

# A Reduced-Order Model for Evaluating the Dynamic Response of Multilayer Plates to Impulsive Loads

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## ABSTRACT

Assessing the dynamic performance of multilayer plates subjected to impulsive loading is of interest for identifying configurations that either absorb energy or transmit the energy in the transverse directions, thereby mitigating the through-thickness energy propagation. A reduced-order modeling approach is presented in this paper for rapidly evaluating the structural dynamic performance of various multilayer plate designs. The new approach is based on the reverberation matrix method (RMM) with the theory of generalized rays for fast analysis of the structural dynamic characteristics of multilayer plates. In the RMM model, the waves radiated from the dynamic load are reflected and refracted at each interface between layers, and the waves within each layer are transmitted with a phase lag. These two phenomena are represented by the global scattering matrix and the global phase matrix, respectively. The product of these two matrices after some mathematical manipulations provides a reverberation matrix that represents the waves within the entire plate. The dynamic response of the plate is calculated by employing the generalized ray theory and an inverse Fourier Transformation. Free response results obtained by the new approach are shown to have good agreement with those obtained by a spectral finite element analysis. Then, different multilayer plate design configurations are ranked by evaluating the dynamic response with the new reduced-order model. These forced response results are validated successfully by comparison to the ranking obtained through much more computationally expensive finite element analysis results.

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## INTRODUCTION

The concepts of energy absorption and re-direction are used in vehicle designs for improving vehicle survivability, as well as force protection by mitigating injury due to underbody blast and vehicle crash events. Full system, end-to-end modeling and simulation methodologies [1, 2, 3, 4] have been used extensively for the development of more blastworthy ground vehicles. However, due to space and packaging constraints, utilizing traditional connected structural elements to distribute energy is often infeasible for vehicle structures subjected to underbody blast loads.

Recently, the concept of using properties of softer structural materials in combination with a damping mechanism has been investigated for a V-hull structure. This was found to reduce the Dynamic Response Index (DRI), and hence the occupant injury risk, by absorbing destructive energy through higher deformation compared to isotropic high strength steel [5-6]. Few monolithic materials appear likely to be able to meeting the conflicting demands of greater survivability and lighter structure weight, which is also an important factor in vehicle design. Thus, innovative multilayer materials or structures have been developed, and the study of the dynamic performance of multilayer plates is of great interest as a mechanism to absorb and spread the energy from an

impulsive load in the lateral direction instead of entering the occupant space [7-8]. To reduce the expensive computational cost by using finite element analysis (FEA) on a large number of alternative configurations, many reduced-order modeling approaches have been applied in preliminary design [9, 10, 11, 12, 13, 14].

In this paper, the reverberation matrix method (RMM) provides the foundation for developing a reduced-order model for studying the dynamic response of multilayer structures [15, 16, 17, 18, 19, 20, 21, 22] subjected to an impulsive load. In this method, the wave potential of each sublayer is transformed into the spectral domain by applying double Fourier Transforms. The waves generated by external loading are reflected and refracted at the interfaces of adjacent sublayers, and a scattering matrix  $\mathbf{S}$  is built to represent the condition of continuity of stresses and displacements at each interface. The phase lag values of the traveling waves within each sublayer are represented by a phase matrix  $\mathbf{P}$ . By inserting a permutation matrix  $\mathbf{U}$ , a reverberation matrix  $\mathbf{R}$  is generated as  $\mathbf{R}=\mathbf{S}\mathbf{P}\mathbf{U}$  to represent the wave propagation and dynamic characteristics of the entire structure. By employing the generalized ray theory and inverse Fourier Transform, the dynamic response can be evaluated very quickly. Thus, the reduced-order dynamic model can be used to rapidly evaluate the performance of

alternative multilayer plate designs and generate information for identifying the preferred directions and amounts of energy transfer. Then, by tuning material and physical properties in each sublayer, the information generated by the reduced-order model can be used as part of a design methodology that can be applied to a large number of candidate configurations of multilayer plates.

