

A Methodology for Fatigue Life Estimation of Linear Vibratory Systems under Non-Gaussian Loads

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ABSTRACT

Fatigue life estimation, reliability and durability are important in acquisition, maintenance and operation of vehicle systems. Fatigue life is random because of the stochastic load, the inherent variability of material properties, and the uncertainty in the definition of the S-N curve. The commonly used fatigue life estimation methods calculate the mean (not the distribution) of fatigue life under Gaussian loads using the potentially restrictive narrow-band assumption. In this paper, a general methodology is presented to calculate the statistics of fatigue life for a linear vibratory system under stationary, non-Gaussian loads considering the effects of skewness and kurtosis. The input loads are first characterized using their first four moments (mean, standard deviation, skewness and kurtosis) and a correlation structure equivalent to a given Power Spectral Density (PSD). Then, the first four moments and the correlation structure of the response process are calculated using the impulse response function of the system. Polynomial Chaos Expansion (PCE) and Karhunen-Loeve (KL) expansion are used to develop stochastic metamodels of the input and output non-Gaussian processes. Subsequently, simulated trajectories from the response stochastic metamodel are rainflow counted to obtain realizations of the fatigue life random variable based on the Miner's damage model. Finally, the Saddlepoint Approximations (SPA) method provides the PDF and percentiles of the fatigue life. The approach can be easily extended to non-stationary loads, and with some modifications, to even non-linear vibratory systems. It can also be used to develop an accelerated testing procedure without increasing the applied load artificially. The proposed methodology is demonstrated with a quarter car example under non-Gaussian load.

CITATION: Tsianika, V., Geroulas, V., Mourelatos, Z., and Baseski, I., "A Methodology for Fatigue Life Estimation of Linear Vibratory Systems under Non-Gaussian Loads," *SAE Int. J. Commer. Veh.* 10(2):2017, doi:10.4271/2017-01-0197.

1. INTRODUCTION

Fatigue life estimation, reliability and durability are important in acquisition, maintenance and operation of vehicle systems. Assessment of fatigue damage and durability in a laboratory environment is also important. The problem is however, challenging both theoretically and practically. It has been shown that damage accumulation is sensitive to non-Gaussian excitation [1] and is also affected by elastoplastic deformation in a multiaxial stress environment under non-stationary and non-Gaussian stress signals [2].

In automotive applications, it is common to experience non-Gaussian excitation such as road irregularities (e.g., bumps and potholes), among others. The traditional Gaussian random test signals cannot represent these road irregularities. Very different field failure rates can be observed if the right type of test signal is not used during vibration testing [1].

There are many fatigue life estimation methods in the frequency and time domains [2, 3, 4, 5, 6, 7, 8]. Frequency domain methods are relatively simple. An overview is provided in [4]. However, they introduce restrictive assumptions as we describe later. Fatigue failure

is a cumulative event due to cyclic loading. The fatigue damage $d = \frac{\alpha}{\alpha_f}$ is the ratio of the current crack length α to the final crack length α_f at failure. The linear damage Palmgren-Miner damage model [2, 9] $d = \frac{n}{N_f}$ is common where n is the number of cycles at a constant amplitude cyclic load (e.g. stress) and N_f is the number of cycles for the specimen to fail at the same stress level.

The Palmgren-Miner damage model has been proved accurate for random loading if stresses are not high enough to induce plastic deformation [2]. One first counts (estimates) the number of cycles n_i at stress level S_i and uses the material S-N curve to calculate the number of cycles N_{fi} the specimen will fail under the cyclic load S_i . The damage model is then utilized to calculate the incremental damage d_i . The cumulative damage D is calculated by summing up the incremental damage d_i over all load (stress) levels i as

$$D = \sum_i d_i = \sum_i \frac{n_i}{N_{fi}}$$

For a zero-mean stationary random stress signal and under the strong assumption of a narrow-band random process, the linear damage model provides the following *expected* damage [2, 9]

$$E(D) = \frac{v_0 T}{A} \int_0^{\infty} s^m f_S(s) ds \quad (2)$$

where v_0 is the mean up-crossing rate, T is the signal duration, A and m are coefficients of the S - N curve ($N_f \cdot S^m = A$) and $f_S(s)$ is the PDF of the peak stress (S) level. The distribution $f_S(s)$ is used for cycle counting under the narrow-band assumption (existence of a negative stress trough equal in magnitude with a positive stress peak to form a load cycle). Note that D is a random variable and its range of values (e.g., 95th percentile minus 5th percentile) can be wide. Thus, the expected value $E(D)$ may be of limited practical significance.

