

Mean-Value Second-Order Saddlepoint Approximation for Reliability Analysis

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ABSTRACT

A new second-order Saddlepoint Approximation (SA) method for structural reliability analysis is introduced. The Mean-value Second-order Saddlepoint Approximation (MVSOSA) is presented as an extension to the Mean-value First-order Saddlepoint Approximation (MVFOA). The proposed method is based on a second-order Taylor expansion of the limit state function around the mean value of the input random variables. It requires not only the first but also the second-order sensitivity derivatives of the limit state function. If sensitivity analysis must be avoided because of computational cost, a quadrature integration approach, based on sparse grids, is also presented and linked to the saddlepoint approximation (SGSA - Sparse Grid Saddlepoint Approximation). The SGSA method is compared with the first and second-order SA methods in terms of accuracy and efficiency. The proposed MVSOSA and SGSA methods are used in the reliability analysis of two examples.

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INTRODUCTION

The Saddlepoint Approximation (SA), based on the Laplace method or the method of steepest descents [1], is an efficient and accurate statistical approach to calculate the PDF and CDF of a random variable. It is a very useful method in reliability analysis because it calculates the tails of the distribution very accurately. A detailed explanation of the mathematics is presented in [2].

The SA approach has been extensively used in statistics and engineering design [3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19]. The first-order saddlepoint approximation (FOA) presented in [3], is based on the approximation of the limit state function at the most likelihood point in the original (not standard normal) space. This is in contrast to the first-order reliability method (FORM) which linearizes the limit state function in the standard normal space. Also, since FOA does not transform the limit state from the original space to the standard normal space, it is expected to be more accurate than FORM in which the accuracy may be reduced when the transformation increases the nonlinearity of the limit state function.

The mean-value first-order saddlepoint approximation (MVFOA) [4] is an alternative to FOA. It is based on a first-order Taylor expansion of the limit state function, at the mean values of the random variables, avoiding the need for solving an optimization problem to locate the most probable point (MPP). As a result, MVFOA is computationally more efficient than FOA and FORM, although it can be less accurate than FOA. Also, MVFOA is more accurate than the mean-value first-order second moment (MVFOSM) method and has similar computational cost.

A unified MVFOA method has been proposed in [5] to simultaneously consider epistemic and aleatory uncertainty. Also, an extension of FOA to reliability-based design optimization (RBDO) has been reported in [6-7] by implementing a sequential optimization and reliability analysis (SORA) using SA. Saddlepoint approximation has been used in [8] for multi-disciplinary RBDO. The extension of FOA from component reliability analysis to system reliability analysis and its relative superiority over FOSM is discussed in [9]. Sensitivity analysis of reliability based on saddlepoint approximations has been presented in [10-11].

A sampling approach to saddlepoint approximation has been presented in [12], while an improved third-moment saddlepoint approximation appears in [13]. The dimension reduction integration method has been used in [14] to efficiently calculate the statistical moments needed to calculate the first four cumulants for estimating the cumulant generating function (CGF) in SA. A reliability analysis based on Taylor expansion and statistical averaging is also presented in [15]. In structural mechanics, the saddlepoint approximation has been used in several applications such as structural reliability analysis [16] and reliability analysis of dynamic systems [17].

Second-order reliability methods (SORM) (e.g., [18, 19]) have been proposed as an extension to FORM using curvature effects to improve accuracy. However, saddlepoint approximation methods based on second-order expansions have not been presented in the literature. This paper proposes a saddlepoint approximation method for reliability analysis based on a second-order expansion of the limit state function around the mean value of the input random variables. The proposed approach uses gradient-based, Hessian-based and sensitivity-free approaches. The novel points of the study are (a) the extension of the first-order saddlepoint approximation (MVFOA) to

second-order saddlepoint approximation (MVSOSA), using a second-order Taylor expansion (b) the introduction of a sparse grid quadrature approach (SGSA) to reliability analysis to calculate the first moments and cumulants needed in SA, and (c) the comparison of the MVFOSA, MVSOSA and SGSA approaches in terms of accuracy and computational cost as well as complexity, parallelization issues and applicability to general (non-Gaussian) random variables.

First, the existing first-order saddlepoint approximation is summarized. Then we present the proposed second-order saddlepoint approximation based on second-order Taylor expansion, and the proposed reliability analysis using sparse grids and saddlepoint approximation. Finally, the proposed methods are applied and compared using two examples.

MEAN-VALUE FIRST-ORDER SADDLEPOINT APPROXIMATION (MVFOSA)

This section summarizes the mean value first-order saddlepoint approximation (MVFOSA) for reliability analysis [4]. It is based on the first-order Taylor expansion of the limit state function around the mean value. The linearization of the limit state function is very convenient, since it allows the computation of its moment generating function (MGF) using the MGFs of the input random variables.