In the next section, the reduced-order modeling method is presented. In the following sections, the accuracy and efficiency of the method is validated for a numerical example of a three-layer plate. First, the free response results are compared with those from spectral FEA [23]. Then, the forced response results and CPU times are compared with those from classical FEA using the commercial code Nastran.

## REDUCED-ORDER MODEL (ROM)

In this section, the five steps for developing the ROM are presented: (1) definition of local coordinate systems, (2) elastic wave propagation in spectral domain, (3) scattering phenomenon and boundary conditions, (4) phase matrix and reverberation matrix generation, and (5) dynamic response calculation.

### Definition of Local Coordinate Systems

To apply the ROM, an infinite multilayer plate with  $N$  layers  $y=y^J$  ( $J=0, 1, 2, \dots, N$ ) is considered. A 2-D coordinate system  $(x, y)$  is constructed in which the wave normal of the p-wave (in-plane longitudinal wave) and s-wave (shear wave) lie. The double local coordinate system  $(x^{IJ}, y^{IJ})$  and  $(x^{I(I-1)}, y^{I(I-1)})$  at each interface and a regular local coordinate system at the top boundary  $(x^{01}, y^{01})$  and the bottom boundary  $(x^{N(N-1)}, y^{N(N-1)})$  are defined respectively, as shown in Figure 1. The material properties in each layer are isotropic, including mass density  $\rho^J$ , elasticity modulus  $\lambda^J$  and  $\mu^J$ , and the thickness  $h^J$ . The force applied at the interface is defined as a vector  $\mathbf{f}^J$ .

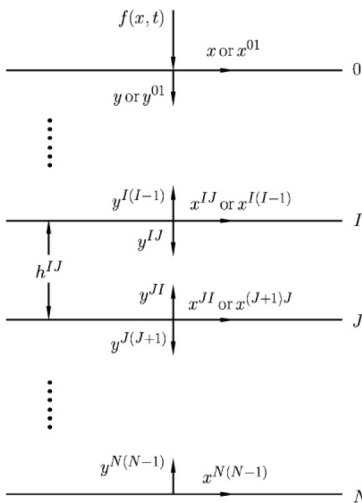


Figure 1. An infinite multilayer plate and its local coordinate systems at the boundaries and interfaces

### Elastic Wave Propagation in Spectral Domain

The p-wave potential  $\phi^{IJ}(x, y, t)$  with p-wave speed  $C_p^{IJ}$  and s-wave potential  $\psi^{IJ}(x, y, t)$  with s-wave speed  $C_s^{IJ}$  are defined in each layer. By applying the double Fourier Transform on the fundamental wave equations in the space and time domains, the wave equations are represented in the wave number  $k$  and frequency  $\omega$  domains:

$$\frac{d^2 \tilde{\phi}}{dy^2} + \alpha^2 \tilde{\phi} = 0; \quad (1a)$$

$$\frac{d^2 \tilde{\psi}}{dy^2} + \beta^2 \tilde{\psi} = 0; \quad (1b)$$

where

$$\alpha = \sqrt{\frac{\omega^2}{C_p^2} - k^2}; \text{ (p-wave number in y direction);}$$

$$\beta = \sqrt{\frac{\omega^2}{C_s^2} - k^2}; \text{ (s-wave number in y direction);}$$

The solutions for the wave equations can be expressed in terms of the arrival wave amplitude and departure wave amplitude as follows:

$$\tilde{\phi}(k, y, \omega) = \tilde{a}_p(k, \omega)e^{-i\alpha y} + \tilde{d}_p(k, \omega)e^{i\alpha y}; \quad (2a)$$

$$\tilde{\psi}(k, y, \omega) = \tilde{a}_s(k, \omega)e^{-i\beta y} + \tilde{d}_s(k, \omega)e^{i\beta y}; \quad (2b)$$

where  $\tilde{a}_p, \tilde{a}_s$  are the unknown arrival wave amplitudes;

