

THEORY OF PLATES AND SHELLS

S. TIMOSHENKO

*Professor Emeritus of Engineering Mechanics
Stanford University*

S. WOINOWSKY-KRIEGER

*Professor of Engineering Mechanics
Laval University*

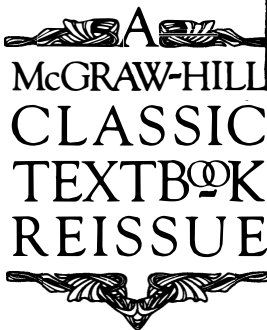
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THEORY OF PLATES AND SHELLS

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PREFACE

Since the publication of the first edition of this book, the application of the theory of plates and shells in practice has widened considerably, and some new methods have been introduced into the theory. To take these facts into consideration, we have had to make many changes and additions. The principal additions are (1) an article on deflection of plates due to transverse shear, (2) an article on stress concentrations around a circular hole in a bent plate, (3) a chapter on bending of plates resting on an elastic foundation, (4) a chapter on bending of anisotropic plates, and (5) a chapter reviewing certain special and approximate methods used in plate analysis. We have also expanded the chapter on large deflections of plates, adding several new cases of plates of variable thickness and some numerical tables facilitating plate analysis.

In the part of the book dealing with the theory of shells, we limited ourselves to the addition of the stress-function method in the membrane theory of shells and some minor additions in the flexural theory of shells.

The theory of shells has been developing rapidly in recent years, and several new books have appeared in this field. Since it was not feasible for us to discuss these new developments in detail, we have merely referred to the new bibliography, in which persons specially interested in this field will find the necessary information.

S. Timoshenko
S. Woinowsky-Krieger

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NOTATION

x, y, z	Rectangular coordinates
r, θ	Polar coordinates
r_x, r_y	Radii of curvature of the middle surface of a plate in xz and yz planes, respectively
h	Thickness of a plate or a shell
q	Intensity of a continuously distributed load
p	Pressure
P	Single load
γ	Weight per unit volume
$\sigma_x, \sigma_y, \sigma_z$	Normal components of stress parallel to x, y , and z axes
σ_n	Normal component of stress parallel to n direction
σ_r	Radial stress in polar coordinates
σ_t, σ_θ	Tangential stress in polar coordinates
τ	Shearing stress
$\tau_{xy}, \tau_{xz}, \tau_{yz}$	Shearing stress components in rectangular coordinates
u, v, w	Components of displacements
ϵ	Unit elongation
$\epsilon_x, \epsilon_y, \epsilon_z$	Unit elongations in x, y , and z directions
ϵ_r	Radial unit elongation in polar coordinates
$\epsilon_t, \epsilon_\theta$	Tangential unit elongation in polar coordinates
$\epsilon_\varphi, \epsilon_\theta$	Unit elongations of a shell in meridional direction and in the direction of parallel circle, respectively
$\gamma_{xy}, \gamma_{xz}, \gamma_{yz}$	Shearing strain components in rectangular coordinates
$\gamma_{r\theta}$	Shearing strain in polar coordinates
E	Modulus of elasticity in tension and compression
G	Modulus of elasticity in shear
ν	Poisson's ratio
V	Strain energy
D	Flexural rigidity of a plate or shell
M_x, M_y	Bending moments per unit length of sections of a plate perpendicular to x and y axes, respectively
M_{xy}	Twisting moment per unit length of section of a plate perpendicular to x axis
M_n, M_{nt}	Bending and twisting moments per unit length of a section of a plate perpendicular to n direction
Q_x, Q_y	Shearing forces parallel to z axis per unit length of sections of a plate perpendicular to x and y axes, respectively
Q_n	Shearing force parallel to z axis per unit length of section of a plate perpendicular to n direction
N_x, N_y	Normal forces per unit length of sections of a plate perpendicular to x and y directions, respectively

