

Bifurcation Analysis of a Car Model Running on an Even Surface - A Fundamental Study for Addressing Autonomous Vehicle Dynamics

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ABSTRACT

The paper deals with the bifurcation analysis of a simple mathematical model describing an automobile running on an even surface. Bifurcation analysis is adopted as the proper procedure for an in-depth understanding of the stability of steady-state motion of cars (either cornering or running straight ahead). The aim of the paper is providing the fundamental information for inspiring further studies on vehicle dynamics with or without a human driver.

The considered mechanical model of the car has two degrees of freedom, nonlinear tire characteristics are included. A simple driver model is introduced. Experimental validations of the model are produced.

As a first step, bifurcation analysis is performed without driver (fixed control). Ten different combinations of front and rear tire characteristics (featuring understeer or oversteer automobiles) are considered. Steering angle and speed are varied. Many different dynamical behaviors of the model are found. Homoclinic bifurcations, stable and unstable limit cycles are found, giving a sound base of knowledge to control engineers who are asked to implement robust algorithms to reach stability. As a second step, bifurcation analysis is performed including the driver control action. Straight ahead motion is studied. Limit cycles exist that may suggest how complicated may be controlling the stability of such relatively simple running condition.

The knowledge of the derived set of bifurcations seems important to fully understand the actual vehicle yaw motions occurring while running on an even surface and for conceiving robust control schemes for autonomous vehicles.

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INTRODUCTION

In the literature few papers and almost no textbooks deal in-depth with the problem of stability of nonlinear car models [1, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16]. A pioneering contribution on this topic was given by Pacejka [2,3]. It is known that the understeering car (at any centripetal acceleration level) is always stable [4]. But the stable behavior may change dramatically if nominal parameters of the vehicle are changed, e.g., if working condition of tires are modified due to a wrongly designed suspension control, or tire wear, or wrong tire pressures, and so on. Understanding the paradigms of vehicle stability in such a non-nominal condition is crucial, especially if the focus is on autonomous or automated vehicles.

The issue that the greatest majority of textbooks neglect is the fact (addressed in [1,2,3]) that, at a given speed, and on a certain curve radius (thus at a certain steady-state centripetal acceleration), a car may run with more than one steering wheel angle. In other words, due to a particular combination of front and rear (nonlinear) tire characteristics, if we consider a car running on a steering pad, at a given speed, more than one steering wheel angle is possible. Usually

only one steady-state running condition is stable, thus the other unstable steady-state equilibrium, is impossible to be found experimentally. But the existence of such unstable steady-state equilibrium do influence the overall stability of the vehicle! Up to five different steady-state equilibria were theoretically found at a given speed [1]. The existence of such steady-state unstable equilibria is extremely relevant to assess the nonlinear dynamic behavior of the vehicle, especially when a disturbance occurs and the vehicle (and the driver) have to get back to the steady-state stable condition previously attained.

The relevant literature on vehicle stability is introduced and commented in [1,17,18]. The paper gathers the results given in [1,17,18] to make an overview of the nonlinear behavior of a car either with or without a driver.

The paper is organized as follows. After the system models of the car and of the driver are introduced and validated, a complete overview of the nonlinear behavior of a car is given, depending on different characteristic at front and rear, and depending on different speed and

steering angle. Such nonlinear behavior is studied without driver control action, but at any possible cornering. Subsequently the nonlinear straight ahead motion is studied, with driver control.

1. SYSTEM MODEL

1.1. Vehicle Model

The mechanical model analyzed in this paper is the well-known 2 d.o.f. single-track model [4, 5, 6] shown in Fig. 1.

The main simplifying hypotheses are:

- the forward speed $u = \sqrt{\dot{X}^2 + \dot{Y}^2}$ is constant;
- the center of gravity lies at the ground level;
- the vehicle body is modelled referring to its longitudinal axis;
- the resultant of the forces acting at the front and rear axles are applied at the centers of the axles;
- the slip angles α_i , $i = 1, 2$, and the steering angle δ (see Fig. 1a) are small;
- no longitudinal forces are acting at the wheels.

