

An Improved Reanalysis Method Using Parametric Reduced Order Modeling for Linear Dynamic Systems

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ABSTRACT

Finite element analysis is a standard tool for deterministic or probabilistic design optimization of dynamic systems. The optimization process requires repeated eigenvalue analyses which can be computationally expensive. Several reanalysis techniques have been proposed to reduce the computational cost including Parametric Reduced Order Modeling (PROM), Combined Approximations (CA), and the Modified Combined Approximations (MCA) method. Although the cost of reanalysis is substantially reduced, it can still be high for models with a large number of degrees of freedom and a large number of design variables. Reanalysis methods use a basis composed of eigenvectors from both the baseline and the modified designs which are in general linearly dependent. To eliminate the linear dependency and improve accuracy, Gram Schmidt orthonormalization is employed which is costly itself. In this paper, we propose a method to reduce the orthonormalization cost and improve the computational efficiency of the PROM reanalysis method. Our method eliminates non-important design variables and/or reduces the basis size by eliminating redundant modes. A vibratory analysis of an automotive door demonstrates the efficiency and accuracy of the proposed method.

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1. INTRODUCTION

Finite element analysis (FEA) is often the main computational tool for Noise, Vibration and Harshness (NVH) studies. Although FEA can handle large-scale models, automotive companies demand even larger detailed models with millions of degrees of freedom (DOFs) to study vibro-acoustic problems. As a result, the computational cost can be prohibitive even for high-end workstations with the most advanced software.

For linear vibration analysis, a modal reduction is commonly used to solve for the system response. An eigenanalysis is performed using the system stiffness and mass matrices and a smaller in size modal model is formed which is solved more efficiently. The computational cost can also be reduced using substructuring (superelement analysis). Modal reduction is applied to each substructure to obtain the component modes and the system level response is obtained using Component Mode Synthesis (CMS).

When design changes are involved (e.g., optimization), the FEA analysis must be repeated many times and the computational cost can become prohibitive. Reanalysis methods are intended to efficiently analyze structures that are repeatedly modified by estimating the response of the modified structure without solving the complete set of modified analysis equations. They use a basis of eigenvectors from the baseline design, augmented sometimes by eigenvectors from other design points as in PROM [1], to estimate the eigenvalues and eigenvectors of the modified design without performing a new eigenvalue analysis. The eigenvectors in the basis are often linearly

dependent. To avoid numerical errors from this dependency, a Gram Schmidt orthonormalization must be performed. In general, this is a very costly operation especially when the basis is large (many eigenvectors are used) and/or the model size is large (many DOFs).

The eigenvalue problem for an undamped system with stiffness $\mathbf{K}_{n \times n}$ and mass $\mathbf{M}_{n \times n}$ is given by

$$\mathbf{K}_{n \times n} \Phi_{n \times m} = \mathbf{M}_{n \times n} \Phi_{n \times m} \Lambda_{m \times m} \quad (1)$$

where n represents the number of degrees of freedom (DOF) and m represents the number of kept modes. If a perturbation is applied on the existing structure, the modified system matrices are

$$\mathbf{K}_p = \mathbf{K}_0 + \Delta \mathbf{K}, \quad \mathbf{M}_p = \mathbf{M}_0 + \Delta \mathbf{M} \quad (2)$$

where $\mathbf{K}_0$, $\mathbf{M}_0$ and $\mathbf{K}_p$, $\mathbf{M}_p$ are the stiffness and mass matrices of the original (baseline) and modified systems, respectively.

In reanalysis, a modal model is formed using the following reduced matrices

$$\mathbf{K}_r = \Phi_0^T \mathbf{K}_p \Phi_0, \quad \mathbf{M}_r = \Phi_0^T \mathbf{M}_p \Phi_0 \quad (3)$$

where Φ_0 are the eigenvectors of the original (baseline) design and the following eigenvalue problem is then solved.

$$\mathbf{K}_r \boldsymbol{\Theta} = \boldsymbol{\Lambda} \mathbf{M}_r \boldsymbol{\Theta} \quad (4)$$

The reduced eigenvector $\boldsymbol{\Theta}$ is then projected back to the physical domain to obtain the eigenvectors $\boldsymbol{\Phi}_p$ of the modified design as

$$\boldsymbol{\Phi}_p = \boldsymbol{\Phi}_0 \boldsymbol{\Theta} \quad (5)$$

In this paper, we present an approach to minimize the computational effort for the eigenvector orthonormalization in PROM by reducing the size of the basis. This is achieved first by eliminating the eigenvectors of non-important design variables (corner points) and second by neglecting redundant modes. In [Section 2](#), we present an overview of reanalysis methods, including PROM. [Section 3](#) provides details on our proposed methodology and [Section 4](#) uses an automotive door example to demonstrate the efficiency and accuracy of the proposed approach. Finally, [Section 5](#) summarizes and concludes.

2. REVIEW OF EXISTING REANALYSIS METHODS

In this section, we briefly review the PROM, CA, and MCA reanalysis methods.