N_{xy}	Shearing force in direction of y axis per unit length of section of a plate perpendicular to x axis
M_r, M_t, M_{rt}	Radial, tangential, and twisting moments when using polar coordinates
Q_r, Q_t	Radial and tangential shearing forces
N_r, N_t	Normal forces per unit length in radial and tangential directions
r_1, r_2	Radii of curvature of a shell in the form of a surface of revolution in meridional plane and in the normal plane perpendicular to meridian, respectively
$\chi_\varphi, \chi_\theta$	Changes of curvature of a shell in meridional plane and in the plane perpendicular to meridian, respectively
$\chi_{\theta\varphi}$	Twist of a shell
X, Y, Z	Components of the intensity of the external load on a shell, parallel to x, y , and z axes, respectively
$N_\varphi, N_\theta, N_{\varphi\theta}$	Membrane forces per unit length of principal normal sections of a shell
M_θ, M_φ	Bending moments in a shell per unit length of meridional section and a section perpendicular to meridian, respectively
χ_x, χ_φ	Changes of curvature of a cylindrical shell in axial plane and in a plane perpendicular to the axis, respectively
$N_\varphi, N_x, N_{x\varphi}$	Membrane forces per unit length of axial section and a section perpendicular to the axis of a cylindrical shell
M_φ, M_x	Bending moments per unit length of axial section and a section perpendicular to the axis of a cylindrical shell, respectively
$M_{x\varphi}$	Twisting moment per unit length of an axial section of a cylindrical shell
Q_φ, Q_x	Shearing forces parallel to z axis per unit length of an axial section and a section perpendicular to the axis of a cylindrical shell, respectively
$\log$	Natural logarithm
$\log_{10}, \text{Log}$	Common logarithm

INTRODUCTION

The bending properties of a plate depend greatly on its thickness as compared with its other dimensions. In the following discussion, we shall distinguish between three kinds of plates: (1) thin plates with small deflections, (2) thin plates with large deflections, (3) thick plates.

Thin Plates with Small Deflection. If deflections w of a plate are small in comparison with its thickness h , a very satisfactory approximate theory of bending of the plate by lateral loads can be developed by making the following assumptions:

1. There is no deformation in the middle plane of the plate. This plane remains *neutral* during bending.
2. Points of the plate lying initially on a normal-to-the-middle plane of the plate remain on the normal-to-the-middle surface of the plate after bending.
3. The normal stresses in the direction transverse to the plate can be disregarded.

Using these assumptions, all stress components can be expressed by deflection w of the plate, which is a function of the two coordinates in the plane of the plate. This function has to satisfy a linear partial differential equation, which, together with the boundary conditions, completely defines w . Thus the solution of this equation gives all necessary information for calculating stresses at any point of the plate.

The second assumption is equivalent to the disregard of the effect of shear forces on the deflection of plates. This assumption is usually satisfactory, but in some cases (for example, in the case of holes in a plate) the effect of shear becomes important and some corrections in the theory of thin plates should be introduced (see Art. 39).

If, in addition to lateral loads, there are external forces acting in the middle plane of the plate, the first assumption does not hold any more, and it is necessary to take into consideration the effect on bending of the plate of the stresses acting in the middle plane of the plate. This can be done by introducing some additional terms into the above-mentioned differential equation of plates (see Art. 90).

Thin Plates with Large Deflection. The first assumption is completely satisfied only if a plate is bent into a developable surface. In other cases bending of a plate is accompanied by strain in the middle plane, but calculations show that the corresponding stresses in the middle plane are negligible if the deflections of the plate are small in comparison with its thickness. If the deflections are not small, these supplementary stresses must be taken into consideration in deriving the differential equation of plates. In this way we obtain nonlinear equations and the solution of the problem becomes much more complicated (see Art. 96). In the case of large deflections we have also to distinguish between immovable edges and edges free to move in the plane of the plate, which may have a considerable bearing upon the magnitude of deflections and stresses of the plate (see Arts. 99, 100). Owing to the curvature of the deformed middle plane of the plate, the supplementary tensile stresses, which predominate, act in opposition to the given lateral load; thus, the given load is now transmitted partly by the flexural rigidity and partly by a membrane action of the plate. Consequently, very thin plates with negligible resistance to bending behave as membranes, except perhaps for a narrow edge zone where bending may occur because of the boundary conditions imposed on the plate.