$\tilde{d}_p, \tilde{d}_s$  are the unknown departure wave amplitudes;

The displacement fields  $u = [u_x, u_y]$  and the stress components  $\sigma = [\sigma_{xy}, \sigma_{yy}]$  can be determined by applying fundamental elastodynamic solutions in each local coordinate system:

$$u_x = \frac{\partial \phi}{\partial x} - \frac{\partial \psi}{\partial y}; \quad (3a)$$

$$u_y = \frac{\partial \phi}{\partial y} + \frac{\partial \psi}{\partial x}; \quad (3b)$$

$$\sigma_{xy} = \mu \left[ 2 \frac{\partial^2 \phi}{\partial x \partial y} + \frac{\partial^2 \psi}{\partial x^2} - \frac{\partial^2 \psi}{\partial y^2} \right]; \quad (3c)$$

$$\sigma_{yy} = \lambda \nabla^2 \phi + 2\mu \left[ \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \psi}{\partial x \partial y} \right]; \quad (3d)$$

### Scattering Phenomenon and Boundary Condition

The scattering phenomenon at each interface is shown in Figure 2, where  $\mathbf{a}$  represents the “arrival waves” and  $\mathbf{d}$  represents the “departure waves”. Also, the subscript  $p$  represents the “p-wave” which is also called the longitudinal wave, and the subscript  $s$

represents the “sv-wave”, which is also called the shear wave. During the scattering, the incident waves will separate into reflected waves and refracted waves, and the wave types will interconvert during scattering. Therefore, a local scattering matrix is built to represent this complicated phenomenon.

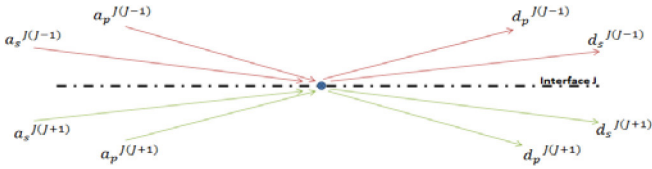


Figure 2. Scattering phenomenon at interface J

Basically, if there are no wave sources inside the multilayer plate, the displacement and stress yield the continuity condition at each interface:

$$\tilde{u}_x^{J(J-1)}(k, \omega) - \tilde{u}_x^{J(J+1)}(k, \omega) = 0; \quad (4a)$$

$$\tilde{u}_y^{J(J-1)}(k, \omega) + \tilde{u}_y^{J(J+1)}(k, \omega) = 0; \quad (4b)$$

$$\tilde{\sigma}_{xy}^{J(J-1)}(k, \omega) + \tilde{\sigma}_{xy}^{J(J+1)}(k, \omega) = 0; \quad (4c)$$

$$\tilde{\sigma}_{yy}^{J(J-1)}(k, \omega) - \tilde{\sigma}_{yy}^{J(J+1)}(k, \omega) = 0; \quad (4d)$$

By substituting the elastodynamic solution into the above equations, a linear system is built to describe the scattering phenomenon, where  $A^J$  and  $D^J$  are both  $4 \times 4$  matrices.

$$A^J \cdot \tilde{\mathbf{a}}^J + D^J \cdot \tilde{\mathbf{d}}^J = \tilde{\mathbf{g}}^J(k, \omega) = 0 \quad (5)$$

And the unknown arrival and departure wave amplitude vectors are:

$$\tilde{\mathbf{a}}^J = (\tilde{a}_p^{J(J-1)}, \tilde{a}_s^{J(J-1)}, \tilde{a}_p^{J(J+1)}, \tilde{a}_s^{J(J+1)});$$

$$\tilde{\mathbf{d}}^J = (\tilde{d}_p^{J(J-1)}, \tilde{d}_s^{J(J-1)}, \tilde{d}_p^{J(J+1)}, \tilde{d}_s^{J(J+1)});$$