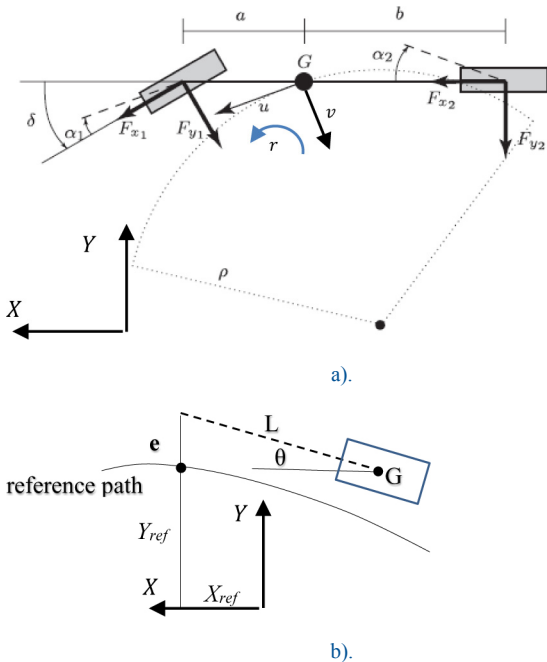


Figure 1. a) Representation of the single-track car model. b) Driver preview distance and reference path, the car is represented by a rectangle.

Under the above hypotheses, the equations of motion with respect to the reference system on the vehicle read [23]

$$\begin{aligned} m(\dot{v} + ur) &= F_{y1} + F_{y2} \\ I_z \dot{r} &= F_{y1} a - F_{y2} b \end{aligned} \quad (1a)$$

Where the dots stand for time-derivative, u and v are the longitudinal and lateral speeds respectively, r is the yaw rate, subscripts 1 and 2 refer, respectively, to the front and rear axle, F_{y_i} is the lateral force on the i^{th} ($i=1,2$) axle, m is the vehicle mass, a and b are the distances, respectively, of the front and the rear axle centre from the vehicle's centre of mass, I_z is the moment of inertia of the vehicle around the vertical axis at the centre of gravity. The front and rear slip angles can be obtained through the following equations:

$$-\alpha_2 = \frac{v-rb}{u}, \quad \delta - \alpha_1 = \frac{v+ra}{u} \quad (1b)$$

The lateral forces $F_{y_i}(\alpha_i)$ ($i=1,2$) can be expressed as functions of the slip angles α_i [4]. The functions $F_{y_i}(\alpha_i)$, namely the tyre characteristics, can have very different shapes [4, 6]. The lateral forces F_{y_i} ($i=1,2$) can be expressed as functions of the slip angles α_i , which are nonlinear and take the analytical form [4]

$$\begin{aligned} F_{y1}(\alpha_1) &= D_1 \sin(C_1 \tan^{-1}(B_1 \alpha_1 - E_1(B_1 \alpha_1 - \tan^{-1}(B_1 \alpha_1)))) \\ F_{y2}(\alpha_2) &= D_2 \sin(C_2 \tan^{-1}(B_2 \alpha_2 - E_2(B_2 \alpha_2 - \tan^{-1}(B_2 \alpha_2)))) \end{aligned}$$

where $D_1 = \mu_1 (mg/l)b$; $D_2 = \mu_2 (mg/l)a$ and μ_1 and μ_2 are the friction coefficients, respectively, of the front and the rear tyres, B_i , C_i , E_i ($i=1,2$) are tyre parameters.