The case of a plate bent into a developable, in particular into a cylindrical, surface should be considered as an exception. The deflections of such a plate may be of the order of its thickness without necessarily producing membrane stresses and without affecting the linear character of the theory of bending. Membrane stresses would, however, arise in such a plate if its edges are immovable in its plane and the deflections are sufficiently large (see Art. 2). Therefore, in "plates with small deflection" membrane forces caused by edges immovable in the plane of the plate can be practically disregarded.

Thick Plates. The approximate theories of thin plates, discussed above, become unreliable in the case of plates of considerable thickness, especially in the case of highly concentrated loads. In such a case the thick-plate theory should be applied. This theory considers the problem of plates as a three-dimensional problem of elasticity. The stress analysis becomes, consequently, more involved and, up to now, the problem is completely solved only for a few particular cases. Using this analysis, the necessary corrections to the thin-plate theory at the points of application of concentrated loads can be introduced.

The main suppositions of the theory of thin plates also form the basis for the usual theory of thin shells. There exists, however, a substantial difference in the behavior of plates and shells under the action of external loading. The static equilibrium of a plate element under a lateral load is only possible by action of bending and twisting moments, usually

accompanied by shearing forces, while a shell, in general, is able to transmit the surface load by “membrane” stresses which act parallel to the tangential plane at a given point of the middle surface and are distributed uniformly over the thickness of the shell. This property of shells makes them, as a rule, a much more rigid and a more economical structure than a plate would be under the same conditions.

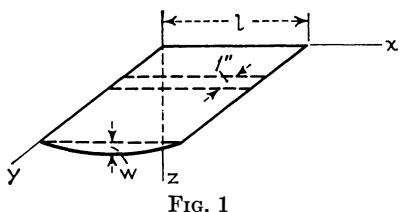
In principle, the membrane forces are independent of bending and are wholly defined by the conditions of static equilibrium. The methods of determination of these forces represent the so-called “membrane theory of shells.” However, the reactive forces and deformation obtained by the use of the membrane theory at the shell’s boundary usually become incompatible with the actual boundary conditions. To remove this discrepancy the bending of the shell in the edge zone has to be considered, which may affect slightly the magnitude of initially calculated membrane forces. This bending, however, usually has a very localized¹ character and may be calculated on the basis of the same assumptions which were used in the case of small deflections of thin plates. But there are problems, especially those concerning the elastic stability of shells, in which the assumption of small deflections should be discontinued and the “large-deflection theory” should be used.

If the thickness of a shell is comparable to the radii of curvature, or if we consider stresses near the concentrated forces, a more rigorous theory, similar to the thick-plate theory, should be applied.

¹ There are some kinds of shells, especially those with a negative Gaussian curvature, which provide us with a lot of exceptions. In the case of developable surfaces such as cylinders or cones, large deflection without strain of the middle surface is possible, and, in some cases, membrane stresses can be neglected and consideration of the bending stresses alone may be sufficient.

BENDING OF LONG RECTANGULAR PLATES TO A CYLINDRICAL SURFACE

1. Differential Equation for Cylindrical Bending of Plates. We shall begin the theory of bending of plates with the simple problem of the bending of a long rectangular plate that is subjected to a transverse load that does not vary along the length of the plate. The deflected surface of a portion of such a plate at a considerable distance from the ends¹ can be assumed cylindrical, with the axis of the cylinder parallel to the length of the plate. We can therefore restrict ourselves to the investigation of the bending of an elemental strip cut from the plate by two planes perpendicular to the length of the plate and a unit distance (say 1 in.) apart. The deflection of this strip is given by a differential equation which is similar to the deflection equation of a bent beam.



