate “shape” and “magnitude” metrics have also been established as more detailed characterizations of the similarity. The “shape” metric has been largely based on crosscorrelation. (2) How to relate the detailed “shape” and “magnitude” metrics to an overall similarity metric, or in other words, how to logically partition the summary metric into the detailed “component” metrics? (3) How to formulate the alignment problem for a set of more than two signals? and what is an efficient and systematic numerical algorithm for optimal alignment in this situation? Minimizing the CV or maximizing the (sum of) crosscorrelation have been proposed in different studies, and most of the numerical schemes have been less than rigorous, in the sense of formal optimization search.

Questions (1) and (2) have been answered by one original study on this subject by Jovanovski [10]. It treated the signals as realizations of a random process and proposed a metric, the Integral Square Error (ISE) which in essence is a version of the CV. A normalized metric NISE was also formulated. Based on these and the crosscorrelation function, the similarity was partitioned into phase and amplitude components. The concepts and approach in [10] can be viewed as the basis of a number of studies cited so far, and it was used in [11]. They are also the starting point of this current study. Question (3) above, however, was not addressed in [10], as formulation and numerical applications all involved only a pair of signals.

The objective of this study is to analyze and answer the three questions above in the context of a set of more than two signals; and propose a general similarity assessment framework containing normalized assessment metrics and an objective alignment method.

In the following, a simple formulation is presented that, for a set of signals, relates a CV-based metric to a magnitude metric and crosscorrelation-based shape metric, and this relationship provides a natural and logical decomposition of the overall CV-based metric into the more detailed magnitude and phase metrics. All these metrics are normalized quantities with known ranges. Furthermore, this formulation shows that, under a typically satisfied condition, minimizing the CV and maximizing the sum of correlation are mathematically equivalent for aligning a set of signals. Finally, a practical algorithm is presented for optimal alignment through formal optimization search.

FORMULATION OF ASSESSMENT METRICS

Let $x_i(t)$, $i = 1, \dots, n$ be a set of impact response time histories. At any given time, t , we have

$$\frac{\sum (x_i - \frac{\sum x_i}{n})^2}{n-1} = \frac{\sum x_i^2}{n} - \frac{\sum \sum_{i \neq j} x_i x_j}{n(n-1)} . \quad \text{Eq. (10)}$$

If we view this set of impact responses as a sample of an underlying random process, then the left side of the equation is the sample variance as a function of time. Since our objective is to construct a summary descriptor throughout the impact duration, the above relationship is time-averaged:

$$\frac{1}{T} \int_T \frac{\sum (x_i - \frac{\sum x_i}{n})^2}{n-1} dt = \frac{1}{n} \sum \frac{1}{T} \int_T x_i^2 dt - \frac{1}{n(n-1)} \sum \sum_{i \neq j} \frac{1}{T} \int_T x_i x_j dt . \quad \text{Eq. (11)}$$

Define the norm, or *magnitude*, of each signal as:

$$\|x_i\| \equiv \sqrt{\frac{1}{T} \int_T x_i^2 dt} . \quad \text{Eq. (12)}$$

Further, define an *overall similarity metric* v , which is a *normalized cumulative variance*, as

$$v \equiv \frac{\frac{1}{T} \int_T \frac{\sum (x_i - \frac{\sum x_i}{n})^2}{n-1} dt}{\frac{\sum \|x_i\|^2}{n}} . \quad \text{Eq. (13)}$$

Then the relationship of Eq. (11) becomes:

$$v = 1 - \frac{\frac{\sum \sum_{i \neq j} \frac{1}{T} \int_T x_i x_j dt}{n(n-1)}}{\frac{\sum \|x_i\|^2}{n}} = 1 - \frac{\frac{\sum \sum_{i \neq j} \|x_i\| \|x_j\|}{n(n-1)}}{\frac{\sum \|x_i\|^2}{n}} \frac{\sum \sum_{i \neq j} \frac{1}{T} \int_T x_i x_j dt}{\sum \sum_{i \neq j} \|x_i\| \|x_j\|} . \quad \text{Eq. (14)}$$

The above can be written as:

$$v = 1 - ms, \quad \text{Eq. (15)}$$

where:

$$m \equiv \frac{\frac{\sum \sum_{i \neq j} \|x_i\| \|x_j\|}{n(n-1)}}{\frac{\sum \|x_i\|^2}{n}} \quad \text{Eq. (16)}$$

$$s \equiv \frac{\sum \sum_{i \neq j} \frac{1}{T} \int_T x_i x_j dt}{\sum \sum_{i \neq j} \|x_i\| \|x_j\|} \quad \text{Eq. (17)}$$

Eq. (17) can be viewed as a generalized version of the two-signal shape metric definition in [10, 2].

