

# Time-Dependent Reliability Analysis Using a Modified Composite Limit State Approach

Monica Majcher  
US Army TARDEC

Zissimos Mourelatos and Vasiliki Tsianika  
Oakland University

## ABSTRACT

Recent developments in time-dependent reliability have introduced the concept of a composite limit state. The composite limit state method can be used to calculate the time-dependent probability of failure for dynamic systems with limit-state functions of input random variables, input random processes and explicit in time. The probability of failure can be calculated exactly using the composite limit state if the instantaneous limit states are linear, forming an open or close polytope, and are functions of only two random variables. In this work, the restriction on the number of random variables is lifted. The proposed algorithm is accurate and efficient for linear instantaneous limit state functions of any number of random variables. An example on the design of a hydrokinetic turbine blade under time-dependent river flow load demonstrates the accuracy of the proposed general composite limit state approach.

**CITATION:** Majcher, M., Mourelatos, Z., and Tsianika, V., "Time-Dependent Reliability Analysis Using a Modified Composite Limit State Approach," *SAE Int. J. Commer. Veh.* 10(1):2017, doi:10.4271/2017-01-0206.

## INTRODUCTION

The time-dependent probability of failure, or cumulative probability of failure [1, 2, 3, 4], is defined as

$$P_f(0, T) = P\{\exists t \in [0, T]: g(\mathbf{X}, \mathbf{Y}(t), t) \leq 0\} \quad (1)$$

where the limit state  $g(\mathbf{X}, \mathbf{Y}(t), t) = 0$  depends on the vector  $\mathbf{X} = [X_1, X_2, \dots, X_n]$  of  $n$  input random variables, the vector  $\mathbf{Y}(t) = [Y_1(t), Y_2(t), \dots, Y_m(t)]$  of  $m$  input random processes, and explicit time  $t$ . Failure occurs if  $g(\cdot) \leq 0$  at any time  $t \in [0, T]$  where  $T$  is the planning horizon (time of interest).

The random process  $g(\mathbf{X}, \mathbf{Y}(t), t)$  can be viewed as a collection of random variables at different time instances  $t$ . Let us consider the output random process,

$$Z(t) = Z(\mathbf{X}, \mathbf{Y}(t), t), \quad 0 \leq t \leq T \quad (2)$$

which may represent the response of a dynamic system such as force, stress, strain or displacement. The time  $t$  can be either literal time or generalized time, such as cycles of fatigue life. A time-dependent limit state is defined as

$$g(\mathbf{X}, \mathbf{Y}(t), t) = \alpha_T - Z(\mathbf{X}, \mathbf{Y}(t), t) \quad (3)$$

where  $\alpha_T$  is a maximum response level which must not be exceeded.

An up-crossing event is defined if we do not have failure at time  $t$  (i.e.,  $g(\mathbf{X}, \mathbf{Y}(t), t) > 0$ ) while we have failure at  $t + \Delta t$  (i.e.,  $g(\mathbf{X}, \mathbf{Y}(t + \Delta t), t + \Delta t) \leq 0$ ). Thus, there is no prior record of failure except at time steps  $t$  and  $t + \Delta t$ . Based on this definition, there can be multiple up-crossings over time.

Equation (1) describes the *time-dependent* or *cumulative* probability of failure, which in general, is different from the *instantaneous* probability of failure

$$P_f^i(t) = P(g(\mathbf{X}, \mathbf{Y}(t), t) < 0), \quad (4)$$

where the probability of failure is calculated in a quasi-steady sense, by fixing time  $t$  and replacing the random process by a random variable. As a result, it does not account for possible failures before time  $t$ .

The computation of the up-crossing rate is commonly used to obtain an upper bound on the time-dependent (cumulative) probability of failure. The up-crossing rate is defined as [5]

$$\nu^+(t) = \lim_{\Delta t \rightarrow 0, \Delta t > 0} \frac{P[g(\mathbf{X}, F(t), t) > 0 \cap g(\mathbf{X}, F(t + \Delta t), t + \Delta t) \leq 0]}{\Delta t} \quad (5)$$

If  $N^+(0, T)$  is the number of out-crossings in the time interval  $[0, T]$ , Ditlevsen and Madsen [6] have shown that the mean number of out-crossings can be found by integrating the out-crossing rate over

the period of interest, i.e.,  $\int_{t_{\min}}^T v^+(t) dt$  and Shinozuka [7] developed an expression for the bounds of the cumulative probability of failure.