To obtain the equation for the deflection, we consider a plate of uniform thickness, equal to h , and take the xy plane as the middle plane of the plate before loading, *i.e.*, as the plane midway between the faces of

the plate. Let the y axis coincide with one of the longitudinal edges of the plate and let the positive direction of the z axis be downward, as shown in Fig. 1. Then if the width of the plate is denoted by l , the elemental strip may be considered as a bar of rectangular cross section which has a length of l and a depth of h . In calculating the bending stresses in such a bar we assume, as in the ordinary theory of beams, that cross sections of the bar remain plane during bending, so that they undergo only a rotation with respect to their neutral axes. If no normal forces are applied to the end sections of the bar, the neutral surface of the bar coincides with the middle surface of the plate, and the unit elongation of a fiber parallel to the x axis is proportional to its distance z

¹ The relation between the length and the width of a plate in order that the maximum stress may approximate that in an infinitely long plate is discussed later, see pp. 118 and 125.

from the middle surface. The curvature of the deflection curve can be taken equal to $-d^2w/dx^2$, where w , the deflection of the bar in the z direction, is assumed to be small compared with the length of the bar l . The unit elongation ϵ_x of a fiber at a distance z from the middle surface (Fig. 2) is then $-z d^2w/dx^2$.

Making use of Hooke's law, the unit elongations ϵ_x and ϵ_y in terms of the normal stresses σ_x and σ_y acting on the element shown shaded in Fig. 2a are

$$\begin{aligned}\epsilon_x &= \frac{\sigma_x}{E} - \frac{\nu\sigma_y}{E} \\ \epsilon_y &= \frac{\sigma_y}{E} - \frac{\nu\sigma_x}{E} = 0\end{aligned}\quad (1)$$

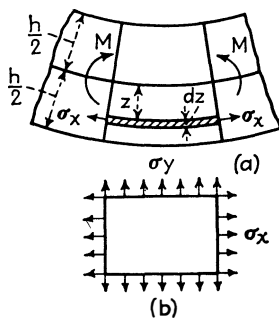


FIG. 2

where E is the modulus of elasticity of the material and ν is Poisson's ratio. The lateral strain in the y direction must be zero in order to maintain continuity in the plate during bending, from which it follows by the second of the equations (1) that $\sigma_y = \nu\sigma_x$. Substituting this value in the first of the equations (1), we obtain

$$\epsilon_x = \frac{(1 - \nu^2)\sigma_x}{E}$$

$$\text{and} \quad \sigma_x = \frac{E\epsilon_x}{1 - \nu^2} = - \frac{Ez}{1 - \nu^2} \frac{d^2w}{dx^2} \quad (2)$$

If the plate is submitted to the action of tensile or compressive forces acting in the x direction and uniformly distributed along the longitudinal sides of the plate, the corresponding direct stress must be added to the stress (2) due to bending.

Having the expression for bending stress σ_x , we obtain by integration the bending moment in the elemental strip:

$$M = \int_{-h/2}^{h/2} \sigma_x z \, dz = - \int_{-h/2}^{h/2} \frac{Ez^2}{1 - \nu^2} \frac{d^2w}{dx^2} \, dz = - \frac{Eh^3}{12(1 - \nu^2)} \frac{d^2w}{dx^2}$$

Introducing the notation

$$\frac{Eh^3}{12(1 - \nu^2)} = D \quad (3)$$

we represent the equation for the deflection curve of the elemental strip in the following form:

$$D \frac{d^2w}{dx^2} = -M \quad (4)$$

in which the quantity D , taking the place of the quantity EI in the case

of beams, is called the *flexural rigidity* of the plate. It is seen that the calculation of deflections of the plate reduces to the integration of Eq. (4), which has the same form as the differential equation for deflection of beams. If there is only a lateral load acting on the plate and the edges are free to approach each other as deflection occurs, the expression for the bending moment M can be readily derived, and the deflection curve is then obtained by integrating Eq. (4). In practice the problem is more complicated, since the plate is usually attached to the boundary and its edges are not free to move. Such a method of support sets up tensile reactions along the edges as soon as deflection takes place. These reactions depend on the magnitude of the deflection and affect the magnitude of the bending moment M entering in Eq. (4). The problem reduces to the investigation of bending of an elemental strip submitted to the action of a lateral load and also an axial force which depends on the deflection of the strip.¹ In the following we consider this problem for the particular case of uniform load acting on a plate and for various conditions along the edges.