It can be shown that for a zero-mean, Gaussian, narrow-band process (e.g., harmonic process with constant frequency and random amplitude), $f_S(s)$ is Rayleigh distributed as

$$f_S(s) = \frac{s}{\sigma_X^2} \exp\left(-\frac{s^2}{2\sigma_X^2}\right) \quad (3)$$

where σ_X^2 is the variance of the underlying Gaussian process $X(t)$. In this case, the linear damage model (1) yields the following expected value $E(D)$ of damage [9]

$$E(D) = \frac{v_0 T}{A} (\sqrt{2}\sigma)^m \Gamma\left(\frac{m}{2} + 1\right). \quad (4)$$

For a zero-mean, Gaussian, wide-band process with irregularity factor $\alpha = \frac{v_0}{v_p}$ (ratio of zero up-crossing frequency to peak frequency), the distribution $f_S(s)$ is provided by the following Rice formula [4, 9]

$$f_S(s) = (1 - \alpha^2) \frac{1}{\sqrt{2\pi(1 - \alpha^2)}\sigma_X} \cdot \exp\left(-\frac{s^2}{2(1 - \alpha^2)\sigma_X^2}\right) + \alpha \Phi\left(\frac{\alpha}{\sqrt{1 - \alpha^2}} \cdot \frac{s}{\sigma_X}\right) \frac{s}{\sigma_X^2} \cdot \exp\left(-\frac{s^2}{2\sigma_X^2}\right) \quad (5)$$

The Powell distribution of peak stresses $f_S(s) = -\frac{1}{v_p} \frac{d(v_{s+})}{ds}$ where v_{s+} is the up-crossing rate for stress level s and v_p is the peak frequency, does not use the Gaussian assumption and is therefore, more general than the Rice formula of Equation (5). However, it cannot be used for cycle counting unless we assume a narrow band process. Both v_{s+} and v_p can be calculated numerically for a non-Gaussian distribution [10, 11].

Dirlik [12, 4] has developed an empirical peak stress distribution using Monte Carlo simulations and rainflow cycle counting as a weighted average of an exponential and two Rayleigh distributions. Dirlik's peak stress distribution is considered more accurate compared to Rayleigh or Rice distributions for wide-band processes [4, 6]. However, it can only be used to estimate the expected (not the actual) cumulative damage using Equation (2).

Fatigue failure occurs if D is greater than one ($D > 1$). The fatigue life T_L is random because of 1) randomness of the input (represented by a provided spectrum), 2) inherent variability of the test component (material properties, dimensions, etc.), and 3) uncertainty in the definition of the S - N curve (randomness of m and A coefficients). The fatigue reliability $R(T)$ at time T is defined as the probability that T_L is greater than T ; i.e.,

$$R(T) = \Pr(T_L > T) \text{ or } R(T) = \Pr(D < 1). \quad (6)$$

The main shortcomings of the described state-of-the-art fatigue reliability methods (Equations 2 to 5) are: 1) Equation (2) provides only the mean value of damage. This paper estimates the entire PDF of damage; 2) The restrictive narrow-band assumption is used to allow cycle counting based on the peak stress distribution; 3) The stress process is assumed Gaussian. A non-Gaussian behavior is common in automotive applications.

Equations (3) to (5) are based on the Gaussian assumption which simplifies the estimation of fatigue life significantly. If the excitation is Gaussian and the system is linear, the output stress is also Gaussian and one of the aforementioned approaches can be applied. However, many loads are non-Gaussian resulting in a non-Gaussian stress. Furthermore, characterizing a non-Gaussian signal is not easy. It involves higher moments such as skewness and kurtosis [1, 7]. This paper presents a theoretically sound approach for random vibrations and fatigue life estimation of linear systems under non-Gaussian loads.