The computation of the out-crossing rate has been extensively studied [8, 9, 10, 11], under the assumption of the out-crossings being independent and Poisson distributed. Methods such as the PHI2 and the Asymptotic Method (AsM) [3] compute an upper bound of the time-dependent probability of failure of Equation (1). Hagen and Tvedt [5] suggested a parallel system reliability formulation to compute the up-crossing rate. It uses two successive time-invariant analyses based on FORM, and the binomial cumulative distribution to calculate the probability of the joint event in Equation (5). This idea has been used and extended in the PHI2 method. The AsM couples analytical results for up-crossings of random processes with an asymptotic integration.

Advanced methods to calculate the up-crossing rate  $v^+(t)$  of Equation (5), and the time-dependent probability of failure, have been recently proposed [12, 13, 14, 15] with superior accuracy compared to both the PHI2 and AsM methods [13]. An overview is provided below.

If the failure event is defined as (see Equation 3)  $E = \{\exists t \in [0, T]: g(\mathbf{X}, \mathbf{Y}(t), t) = \alpha_T(t) - Z(\mathbf{X}, \mathbf{Y}(t), t) \leq 0\}$ , the probability  $P_f[0, t]$ ,  $0 \leq t \leq T$  is given by

$$P_f[0, t] = P_f^0 + (1 - P_f^0) \int_0^t f(\tau) d\tau \quad (6)$$

where  $f(t)$  is the PDF of the first time to failure and

$$P_f^0 = P\{Z(\mathbf{X}, \mathbf{Y}(0), 0) > \alpha_T(0)\} \quad (7)$$

is the instantaneous probability of failure at  $t = 0$ . The level crossing threshold  $\alpha_T(t)$  is in general, time-dependent. The integral equation approach to calculate  $P(E)$  has been first introduced in [16] and has been also used in [12].

The up-crossing rate at time  $t$  is the probability that the first up-crossing occurred at  $t$  or at a previous time  $\tau$ , and can be expressed as [12, 16]

$$v^+(t) = f(t) + \int_0^t v^+(t|\tau) f(\tau) d\tau \quad (8)$$

where  $v^+(t|\tau)$  is the up-crossing rate at  $t$  conditioned on the time  $\tau$  of the first up-crossing.

Equation (8) can be modified to

$$v^+(t) = f(t) + \int_0^t \frac{v^{++}(t, \tau)}{v^+(\tau)} f(\tau) d\tau \quad (9)$$

in terms of the up-crossing rate  $v^+(\tau)$  and the joint up-crossing rate  $v^{++}(t, \tau)$ . Based on the definition of Equation (5), the up-crossing rate  $v^+(t)$  is expressed as  $P(\bar{E}_3 \cap E_4) = P(E_4) - P(E_3 \cap E_4)$ . This probability is calculated exactly for a chosen  $\Delta t$ . Subscripts 3 and 4 designate the safe ( $\bar{E}_3$ ) and failure ( $E_4$ ) events at times  $t_3 = t - \Delta t$  and  $t_4 = t$ , respectively.  $P(E_4)$  is equal to  $\Phi(-\beta_4)$  and Equation (10) can be used to find the joint probability of the two events [17-18]:

$$P_{ij}^f = \Phi_2(-\beta_i, -\beta_j, \rho) = \frac{1}{2\pi} \int_{\arccos(\rho)}^{\pi} \exp\left[-\frac{1}{2}(\beta_i^2 + \beta_j^2 - 2\beta_i\beta_j \cos \theta)\right] \cos \theta d\theta \quad (10)$$

where  $\Phi_2$  is the bivariate standard normal CDF. To calculate the joint up-crossing rate