2. Cylindrical Bending of Uniformly Loaded Rectangular Plates with Simply Supported Edges. Let us consider a uniformly loaded long rectangular plate with longitudinal edges which are free to rotate but cannot move toward each other during bending. An elemental strip cut out

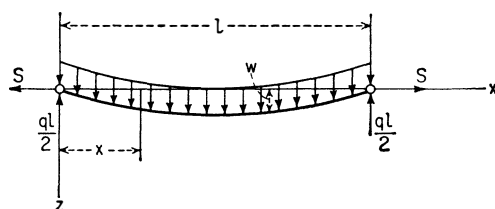


FIG. 3

from this plate, as shown in Fig. 1, is in the condition of a uniformly loaded bar submitted to the action of an axial force S (Fig. 3). The magnitude of S is such as to prevent the ends of the bar from moving along the x axis. Denoting by q the intensity of the uniform load, the bending moment at any cross section of the strip is

$$M = \frac{ql}{2}x - \frac{qx^2}{2} - Sw$$

¹ In such a form the problem was first discussed by I. G. Boobnov; see the English translation of his work in *Trans. Inst. Naval Architects*, vol. 44, p. 15, 1902, and his "Theory of Structure of Ships," vol. 2, p. 545, St. Petersburg, 1914. See also the paper by Stewart Way presented at the National Meeting of Applied Mechanics, ASME, New Haven, Conn., June, 1932; from this paper are taken the curves used in Arts. 2 and 3.

Substituting in Eq. (4), we obtain

$$\frac{d^2w}{dx^2} - \frac{Sw}{D} = -\frac{qlx}{2D} + \frac{qx^2}{2D} \quad (a)$$

Introducing the notation

$$\frac{S}{D} \frac{l^2}{4} = u^2 \quad (5)$$

the general solution of Eq. (a) can be written in the following form:

$$w = C_1 \sinh \frac{2ux}{l} + C_2 \cosh \frac{2ux}{l} + \frac{ql^3x}{8u^2D} - \frac{ql^2x^2}{8u^2D} - \frac{ql}{16u} \frac{l}{D} \quad (b)$$

The constants of integration C_1 and C_2 will be determined from the conditions at the ends. Since the deflections of the strip at the ends are zero, we have

$$w = 0 \quad \text{for } x = 0 \text{ and } x = l \quad (c)$$

Substituting for w its expression (b), we obtain from these two conditions

$$C_1 = \frac{ql^4}{16u^4D} \frac{1 - \cosh 2u}{\sinh 2u} \quad C_2 = \frac{ql^4}{16u^4D}$$

and the expression (b) for the deflection w becomes

$$w = \frac{ql^4}{16u^4D} \left(\frac{1 - \cosh 2u}{\sinh 2u} \sinh \frac{2ux}{l} + \cosh \frac{2ux}{l} - 1 \right) + \frac{ql^3x}{8u^2D} - \frac{ql^2x^2}{8u^2D}$$

Substituting

$$\begin{aligned} \cosh 2u &= \cosh^2 u + \sinh^2 u & \sinh 2u &= 2 \sinh u \cosh u \\ \cosh^2 u &= 1 + \sinh^2 u \end{aligned}$$

we can represent this expression in a simpler form:

$$w = \frac{ql^4}{16u^4D} \left(\frac{-\sinh u \sinh \frac{2ux}{l} + \cosh u \cosh \frac{2ux}{l}}{\cosh u} - 1 \right) + \frac{ql^2x}{8u^2D} (l - x)$$

or

$$w = \frac{ql^4}{16u^4D} \left[\frac{\cosh u \left(1 - \frac{2x}{l} \right)}{\cosh u} - 1 \right] + \frac{ql^2x}{8u^2D} (l - x) \quad (6)$$