Parametric Reduced-Order Modeling (PROM)

The PROM method approximates the mode shapes (eigenvectors) of a new design in the subspace spanned by the dominant mode shapes of some representative designs. The latter are selected such that the formed basis captures the dynamic characteristics of the modified design due to changes in each dimension of the parameter space ([Figure 1](#)). Balmes [2, 3] suggested that these representative designs should correspond to the mid-range of design parameters in the parameter space.

For a structure with m design variables, Zhang [4] suggested that the representative designs include a *baseline* design for which all parameters are at their lower limits plus m designs obtained by perturbing the design variables from their lower limits to their upper limits, one at a time. The points representing these designs in the space of the design parameters will be called *corner points* ([Figure 1](#)).

The mode shapes $\boldsymbol{\Phi}$ of a new design are approximated in the space of the mode shapes of the corner points as

$$\boldsymbol{\Phi} \approx \tilde{\boldsymbol{\Phi}}_p = \mathbf{P} \boldsymbol{\Theta} \quad (6)$$

where the modal matrix $\mathbf{P}$ contains the basis vectors according to [Equation \(7\)](#)

$$\mathbf{P} = [\boldsymbol{\Phi}_0 \quad \boldsymbol{\Phi}_1 \quad \dots \quad \boldsymbol{\Phi}_m] \quad (7)$$

and $\boldsymbol{\Theta}$ represents the participation factors of these vectors. The columns of $\mathbf{P}$ are the dominant mode shapes of the above $(m+1)$ designs. [Figure 1](#) shows a design space for three design parameters.

Point $\mathbf{P}_0$ represents the baseline design with all design parameters at their minimum values, and point $\mathbf{P}_i$, $i = 1, 2, 3$ represents a modified design with all parameters at their minimum values except parameter i which is at its maximum (perturbed) value.

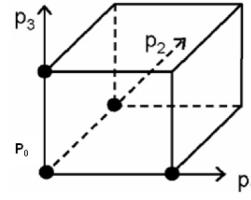


Figure 1. Design space with three parameters

In [Equation \(7\)](#), $\boldsymbol{\Phi}_0$ is the modal matrix composed of the dominant mode shapes of the baseline design, and $\boldsymbol{\Phi}_i$ is the modal matrix of the i^{th} corner point. The mode shapes of the new design satisfy the following eigenvalue problem,

$$\mathbf{K} \tilde{\boldsymbol{\Phi}}_p = \boldsymbol{\Lambda} \mathbf{M} \tilde{\boldsymbol{\Phi}}_p \Leftrightarrow \mathbf{K} \mathbf{P} \boldsymbol{\Theta} = \boldsymbol{\Lambda} \mathbf{M} \mathbf{P} \boldsymbol{\Theta} \quad (8)$$

where $\mathbf{K}$ and $\mathbf{M}$ are the stiffness and mass matrices of the modified design and $\boldsymbol{\Lambda}$ is a diagonal matrix of the first m eigenvalues.

A reduced eigenvalue problem is obtained by pre-multiplying both sides of [Equation \(8\)](#) by $\mathbf{P}^T$ to obtain

$$\mathbf{K}_R \boldsymbol{\Theta} = \boldsymbol{\Lambda} \mathbf{M}_R \boldsymbol{\Theta} \quad (9)$$

where the reduced stiffness and mass matrices are

$$\mathbf{K}_R = \mathbf{P}^T \mathbf{K} \mathbf{P} \text{ and } \mathbf{M}_R = \mathbf{P}^T \mathbf{M} \mathbf{P} \quad (10)$$

The matrix $\boldsymbol{\Theta}$ in [Equation \(9\)](#) contains the eigenvectors of the reduced stiffness and mass matrices $\mathbf{K}_R$ and $\mathbf{M}_R$.

For m design variables, $(m+1)$ eigenvalue problems must be solved in order to form the basis $\mathbf{P}$ of [Equation \(7\)](#). Therefore, both the cost of obtaining the modal matrices $\boldsymbol{\Phi}_i$ and the size of matrix $\mathbf{P}$ increase linearly with m . The PROM approach uses the following algorithm to compute the mode shapes of a new design:

1. Calculate the mode shapes of the baseline design and the designs corresponding to the m corner points in the design space, and form subspace basis $\mathbf{P}$.
2. Calculate the reduced stiffness and mass matrices $\mathbf{K}_R$ and $\mathbf{M}_R$ using [Equation \(10\)](#).
3. Solve the eigen-problem of [Equation \(9\)](#) for matrix $\boldsymbol{\Theta}$.
4. Reconstruct the approximated eigenvectors in $\tilde{\boldsymbol{\Phi}}_p$ using [Equation \(6\)](#).

In the above procedure, step 1 is performed only once. A reanalysis requires only steps 2 to 4. For a small number of mode shapes and a small number of design parameters, the cost of this reanalysis is smaller compared to the cost of a full analysis.

The computational cost of PROM consists of a) the cost of performing $(m + 1)$ full eigen-analyses to form the subspace basis $\mathbf{P}$ in Equation (7), and b) the cost of reanalysis of each new design in steps 2 to 4. The former is considered the fixed cost of PROM because it does not depend on the number of reanalyses and the latter represents the variable cost of PROM since it is proportional to the number of reanalyses. The fixed cost is not attributed to the calculation of the response of a particular design. Simply, it is required to obtain the needed information to apply PROM. The variable cost (cost of reanalysis of a new design in part b) is small compared to the fixed cost.