The parameter set can be divided into three subsets: the vehicle parameters (m , I_z , a , b , $l=a+b$), whose values are reported in Table 1, the tyre parameters (μ_i , B_i , C_i , E_i , $i=1,2$), and the parameters (u , δ) which represent the running conditions. For straight ahead running, we ask the driver to follow a specific trajectory, i.e. running along a straight line. Under this hypothesis, model (1) can be rewritten as [18]

$$\begin{aligned} m \ddot{Y} &= F_{y1} + F_{y2} \\ I_z \ddot{\theta} &= F_{y1} a - F_{y2} b \end{aligned} \quad (2a)$$

1.2. Driver Model

A very simple driver model is proposed, inspired to what was proposed in [6,17,18,21]. Let us refer to Fig.1b, we assume that the driver inside the car is able to see a point L forward beyond the current position of the vehicle and is able to estimate the deviation of the lateral position of the vehicle (X , Y) with respect to a reference path (X_{ref} , Y_{ref})

$$\mathbf{e} = \begin{pmatrix} X \\ Y \end{pmatrix} + L \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix} - \begin{pmatrix} X_{ref} \\ Y_{ref} \end{pmatrix}$$

(note that in general, $\mathbf{e}$, is a two dimensional vector) and that the driver reacts with a gain h to this error with a delay τ_r , i.e.

$$\delta(t + \tau_r) = h \mathbf{e}(t) \quad (3a)$$

Since we want to analyze only the straight ahead running, at steady state we can use a reference system in which X_{ref} can assume any value, while Y_{ref} has a precise constant value. Moreover, it is possible to approximate the driver reaction delay with a Taylor expansion since the frequency at which a human can vary the steering angle does not exceed 3 Hz. So the final model of the driver simplifies to the differential equation:

$$\tau_r \dot{\delta} = -\delta + h(Y + L \sin(\theta) - Y_{ref}) \quad (3b)$$

The proposed model, known as *stable path follower*, [5] is the simplest driver model. However ([5]) it is quite common to consider the reaction of the driver not only proportional to the error, but also considering how the error is changing (derivative feedback). Equations (3) therefore can be rewritten as

$$\begin{aligned} \delta(t + \tau_r) &= h e(t) + k_d \dot{e}(t) \\ \tau_r \dot{\delta} &= -\delta + h(Y + L \sin(\theta) - Y_{ref}) + k_d(\dot{Y} + L\dot{\theta} \cos \theta) \end{aligned} \quad (4)$$

1.3. Car+Driver Mathematical Model

The complete model comprises Eq. (1) (or Eq. 2 for straight ahead running) coupled with Eq. (3). We have a five states model, which reads

$$\begin{aligned} m \ddot{Y} &= F_{y1} + F_{y2} \\ I_z \ddot{\theta} &= F_{y1} a - F_{y2} b \\ \tau_r \dot{\delta} &= -\delta + h(Y + L \sin(\theta) - Y_{ref}) \end{aligned} \quad (5)$$

Considering the driver model described by (4), the car+driver equations become

$$\begin{aligned} m \ddot{Y} &= F_{y1} + F_{y2} \\ I_z \ddot{\theta} &= F_{y1} a - F_{y2} b \\ \tau_r \dot{\delta} &= -\delta + h(Y + L \sin(\theta) - Y_{ref}) + k_d(\dot{Y} + L\dot{\theta} \cos \theta) \end{aligned} \quad (5a)$$

For straight ahead running, the steady-state equilibrium is of the type $[Y, \dot{Y}, \theta, \dot{\theta}, \delta] = [Y_{ref}, 0, 0, 0, 0]$, and represents a vehicle that follows a precise straight line trajectory with increasing abscissa (with velocity $\dot{X} = u$) and constant ordinate Y.