Thus, deflections of the elemental strip depend upon the quantity u , which, as we see from Eq. (5), is a function of the axial force S . This force can be determined from the condition that the ends of the strip (Fig. 3) do not move along the x axis. Hence the extension of the strip produced by the forces S is equal to the difference between the length of the arc along the deflection curve and the chord length l . This difference

for small deflections can be represented by the formula¹

$$\lambda = \frac{1}{2} \int_0^l \left(\frac{dw}{dx} \right)^2 dx \quad (7)$$

In calculating the extension of the strip produced by the forces S . we assume that the lateral strain of the strip in the y direction is prevented and use Eq. (2). Then

$$\lambda = \frac{S(1 - \nu^2)l}{hE} = \frac{1}{2} \int_0^l \left(\frac{dw}{dx} \right)^2 dx \quad (d)$$

Substituting expression (6) for w and performing the integration, we obtain the following equation for calculating S :

$$\frac{S(1 - \nu^2)l}{hE} = \frac{q^2 l^7}{D^2} \left(\frac{5}{256} \frac{\tanh u}{u^7} + \frac{1}{256} \frac{\tanh^2 u}{u^6} - \frac{5}{256u^6} + \frac{1}{384u^4} \right)$$

or substituting $S = 4u^2 D/l^2$, from Eq. (5), and the expression for D , from Eq. (3), we finally obtain the equation

$$\frac{E^2 h^8}{(1 - \nu^2)^2 q^2 l^8} = \frac{135 \tanh u}{16 u^9} + \frac{27 \tanh^2 u}{16 u^8} - \frac{135}{16u^8} + \frac{9}{8u^6} \quad (8)$$

For a given material, a given ratio h/l , and a given load q the left-hand side of this equation can be readily calculated, and the value of u satisfying the equation can be found by a trial-and-error method. To simplify this solution, the curves shown in Fig. 4 can be used. The abscissas of these curves represent the values of u and the ordinates represent the quantities $\log_{10} (10^4 \sqrt{U_0})$, where U_0 denotes the numerical value of the right-hand side of Eq. (8). $\sqrt{U_0}$ is used because it is more easily calculated from the plate constants and the load; and the factor 10^4 is introduced to make the logarithms positive. In each particular case we begin by calculating the square root of the left-hand side of Eq. (8), equal to $Eh^4/(1 - \nu^2)ql^4$, which gives $\sqrt{U_0}$. The quantity $\log_{10} (10^4 \sqrt{U_0})$ then gives the ordinate which must be used in Fig. 4, and the corresponding value of u can be readily obtained from the curve. Having u , we obtain the value of the axial force S from Eq. (5).

In calculating stresses we observe that the total stress at any cross section of the strip consists of a bending stress proportional to the bending moment and a tensile stress of magnitude S/h which is constant along the length of the strip. The maximum stress occurs at the middle of the strip, where the bending moment is a maximum. From the differential equation (4) the maximum bending moment is

$$M_{\max} = -D \left(\frac{d^2 w}{dx^2} \right)_{x=l/2}$$

¹ See Timoshenko, "Strength of Materials," part I, 3d ed., p. 178, 1955.