The following uses an example to highlight the proposed developments. Details are presented in subsequent sections. Assume the fatigue/durability of the coil spring connecting the sprung and un-sprung masses of a quarter car model is of interest. This example is used in Section 3 to demonstrate the benefits of the proposed approach. The vehicle is traveling over a road profile whose vertical elevation is the input random process. The shear stress of the coil spring is the output random process. To assess the spring's fatigue life one must first calculate (or measure) many trajectories (time series) of the spring stress process. Then for each trajectory, rainflow counting estimates the damage. Damage is a random variable because it is different for each trajectory. The PDF of this random variable determines the fatigue reliability (probability that fatigue life will exceed a specified level).

For an analytical implementation of the above, the following steps are executed:

1. Generate road elevation profiles (trajectories). If the road elevation is Gaussian, only its PSD is needed [9]. For non-Gaussian elevation, road profiles are generated using the proposed PCE approach of [Section 2.1.1](#).
2. For a given road profile, calculate the corresponding coil spring stress trajectory. Repeat this a few times and use the generated stress trajectories to develop a stochastic model of the stress process ([Section 2.1.1](#)).
3. Generate many stress trajectories using the PCE stochastic metamodel. For each trajectory, calculate damage using rainflow counting and a fatigue model (Palmgren-Miner). The generated damage realizations are then used to calculate the PDF of damage using Saddlepoint Approximations (SA) ([Section 2.3.1](#)).

For experimental implementation of the above steps, measuring many stress trajectories in step 2 is impractical. This issue is addressed if the quarter car model is linear. In this case, we only need to measure its impulse response function. Then, the correlation structure and first four moments of the stress process can be analytically calculated using the four moments and correlation structure of the road elevation. [Section 2.1.1](#) provides details.

The paper is arranged as follows. [Section 2](#) describes the proposed methodology with all necessary details. [Section 3](#) uses a quarter car example to demonstrate all developments and [Section 4](#) summarizes and concludes.

2. PROPOSED APPROACH

Consider the following n degree-of-freedom (DOF) linear vibratory system with constant parameters

$$[m]\ddot{\mathbf{X}}(t) + [c]\dot{\mathbf{X}}(t) + [k]\mathbf{X}(t) = \mathbf{F}(t) \quad (7)$$

where $\mathbf{X}(t) = [X_1(t) \ X_2(t) \ \dots \ X_n(t)]$ is the vector of response (output) random processes and $\mathbf{F}(t) = [F_1(t) \ F_2(t) \ \dots \ F_n(t)]$ is the vector of input (force) random processes ([Figure 1](#)). The mass $[m]$, damping $[c]$ and stiffness $[k]$ matrices have constant elements. Although there are n random processes in the vectors $\mathbf{X}(t)$ and $\mathbf{F}(t)$, this paper considers only one input $F_i(t)$, $1 \leq i \leq n$ and one output $X_i(t)$, $1 \leq i \leq n$ for simplicity. [Figure 1](#) provides a schematic of the input-output notation. The approach can be easily extended to linear vibratory systems with random parameters, using the total probability theorem [10].

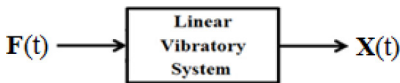


Figure 1. Schematic of input-output notation

2.1 Characterization of Non-Gaussian Input Process

The term “characterization” of a random process indicates the development of a stochastic metamodel for generating trajectories (time series) of the process. Each trajectory is cycle counted and the Miner’s model is used to estimate the corresponding realization of the damage random variable. For a stationary and Gaussian process, the Karhunen Loeve (K-L) expansion [13, 14], the Shinozuka method

[15], the Expansion Optimal Linear Estimation method (EOLE) [13] or the Orthogonal Series Expansion method (OSE) [16] can characterize the process in the time domain. For a non-Gaussian process, the characterization is not easy. Below is the proposed approach to do so using a combination of Polynomial Chaos Expansion (PCE) [17, 18, 19] and K-L expansion. The input random process $F(t)$ is assumed stationary and non-Gaussian.