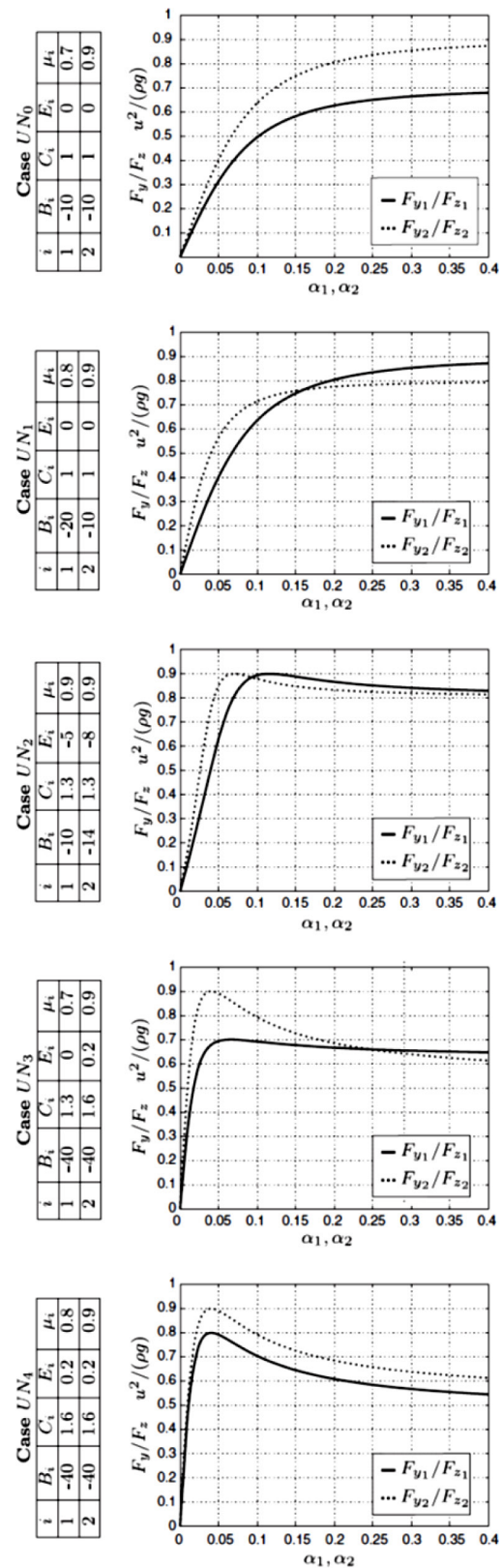


Figure 2. Tire parameters that refer to different tire characteristics. (Adapted from [1]). The lateral force at the tire F_{y1} is divided by the vertical force at the tire F_{z1} . Handling Diagram Theory [4], states that the ratio F_{y1}/F_{z1} is equal to the steady-state centripetal acceleration divided by gravity $u^2/(pg)$. UN stands for understeering. Actually the vehicle is understeering only at low lateral centripetal acceleration.