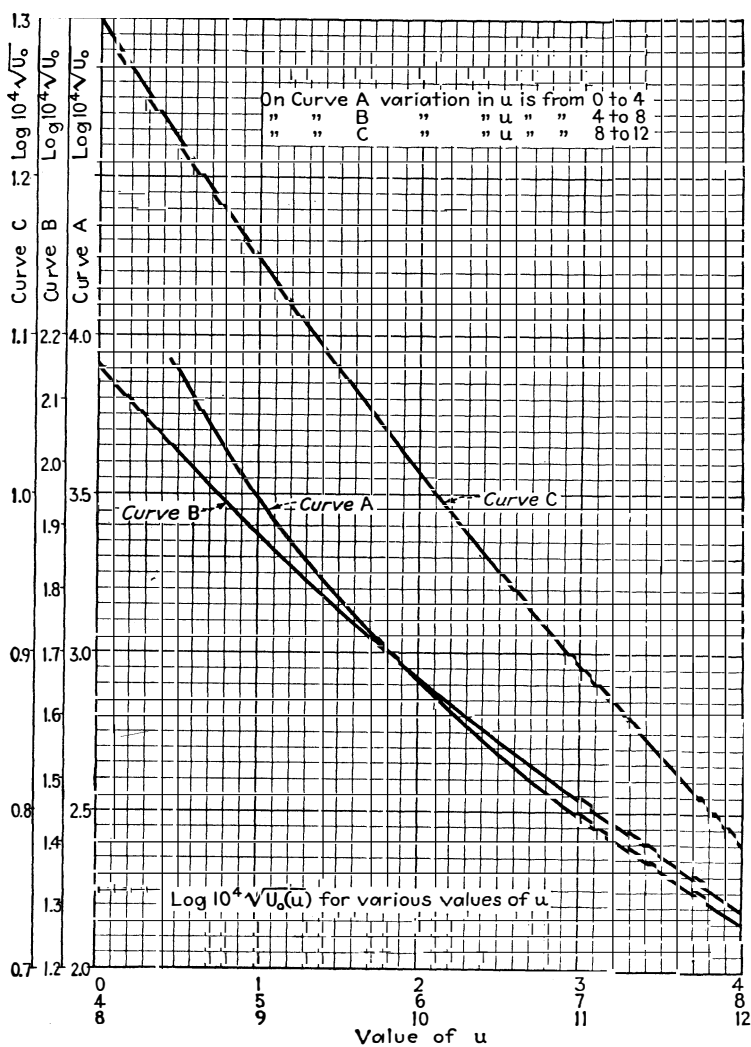


FIG. 4

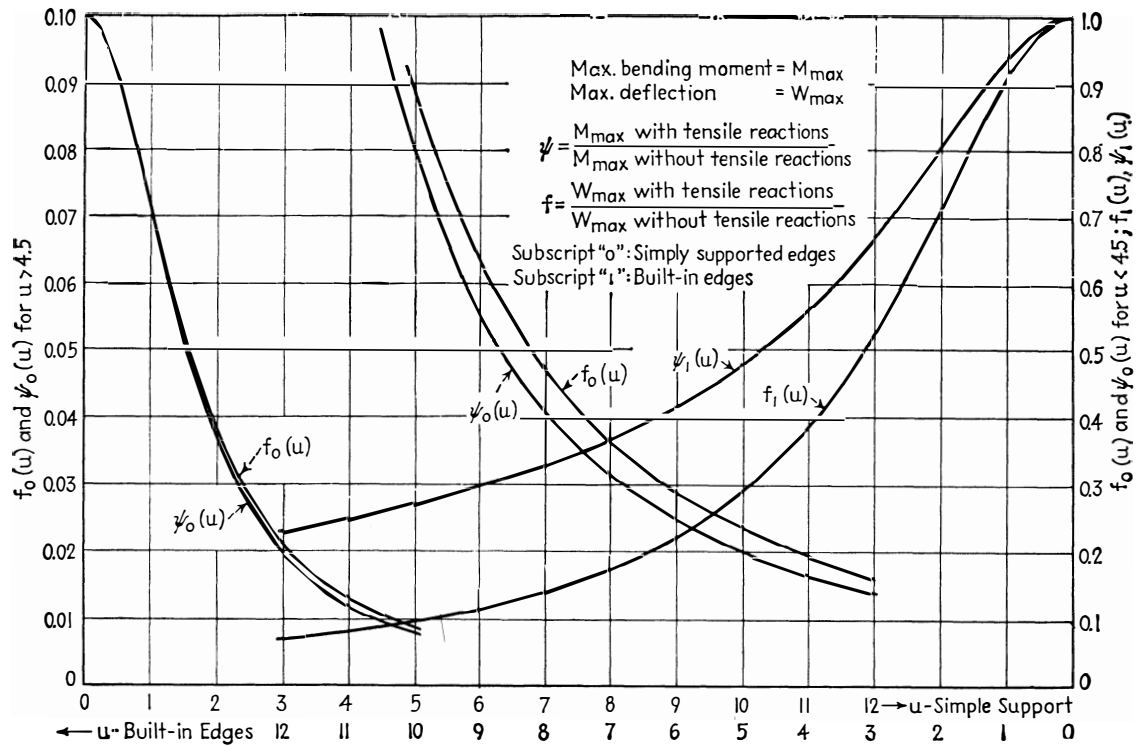
Substituting expression (6) for w , we obtain