2.1.1 Polynomial Chaos Expansion (PCE)

A normal or non-normal random variable Z can be approximated using one-dimensional Hermite orthogonal polynomials Ψ_i of the standard normal random variables ξ as [17, 18]

$$Z = \sum_{i=0} b_i \Psi_i = b_0 + b_1 \xi + b_2 (\xi^2 - 1) + b_3 (\xi^3 - 3\xi) + b_4 (\xi^4 - 6\xi^2 + 3) + \dots \quad (8)$$

The Hermite polynomials Ψ_i are defined as $\Psi_i(\xi) = (-1)^i \frac{\phi^{(i)}(\xi)}{\phi(\xi)}$ where $\phi^{(i)}(\xi)$ is the i^{th} derivative of the standard normal PDF $\phi(\xi)$. The first four moments of random variable Z (mean, standard deviation, skewness and kurtosis) can be expressed in terms of the expansion coefficients b_i , $i = 0, 1, 2, \dots$. Given the four moments, a simple optimization problem is then solved to estimate the coefficients in a least squares approximation sense.

[Equation \(8\)](#) can be extended in time as

$$Z(t) = \sum_{i=0} b_i(t) \Psi_i(t) = b_0(t) + b_1(t) \xi(t) + b_2(t) (\xi^2(t) - 1) + b_3(t) (\xi^3(t) - 3\xi(t)) + b_4(t) (\xi^4(t) - 6\xi^2(t) + 3) + \dots \quad (9)$$

to represent a random process $Z(t)$ as a function of the standard normal process $\xi(t)$. This is the proposed stochastic metamodel of $Z(t)$. Using the orthogonality properties of the Hermite polynomials, the covariance $C_{ZZ}(t_1, t_2)$ of $Z(t)$ between times t_1 and t_2 is equal to

$$C_{ZZ}(t_1, t_2) = \sum_{i=1} b_i(t_1) b_i(t_2) \cdot (i!) \cdot (E[\xi(t_1) \xi(t_2)])^i \quad (10)$$

For a given $C_{ZZ}(t_1, t_2)$, [Equation \(10\)](#) is used to calculate the covariance matrix $C_{\xi\xi}(t_1, t_2) = E[\xi(t_1) \xi(t_2)]$ of $\xi(t)$. Based on $C_{\xi\xi}(t_1, t_2)$, a Karhunen-Loeve (K-L) expansion [13] is used to express $\xi(t)$ in terms of independent standard normal variables ξ_i , $i = 0, 1, 2, \dots, N$ as

$$\xi(t) = \sum_{i=1}^N \sqrt{\lambda_i} \cdot f_i(t) \cdot \xi_i \quad (11)$$

where λ_i and $f_i(t)$ are the eigenvalues and eigenvectors of $C_{\xi\xi}(t_1, t_2)$. If [Equation \(11\)](#) is substituted in [Equation \(9\)](#), one obtains a series expansion of $Z(t)$ in terms of the independent standard normal

variables ξ_i . This series expansion is used to create trajectories (sample functions) of the non-stationary and/or non-Gaussian process $Z(t)$ by sampling the N random variables ξ_i , $i=1,2,\dots,N$.

2.2 Characterization of Non-Gaussian Output Process

This section describes how the input uncertainty is propagated through the linear system to calculate the output uncertainty and thus characterize the output random process.

For a linear vibratory system under a stationary, non-Gaussian random load $F(t)$, the output process $X(t)$ is also stationary and non-Gaussian. It is characterized by developing a PCE stochastic metamodel, similarly to [Section 2.1.1](#), using the first four moments (mean, standard deviation, skewness, kurtosis) and the correlation structure of the output process.

The calculation of the four moments and the autocorrelation function of the output process can be performed analytically or using simulations. For the analytical calculation, only the impulse response function of the linear system is needed. [Sections 2.1.1](#) and [2.2.2](#) provide details on how to obtain the four moments analytically or through simulations.

2.2.1 Analytical Calculation of Output Moments

[Equations \(12\)](#) to [\(15\)](#) are used to calculate the first four moments and the correlation of the output process $X(t)$ in terms of moments of the input process $F(t)$. All equations use the principle of superposition and a convolution integral to calculate the expected response as a function of the expected input and the impulse response function of the system [\[9, 15\]](#).