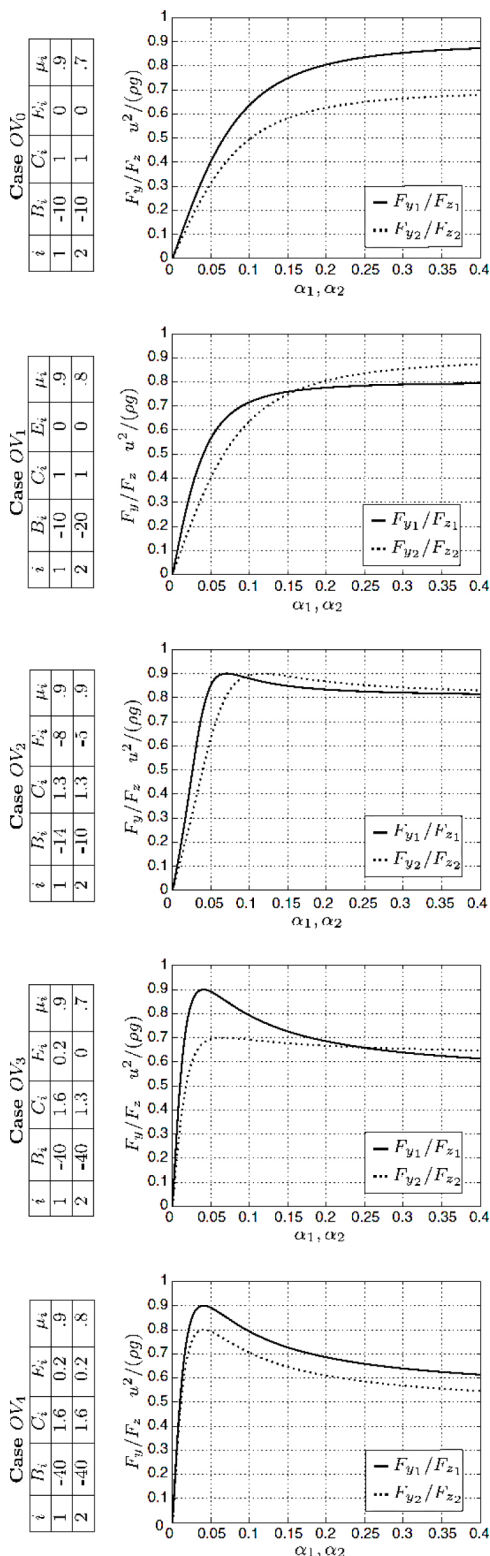


Figure 3. Tire parameters that refer to different tire characteristics. (Adapted from [1]). *OV* stands for oversteering. Actually the vehicle is oversteering only at low lateral centripetal acceleration. Notice that the *OV* cases are just the *UN* ones after index 1 is replaced by index 2 (and vice-versa) in tire characteristics F_{y1}/F_{z1}

Table 1. Vehicle and driver parameters. The tire parameters can be found in Figure 2.

Parameter	Value	Unit
Vehicle mass m	550	kg
Vehicle yaw moment of inertia I_z	750	kg m ²
Distance from front axle to c.g. a	1.64	m
Distance from rear axle to c.g. b	0.98	m
Delay in control τ_r	0.2	s
Reaction h	0.02	rad/s
Ability of prevision L	12	m

2. VALIDATION

Two different validations of the simple models introduced in Section 1 have been performed.

The first validation refers to a sports car running on a race track (for more details see [17]), results are given in Figure 4.

The second validation refers to tests performed at the ACI test track in Milan (Lainate), Italy. A severe yaw vehicle motion is excited by a kick-plate acting at the rear axle of the car at a certain time instant as the car is running straight ahead. The driver is heavily involved in controlling the yaw motion, to avoid a complete spin. Figure 5 reports the results (for more details see [18]).

The models have been written in Matlab/Simulink. The validation was aimed to check that the difference between measured and computed data was not too high.

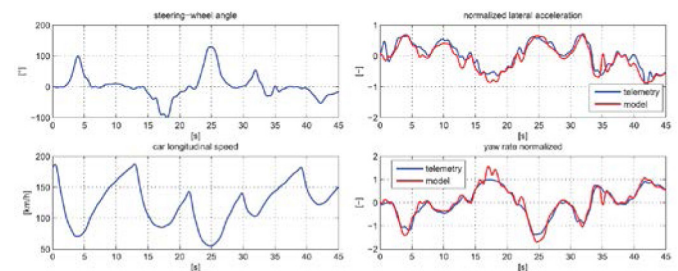


Figure 4. Validation of system model (1). Adapted from [17].

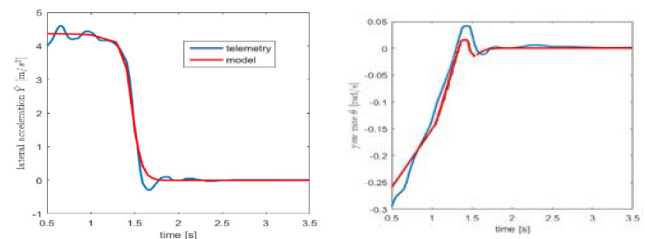


Figure 5. Validation of system model (1) and driver model (3). Adapted from [18].