The moment generating function (MGF) of a random variable X is defined as

$$M_X(t) = \int_{-\infty}^{\infty} e^{tx} f_X(x) dx \quad (1)$$

where $f_X(x)$ is the probability density function (PDF) of X . The cumulant generating function (CGF) is the logarithm of the MGF

$$K_X(t) = \ln[M_X(t)] \quad (2)$$

The first-order expansion of the limit state function $g(\mathbf{X})$ at the mean values μ_X of $\mathbf{X} = \{X_1, X_2, \dots, X_n\}$ is written as

$$Y = g(\mu_X) + \sum_{i=1}^n \left. \frac{\partial g}{\partial X_i} \right|_{\mu_X} (X_i - \mu_{X_i}) \quad (3)$$

where $\left. \frac{\partial g}{\partial X_i} \right|_{\mu_X}$ are the derivatives of the limit state function g with respect to $\mathbf{X}$, computed at the mean value μ_X . Rearrangement of terms yields

$$Y = \left(g(\mu_X) - \sum_{i=1}^n \left. \frac{\partial g}{\partial X_i} \right|_{\mu_X} \mu_{X_i} \right) + \sum_{i=1}^n \left. \frac{\partial g}{\partial X_i} \right|_{\mu_X} X_i = a_0 + \sum_{i=1}^n a_i X_i \quad (4)$$

$$\text{where } a_0 = g(\mu_X) - \sum_{i=1}^n \left. \frac{\partial g}{\partial X_i} \right|_{\mu_X} \mu_{X_i} \text{ and } a_i = \left. \frac{\partial g}{\partial X_i} \right|_{\mu_X}.$$

By definition, the MGF of Y is given by

$$\begin{aligned} M_Y(t) &= \int_{-\infty}^{\infty} e^{t \left(a_0 + \sum_{i=1}^n a_i X_i \right)} f_X(x_1, \dots, x_n) dx \\ &= e^{ta_0} \int_{-\infty}^{\infty} e^{t \left(\sum_{i=1}^n a_i X_i \right)} f_X(x_1, \dots, x_n) dx \end{aligned} \quad (5)$$

If the input random variables $\mathbf{X} = \{X_1, X_2, \dots, X_n\}$ are independent, the MGF is equal to

$$\begin{aligned} M_Y(t) &= e^{ta_0} \int_{-\infty}^{\infty} e^{t a_1 x_1} f_{X_1}(x_1) dx_1 \dots \\ &\int_{-\infty}^{\infty} e^{t a_n x_n} f_{X_n}(x_n) dx_n = e^{ta_0} M_{X_1}(a_1 t) \dots M_{X_n}(a_n t) \end{aligned} \quad (6)$$

The CGF of Y can be finally computed as

$$\begin{aligned} K_Y(t) &= \ln[M_Y(t)] = a_0 t + \ln M_{X_1}(a_1 t) + \dots + \ln M_{X_n}(a_n t) \\ &= a_0 t + \sum_{i=1}^n K_{X_i}(a_i t) \\ &= \left(g(\mu_X) - \sum_{i=1}^n \left. \frac{\partial g}{\partial X_i} \right|_{\mu_X} \mu_{X_i} \right) t + \sum_{i=1}^n K_{X_i} \left(\left. \frac{\partial g}{\partial X_i} \right|_{\mu_X} t \right) \end{aligned} \quad (7)$$

where $K_{X_i}(t)$ are the CGFs of the random variables X_i .

The probability of failure can be calculated as [20]

$$P_f = P[Y \leq y] \cong \Phi(w) + \phi(w) \left(\frac{1}{w} - \frac{1}{v} \right) \quad (8)$$

where Φ and ϕ are the CDF and PDF of the standard normal distribution and

$$w = \text{sgn}(t_s) \{ 2 [t_s y - K_Y(t_s)] \}^{1/2} \quad (9)$$

where

$$v = t_s [K''(t_s)]^{1/2} \quad (10)$$

In Equations (9) and (10), t_s is the saddlepoint computed by solving the equation

$$K'(t_s) = y \quad (11)$$

The first and second-order derivatives of the CGF are obtained by differentiating Equation (4) as

$$\begin{aligned} K'_Y(t) &= \left(g(\mu_X) - \sum_{i=1}^n \frac{\partial g}{\partial X_i}(\mu_X) \mu_{X_i} \right) \\ &+ \sum_{i=1}^n \frac{\partial g}{\partial X_i}(\mu_X) K'_{X_i} \left(\frac{\partial g}{\partial X_i}(\mu_X) t \right) \\ &= a_0 + \sum_{i=1}^n a_i K'_{X_i}(a_i t) \end{aligned} \quad (12)$$

and

$$K''_Y(t) = \sum_{i=1}^n \left[\frac{\partial g}{\partial X_i}(\mu_X) \right]^2 K''_{X_i} \left(\frac{\partial g}{\partial X_i}(\mu_X) t \right) = \sum_{i=1}^n (a_i)^2 K''_{X_i}(a_i t) \quad (13)$$

The major advantage of the MVFOSA approach is its low computational cost especially if an adjoint approach is used to compute the gradient of the limit state function, since the cost for this computation is independent of the number of random variables [21].