For convenience of notation and discussion, the equations have been written for continuous time signals. The discrete signal counterparts of Eqs. (14), (16), and (17) are obtained by replacing the integrations with summation, and these are listed in Appendix A.

The metrics m and s reflect different aspects of the similarity of the set of time histories. The value of the metric m , is only dependent on the relative magnitudes of the time histories regardless of their shapes. Therefore, we call m the *magnitude similarity metric*. Note that when the norms of the signals are identical, m achieves a value of 1.

When the magnitude of each signal maintains but its exact time history changes, i.e., when its 'shape' changes, the metric s for the set generally changes. Therefore, we call s the *shape similarity metric*. Note that when all the time histories are strictly proportional (by definition, aligned in time), i.e., have the same 'shape', s achieves a value of 1, regardless of their magnitudes.

The values of both the overall similarity metric v and the shape similarity metric s of a given set depend on the relative shifts in time of the signals, i.e., the alignment. However, the value of the magnitude metric m is independent of the alignment, under the condition that the time duration T is chosen sufficiently large to cover the non-zero part of the response for any meaningful relative shifts between the signals of the set (see Eq. (12)). It then follows that *the dependences on the alignment by the overall similarity metric v and that by the shape similarity metric s are strictly equivalent*. It is logical to define the *optimal alignment* for a set as that which maximizes the shape metric s ; and the equivalency means that the optimal alignment also minimizes the overall metric v .

The definition of the metrics and alignment presented above also leads to another natural 'phase' metric which quantifies the degree of (mis)alignment of the set of signals using the optimally aligned configuration as a reference:

$$p = s_a - s, \quad \text{Eq. (18)}$$

where the subscript 'a' denotes the optimally aligned state, and $s_a \equiv s(\tau_a)$ is evaluated under the optimal set of shifts τ_a .

Therefore,

$$v = 1 - m(p + s_a), \quad \text{Eq. (19)}$$

This relationship may be thought of as a natural *decomposition of the overall similarity metric into three metrics, magnitude (m), and inherent shape (s_a), and phase (p)*, in the current framework of quantifying similarity. This decomposition scheme can be viewed as an extension of that in [10].

Simple two-signal examples can be used to illustrate the meaning of the above. Eqs. (16) and (17) simplify to the following in this case:

$$m = \frac{2}{\frac{\|x_1\|}{\|x_2\|} + \frac{\|x_2\|}{\|x_1\|}}, \quad \text{Eq. (20)}$$

$$s = \frac{\frac{1}{T} \int_T x_1 x_2 dt}{\|x_1\| \|x_2\|} = \frac{\int_T x_1 x_2 dt}{\sqrt{\int_T x_1^2 dt} \sqrt{\int_T x_2^2 dt}}. \quad \text{Eq. (21)}$$

More specifically, let the two signals to be directly related by a scale factor:

$$x_2(t) = k x_1(t), \quad \text{Eq. (22)}$$

then the metrics are functions of the scale factor k ($p = 0$ in this case because the signals as given are always aligned):

$$m = \frac{2|k|}{1+k^2} \quad \text{Eq. (23)}$$

$$s = \frac{k}{|k|} \quad \text{Eq. (24)}$$

$$v = 1 - ms = 2 - \frac{(1+k)^2}{1+k^2} = \frac{(1-k)^2}{1+k^2}. \quad \text{Eq. (25)}$$

These are graphically shown in Figure 1.

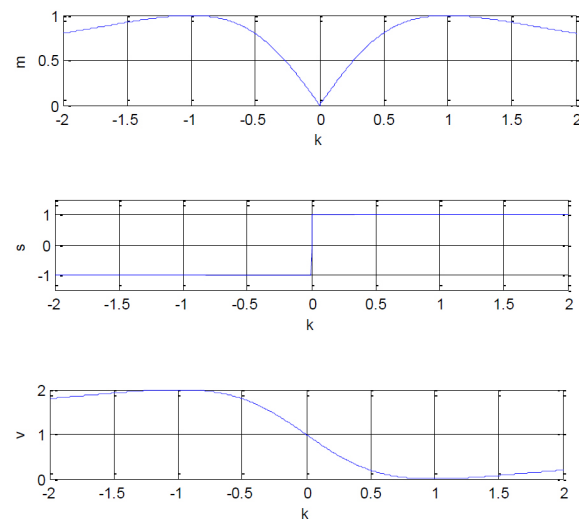


Figure 1. Magnitude, shape, and overall metrics of a scaled pair of time histories. Note the influence of the sign of the scale factor k on the range of values of the metrics: when $k < 0$, $1 < v < 2$, $s = -1$; and when $k > 0$, $0 < v < 1$, $s = 1$.