$$v^{++}(t, \tau) = \frac{P(\bar{E}_1 \cap E_2 \cap \bar{E}_3 \cap E_4)}{(\Delta t)^2} \quad (11)$$

we need the joint probability [13]

$$P(\bar{E}_1 \cap E_2 \cap \bar{E}_3 \cap E_4) = P(E_2 \cap E_4) - P(E_1 \cap E_2 \cap E_4) - P(E_2 \cap E_3 \cap E_4) + P(E_1 \cap E_2 \cap E_3 \cap E_4) \quad (12)$$

where subscripts 1 and 2 designate the safe and failure events at times  $t_1 = t - \tau - \Delta t$  and  $t_2 = t - \tau$ , respectively ( $\bar{E}_1$  is the safe event at  $t - \tau - \Delta t$  and  $E_2$  is the failure event at  $t - \tau$ ).

The joint probabilities can be calculated using equivalent reliability indices, equivalent correlation coefficients, and Equation (10) [13]. Once the up-crossing and joint up-crossing rates are known, the integral Equation (9) is solved numerically.

Monte Carlo simulation has also been used to calculate the time-dependent probability of failure [4, 13, 19]. The Expansion Optimal Linear Estimation (EOLE) method [19] is commonly used for stationary and Gaussian processes. The random process is discretized in time and a series expansion represents the process by a finite set of deterministic functions weighted by a finite set of random variables. A spectral representation [13] and the Karhunen Loeve (K-L) expansion [13, 19] have also been used.

In this paper, we present a time-dependent reliability analysis using a modified composite limit state (CLS) method. The concept of composite limit state is not new. Ref. [14] introduced an exact algorithm to calculate the time-dependent probability of failure of dynamic systems using a CLS method which is called herein "original." The algorithm is exact for linear limit states in two dimensions only. In this paper, we extend it to more than two

dimensions. For completeness, the next section summarizes the original CLS algorithm. Then, the proposed modified CLS algorithm is presented and demonstrated with a hydrokinetic turbine example.

## ORIGINAL COMPOSITE LIMIT STATE METHOD

According to the total probability theorem, the time-dependent probability of failure of Equation (1) can be expressed as

$$P_f(0, T) = P(F) = \int_{\Omega} P(F/\mathbf{W}) f_{\mathbf{W}}(\mathbf{w}) d\mathbf{w} \quad (13)$$

where  $F = \{\mathbf{x}, \mathbf{y}(t): g(\mathbf{X}, \mathbf{Y}(t), t) \leq 0, t \in [0, T]\}$  denotes the failure event where  $\mathbf{x}, \mathbf{y}(t)$  are realizations of  $\mathbf{X}$  and  $\mathbf{Y}(t)$ ,  $P(F/\mathbf{W})$  is a time-dependent conditional probability of failure,  $f_{\mathbf{W}}(\mathbf{w})$  is the joint PDF of the input random variables  $\mathbf{W}$  and  $\Omega$  is the support of  $f_{\mathbf{W}}(\mathbf{w})$ . The set  $\mathbf{W}$  is a proper subset of  $[\mathbf{X} \mathbf{Y}(t)]$  (i.e.,  $\mathbf{W} \subset [\mathbf{X} \mathbf{Y}(t)]$ ) and is chosen so that the limit state function to calculate the time-dependent  $P(F/\mathbf{W})$  becomes linear. This allows for an easy and accurate calculation of  $P(F/\mathbf{W})$ .

For  $t_i = (i - 1)\Delta t, i = 1, \dots, N$ , the time-dependent conditional probability  $P(F/\mathbf{W})$  is given by

$$P(F/\mathbf{W}) = P\left\{\bigcup_{i=0}^N (g(\mathbf{Z}, \mathbf{w}, t_i) \leq 0)\right\}. \quad (14)$$

The composite limit state is defined by the boundary of the domain  $\bigcup_{i=0}^N \bar{E}_i$  where the set  $E_i = \{\mathbf{z} \in \mathbf{Z}, g(\mathbf{Z}, \mathbf{w}, t_i) \leq 0\}$  represents the instantaneous failure region at time  $t_i = (i - 1)\Delta t$  and  $\bar{E}_i$  is the complement of  $E_i$ . This is similar to defining the boundary of the feasible domain for an optimization problem. Figure 1 shows a hypothetical composite limit state function defined by specific instantaneous limit states.