The first two moments (mean and variance) are calculated as

$$E[X(t)] = E\left[\int_{-\infty}^{\infty} h(\xi)F(t-\xi)d\xi\right] = \int_{-\infty}^{\infty} h(\xi)E[F(t-\xi)]d\xi = \bar{F} \int_{-\infty}^{\infty} h(\xi)d\xi \quad (12)$$

and

$$\begin{aligned} E[X^2(t)] &= E\left[\int_{-\infty}^{\infty} h(\xi_1)F(t-\xi_1)d\xi_1 \int_{-\infty}^{\infty} h(\xi_2)F(t-\xi_2)d\xi_2\right] = \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} E[F(t-\xi_1)F(t-\xi_2)]h(\xi_1)h(\xi_2)d\xi_1d\xi_2 = \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} R_{FF}(\xi_1-\xi_2)h(\xi_1)h(\xi_2)d\xi_1d\xi_2 \end{aligned} \quad (13)$$

where $h(t)$ is the impulse response function of the system and E denotes the expected value of a random quantity.

The third (skewness) and fourth (kurtosis) moments are calculated as

$$\begin{aligned} E[X^3(t)] &= E\left[\int_{-\infty}^{\infty} h(\xi_1)F(t-\xi_1)d\xi_1 \int_{-\infty}^{\infty} h(\xi_2)F(t-\xi_2)d\xi_2 \right. \\ &\quad \left. \int_{-\infty}^{\infty} h(\xi_3)F(t-\xi_3)d\xi_3\right] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} E[F(t-\xi_1)F(t-\xi_2) \\ &\quad F(t-\xi_3)]h(\xi_1)h(\xi_2)h(\xi_3)d\xi_1d\xi_2d\xi_3 \end{aligned} \quad (14)$$

and

$$\begin{aligned} E[X^4(t)] &= E\left[\int_{-\infty}^{\infty} h(\xi_1)F(t-\xi_1)d\xi_1 \int_{-\infty}^{\infty} h(\xi_2)F(t-\xi_2)d\xi_2 \right. \\ &\quad \left. \int_{-\infty}^{\infty} h(\xi_3)F(t-\xi_3)d\xi_3 \int_{-\infty}^{\infty} h(\xi_4)F(t-\xi_4)d\xi_4\right] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} E[F(t-\xi_1) \\ &\quad F(t-\xi_2)F(t-\xi_3)F(t-\xi_4)]h(\xi_1)h(\xi_2)h(\xi_3)h(\xi_4)d\xi_1d\xi_2d\xi_3d\xi_4 \end{aligned} \quad (15)$$

In [Equations \(14\)](#) and [\(15\)](#), $E[F(t-\xi_1)F(t-\xi_2)\dots F(t-\xi_k)]$, $k=1,2,3,4$ represents the expected value of the product of correlated random variables $F(t-\xi_1)$, $F(t-\xi_2)$, $F(t-\xi_3)$ and $F(t-\xi_4)$ within the correlation length of process $F(t)$. These expectations are calculated numerically. A sufficient number n of trajectories is generated for the input process $F(t)$ using its PCE stochastic metamodel ([Section 2.1](#)). The n samples of $F(t)$ at each time instant t are then used to estimate the expectations using an unbiased estimator. The integrals in [Equations \(12\)](#) to [\(15\)](#) are calculated using the trapezoidal rule.

The autocorrelation function of the output process $X(t)$ is calculated as [\[15\]](#)

$$R_{XX}(\tau) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} R_{FF}(\tau+\xi_1-\xi_2)h(\xi_1)h(\xi_2)d\xi_1d\xi_2. \quad (16)$$

2.2.2 Simulation-Based Calculation of Output Moments

A simulation-based approach can also estimate the expectations of output quantities in [Equations \(12\)](#) to [\(15\)](#). The equations of motion are first integrated numerically for each of n trajectories of the input process $F(t)$ to obtain n trajectories of the output process $X(t)$. A fourth-order Runge-Kutta integration may be used based on a state-space representation of the equations of motion.

The expectation in the autocorrelation function

$R_{XX}(t, t+\tau) = E[X(t)X(t+\tau)]$ of the output process $X(t)$ can be similarly estimated using the n output trajectories. Because the process $X(t)$ is stationary, the autocorrelation $R_{XX}(t, t+\tau) = R_{XX}(\tau)$ is a function of the relative time τ only.

To calculate the expectations

$E[F(t-\xi_1)F(t-\xi_2)\dots F(t-\xi_k)]$, $k=1,2,3,4$ of [Equations \(14\)](#) and [\(15\)](#) accurately, a very large number n of trajectories is required. This however, becomes computationally impractical even for systems