The most notable drawback of the MVFOSA approach is its poor accuracy. Because of the linear approximation of the limit state function using a first-order Taylor expansion, MVFOSA is accurate only for problems with low uncertainty where the standard deviations of the random variables are small.

MEAN-VALUE SECOND-ORDER SADDLEPOINT APPROXIMATION (MVSOSA)

To improve the accuracy of MVFOSA, we propose using a second-order expansion to obtain a quadratic approximation of the limit state function. The quadratic approximation is expected to provide a more accurate estimation of the probability of failure for problems with larger standard deviations of the random variables. This approach however, requires the computation of the Hessian matrix (second-order derivatives) of the limit state function with respect to the random variables. Although not presented in this paper, the Hessian matrix can be calculated using efficient second-order adjoint methods, making the proposed MVSOSA approach more efficient than other derivative-free approaches.

The second-order saddlepoint approximation is based on a second-order Taylor expansion of the limit state function as

$$\begin{aligned} Y &= g(\mu_X) + \sum_{i=1}^n \frac{\partial g}{\partial X_i} \bigg|_{\mu_X} (X_i - \mu_{X_i}) \\ &+ \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \frac{\partial^2 g}{\partial X_i \partial X_j} \bigg|_{\mu_X} (X_i - \mu_{X_i})(X_j - \mu_{X_j}) \end{aligned} \quad (14)$$

Upon algebraic rearrangements, we have

$$\begin{aligned} Y &= \left(g(\mu_X) - \sum_{i=1}^n \frac{\partial g}{\partial X_i} \bigg|_{\mu_X} \mu_{X_i} + \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \frac{\partial^2 g}{\partial X_i \partial X_j} \bigg|_{\mu_X} \mu_{X_i} \mu_{X_j} \right) + \\ &\sum_{i=1}^n \left(\frac{\partial g}{\partial X_i} \bigg|_{\mu_X} + \sum_{j=1}^n \frac{\partial^2 g}{\partial X_i \partial X_j} \bigg|_{\mu_X} \mu_{X_j} \right) X_i + \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \frac{\partial^2 g}{\partial X_i \partial X_j} \bigg|_{\mu_X} X_i X_j \end{aligned} \quad (15)$$

The response Y can be therefore, expressed as a quadratic function of $\mathbf{X}$ as

$$Y = Q(\mathbf{X}) = \mathbf{X}^T \mathbf{A} \mathbf{X} + \mathbf{b}^T \mathbf{X} + c \quad (16)$$

where

$$\begin{aligned} A_{ij} &= \frac{1}{2} \frac{\partial^2 g}{\partial X_i \partial X_j} \bigg|_{\mu_X} \\ \mathbf{b}_i &= \frac{\partial g}{\partial X_i} \bigg|_{\mu_X} + \sum_{j=1}^n \frac{\partial^2 g}{\partial X_i \partial X_j} \bigg|_{\mu_X} \mu_{X_j} \\ c &= g(\mu_X) - \sum_{i=1}^n \frac{\partial g}{\partial X_i} \bigg|_{\mu_X} \mu_{X_i} + \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \frac{\partial^2 g}{\partial X_i \partial X_j} \bigg|_{\mu_X} \mu_{X_i} \mu_{X_j} \end{aligned} \quad (17)$$

The MGF of the response is then given by

$$M_Y(t) = \int_{-\infty}^{\infty} e^{tQ(\mathbf{x})} f_{\mathbf{X}}(\mathbf{x}) d\mathbf{x} \quad (18)$$

In MVFOSA, the expression for the MGF of the output Y can be expressed in terms of the MGFs of the input random variables. However, this is not true in MVSOSA except for each case of Gaussian input distributions which is presented hereafter.

If $\mathbf{X}$ is a normal vector with mean $\boldsymbol{\mu}$ and covariance Σ , $\mathbf{X} \sim N(\boldsymbol{\mu}, \Sigma)$ the MGF of the quadratic function $Q(\mathbf{x})$ is given by [22]

$$\begin{aligned} M_Y(t) &= \int_{-\infty}^{\infty} e^{tQ(\mathbf{x})} f_{\mathbf{X}}(\mathbf{x}) d\mathbf{x} \\ &= |B|^{-1/2} \exp \left[\left(\boldsymbol{\mu}^T \mathbf{A} \boldsymbol{\mu} + \mathbf{b}^T \boldsymbol{\mu} + c \right) + \left(t^2/2 \right) \mathbf{d}^T B^{-1} \mathbf{d} \right] \end{aligned} \quad (19)$$

where

$$B = I - 2t\Sigma^{1/2} \mathbf{A} \Sigma^{1/2} \quad (20)$$

and

$$\mathbf{d} = \Sigma^{1/2} \mathbf{b} + 2\Sigma^{1/2} \mathbf{A} \boldsymbol{\mu}$$