The fixed cost of PROM is proportional to the number of design variables m since the basis $\mathbf{P}$ consists of the dominant eigenvectors Φ_0 of the baseline design, and the dominant eigenvectors Φ_p , $i = 1, \dots, m$ of the m corner design points as shown in Equation (7). As the size of $\mathbf{P}$ increases, so does the fixed cost because more eigenvalue problems and mode shapes must be calculated. The PROM method results in significant cost savings when applied to problems that involve few design variables (less than 20) and require many analyses (e.g. Monte Carlo simulation or gradient-free optimization using genetic algorithms).

The efficiency of the PROM reanalysis method can be further improved by integrating it with the MCA method [5].

Combined Approximations (CA)

In the CA method [6, 7, 8, 9, 10, 11, 12, 13, 14, 15], the eigenvectors at a new design are approximated using a linear combination of basis vectors. A subspace basis is formed as

$$\mathbf{R}_{n \times s} = [\mathbf{R}_{n \times m}^1 \quad \mathbf{R}_{n \times m}^2 \quad \dots \quad \mathbf{R}_{n \times m}^s] \quad (11)$$

where s is the number of basis vectors normally in the range of 3-6 depending on accuracy and efficiency [7, 10].

For a modified design, the stiffness and mass matrices are given by

$$\mathbf{K}_p = \mathbf{K}_0 + \Delta\mathbf{K}, \quad \mathbf{M}_p = \mathbf{M}_0 + \Delta\mathbf{M} \quad (12)$$

where $\mathbf{K}_0$ and $\mathbf{M}_0$ are the stiffness and mass matrices of the baseline (original) design and $\Delta\mathbf{K}$ and $\Delta\mathbf{M}$ represent large perturbations. The basis vectors in Equation (11) are provided by Equations (13) and (14) as

$$\mathbf{R}_{n \times m}^1 = \mathbf{K}_0^{-1} \mathbf{M}_p \Phi_0 \quad (13)$$

$$\mathbf{R}_{n \times m}^j = -\mathbf{K}_0^{-1} \Delta\mathbf{K} \mathbf{R}_{n \times m}^{j-1}, j = 2, 3, \dots, s \quad (14)$$

where Φ_0 are the eigenvectors of the baseline design. For the perturbed (new) design, a reduced model is obtained with the following stiffness and mass matrices

$$\mathbf{K}_{n \times n}^R = \mathbf{R}_{n \times m}^T \mathbf{K}_p^R \mathbf{R}_{n \times m}, \quad \mathbf{M}_{n \times n}^R = \mathbf{R}_{n \times m}^T \mathbf{M}_p^R \mathbf{R}_{n \times m} \quad (15)$$

and an eigenvalue problem is solved using the $\mathbf{K}_{n \times n}^R$ and $\mathbf{M}_{n \times n}^R$ matrices of Equation (15) to calculate the eigenvectors Θ of the reduced model. These eigenvectors are then projected back to the physical domain to obtain the following approximate eigenvectors Φ_p of the new design as

$$\Phi_p = \mathbf{R}_{n \times s} \Theta. \quad (16)$$

Because the basis vectors are linearly dependent, a Gram Schmidt orthonormalization is performed to avoid numerical errors. The CA method is efficient for problems where the number of desired modes is low. In such a case, the computational effort of stiffness matrix decomposition (Equations 13 and 14) dominates.

The CA method has the following three main advantages:

1. It only requires a single matrix decomposition of stiffness matrix $\mathbf{K}_0$ to calculate the subspace basis $\mathbf{R}$,
2. It is accurate because the basis is updated for every new design, and
3. The eigenvectors of a new design are efficiently approximated using Equation (16) where the eigenvectors Θ correspond to a much smaller eigen-problem.

It should be noted that the CA method is more suitable than PROM, if the number of reanalyses is less than the number of design parameters. However, it is efficient only for problems with a small number of retained modes.

Modified Combined Approximations (MCA)

A modified combined approximations (MCA) method was proposed in [5, 16] where the columns of the subspace basis are obtained from by the following recursive equations

$$\mathbf{T}_{n \times m}^1 = \mathbf{K}_p^{-1} \mathbf{M}_p \Phi_0 \quad (17)$$

$$\mathbf{T}_{n \times m}^j = \mathbf{K}_p^{-1} \mathbf{M}_p \mathbf{T}_{n \times m}^{j-1}, j = 2, 3, \dots, s \quad (18)$$

where $\mathbf{K}_p$ and $\mathbf{M}_p$ are the stiffness and mass matrices of the modified design. The subspace basis is defined as

$$\mathbf{T} = [\Phi_0 \quad \mathbf{T}_1 \quad \dots \quad \mathbf{T}_s] \quad (19)$$

where s is usually equal to 1 or 2. A Gram Schmidt orthonormalization is again performed to avoid numerical errors from linear dependency. A reduced model is defined using the following reduced stiffness and mass matrices