The linear system can be also written in the form of equation (6), by multiplying  $(D^J)^{-1}$  to equation (5):

$$\tilde{\mathbf{d}}^J = S^J \cdot \tilde{\mathbf{a}}^J + \mathbf{s}^J \quad (6)$$

where

$S^J = -(D^J)^{-1} \cdot A^J$ , is the local scattering matrix at interface J

$\mathbf{s}^J = -(D^J)^{-1} \cdot \tilde{\mathbf{g}}^J$ , is the local source vector at interface J

In fact, all of the elements in the local source vector are zero, because in our configuration there are no wave sources inside the multilayer plate. But the condition would be different at the top and bottom

boundaries. First of all, the local scattering matrix will degenerate into  $2 \times 2$  instead of  $4 \times 4$ , because there is only one local coordinate and no continuous displacement relationship exists at the boundary. Second, the external excitation is applied at the top of the plate, so the elements in the local source vector  $\mathbf{s}^0$  are nonzero.

The boundary conditions are:

$$\tilde{\sigma}_{xy}^{01}(k, \omega) = 0; \quad (7a)$$

$$\tilde{\sigma}_{yy}^{01}(k, \omega) = F(\omega); \quad (7b)$$

$$\tilde{\sigma}_{xy}^{N(N+1)}(k, \omega) = 0; \quad (7c)$$

$$\tilde{\sigma}_{yy}^{N(N+1)}(k, \omega) = 0; \quad (7d)$$

$S^0$  and  $S^N$  now degenerate into  $2 \times 2$  matrices and the source vector  $\mathbf{s}^0 \neq 0$  if any loads apply on top surface of the plate.

By combining all the local scattering matrices at interfaces and the local scattering matrices at boundary, a  $4N \times 4N$  global scattering matrix  $S$  is constructed and a global linear system for the entire multilayer plate is represented in the following form:

$$\tilde{\mathbf{d}} = S \cdot \tilde{\mathbf{a}} + \tilde{\mathbf{s}} \quad (8)$$

This could also be shown in expanded form as:

$$\begin{pmatrix} \tilde{d}^0 \\ \tilde{d}^1 \\ \tilde{d}^2 \\ \vdots \\ \tilde{d}^{N-2} \\ \tilde{d}^{N-1} \\ \tilde{d}^N \end{pmatrix} = \begin{pmatrix} S^0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & S^1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & S^2 & \ddots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & S^{N-2} & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & S^{N-1} & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & S^N \end{pmatrix} \begin{pmatrix} \tilde{a}^0 \\ \tilde{a}^1 \\ \tilde{a}^2 \\ \vdots \\ \tilde{a}^{N-2} \\ \tilde{a}^{N-1} \\ \tilde{a}^N \end{pmatrix} + \begin{pmatrix} \tilde{s}^0 \\ \tilde{s}^1 \\ \tilde{s}^2 \\ \vdots \\ \tilde{s}^{N-2} \\ \tilde{s}^{N-1} \\ \tilde{s}^N \end{pmatrix}$$

### Phase Matrix and Reverberation Matrix

An additional equation is defined by noting that a wave departing from one side of the layer becomes the wave arriving at another side of the same layer, with a phase lag.

$$\tilde{a}_p^{J(J-1)} = e^{i\alpha_J h_J} \cdot \tilde{d}_p^{(J-1)J}; \quad (9a)$$

$$\tilde{a}_s^{J(J-1)} = -e^{i\beta_J h_J} \cdot \tilde{d}_s^{(J-1)J}; \quad (9b)$$

$$\tilde{d}_p^{J(J-1)} = e^{-i\alpha_J h_J} \cdot \tilde{a}_p^{(J-1)J}; \quad (9c)$$

$$\tilde{d}_s^{J(J-1)} = -e^{-i\beta_J h_J} \cdot \tilde{a}_s^{(J-1)J};$$