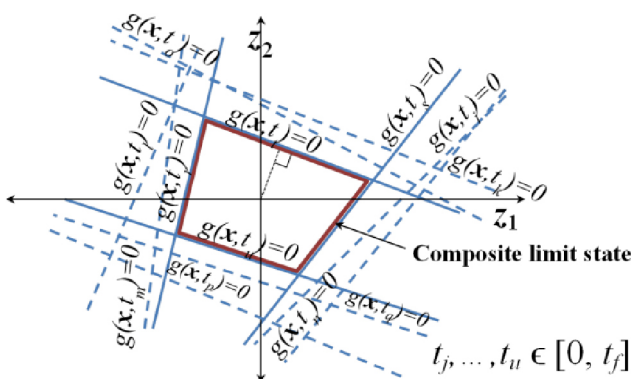


Figure 1. Hypothetical composite limit state function

The original composite limit state method, introduced in [14], calculates the conditional time-dependent probability of failure  $P(F/\mathbf{W})$ . It is based on the premise that the composite limit state will form a convex polytope. It comprises two general steps. In the first step, highly correlated instantaneous limit states are deleted. If two limit states are highly correlated (correlation coefficient greater than 0.99), the limit state with the lower reliability index (higher instantaneous probability of failure) dominates the other one. Thus, only the limit state with the lower reliability index needs to be used to define the failure region.

The second step to identify the composite limit state method involves solving a series of linear programming problems. If the set  $\left\{ \mathbf{Z}: g(\mathbf{Z}, t_i) = 0 \text{ and } \bigcap_{j=1, j \neq i}^{N_{G1}} g(\mathbf{Z}, t_j) \geq 0 \right\}$  is a null set, the instantaneous limit state  $g = 0$  from the remaining limit states from Step 1 is deleted. All other limit states make up the composite limit state. Details are found in [14].

Once the composite limit state is identified, the time-dependent probability of failure for the limit states in CLS is given by

$$P\left(\bigcup_{i=1}^l E_i\right) = \sum_{i \leq l} P(E_i) - \sum_{i_1 < i_2} P(E_{i_1} \cap E_{i_2}) + \sum_{i_1 < i_2 < i_3} P(E_{i_1} \cap E_{i_2} \cap E_{i_3}) - \dots + (-1)^l \sum_{i_1 < i_2 < \dots < i_{l-1}} P(E_{i_1} \cap \dots \cap E_{i_{l-1}}) + (-1)^{l+1} P(E_1 \cap E_2 \cap \dots \cap E_l) \quad (15)$$

The first term of the right hand side calculates the individual probabilities of failure  $P(E_i)$  which are equal to  $\Phi(-\beta_i) = \Phi(-\beta(t_i))$ . The second term calculates the joint probabilities of failure using the analytical expression of Equation (10). The third term calculates the probability of intersection of three events whose limit states form a triangular region (Figure 2) using the following equation

$$P_{ABG}^f = P(g_1 \geq 0, g_2 \leq 0, g_3 \geq 0) = \Phi(-\beta_2) - P_{12}^f - P_{23}^f + P_{13}^f. \quad (16)$$

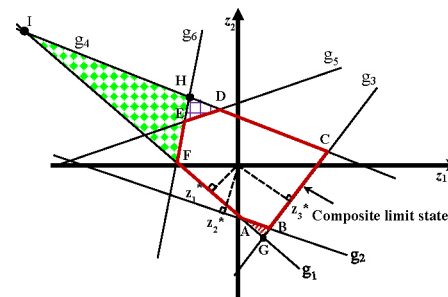


Figure 2. Triangular regions formed by limit states of the CLS

The probability of failure for one triangular region is saved and one of the three instantaneous limit states making up the triangular region is eliminated. The safe region is progressively enlarged by this elimination until only a three-edged convex polytope remains. Details are provided in [14].