$$M_{\max} = \frac{ql^2}{8} \psi_0(u) \quad (9)$$

where

$$\psi_0 = \frac{1 - \operatorname{sech} u}{\frac{u^2}{2}} \quad (e)$$

The values of ψ_0 are given by curves in Fig. 5. It is seen that these values diminish rapidly with increase of u , and for large u the maximum



bending moment is several times smaller than the moment $ql^2/8$ which would be obtained if there were no tensile reactions at the ends of the strip.

The direct tensile stress σ_1 and the maximum bending stress σ_2 are now readily expressed in terms of u , q , and the plate constants as follows:

$$\sigma_1 = \frac{S}{h} = \frac{4u^2 D}{hl^2} = \frac{Eu^2}{3(1-\nu^2)} \left(\frac{h}{l}\right)^2 \quad (10)$$

$$\sigma_2 = \frac{6}{h^2} M_{\max} = \frac{3}{4} q \left(\frac{l}{h}\right)^2 \psi_0 \quad (11)$$

The maximum stress in the plate is then

$$\sigma_{\max} = \sigma_1 + \sigma_2$$

To show how the curves in Figs. 4 and 5 can be used in calculating maximum stresses, let us take a numerical example and assume that a long rectangular steel plate 50 in. wide and $\frac{1}{2}$ in. thick carries a uniformly distributed load $q = 20$ psi. We start with a computation of $\sqrt{U_0}$:

$$\sqrt{U_0} = \frac{E}{(1-\nu^2)q} \left(\frac{h}{l}\right)^4 = \frac{30 \cdot 10^6}{(1-0.3^2)20} \frac{1}{10^8} = 0.01648$$

Then, from tables,

$$\log_{10} (10^{-4} \sqrt{U_0}) = 2.217$$

From the curve *A* in Fig. 4 we find $u = 3.795$, and from Fig. 5 we obtain $\psi_0 = 0.1329$.

Now, computing stresses by using Eqs. (10) and (11), we find

$$\begin{aligned} \sigma_1 &= \frac{30 \cdot 10^6 \cdot 3.795^2}{3(1-0.3^2)} \frac{1}{10^4} = 15,830 \text{ psi} \\ \sigma_2 &= \frac{3}{4} \cdot 20 \cdot 10^4 \cdot 0.1329 = 19,930 \text{ psi} \\ \sigma_{\max} &= \sigma_1 + \sigma_2 = 35,760 \text{ psi} \end{aligned}$$

In calculating the maximum deflection we substitute $x = l/2$ in Eq. (6) of the deflection curve. In this manner we obtain

$$w_{\max} = \frac{5ql^4}{384D} f_0(u) \quad (12)$$

where

$$f_0(u) = \frac{\operatorname{sech} u - 1 + \frac{u^2}{2}}{\frac{5u^4}{24}}$$

To simplify calculations, values of $f_0(u)$ are given by the curve in Fig. 5. If there were no tensile reactions at the ends of the strip, the maximum

deflection would be $5ql^4/384D$. The effect of the tensile reactions is given by the factor $f_0(u)$, which diminishes rapidly with increasing u .

Using Fig. 5 in the numerical example previously discussed, we find that for $u = 3.795$ the value of $f_0(u)$ is 0.145. Substituting this value in Eq. (12), we obtain

$$w_{\max} = 4.74 \cdot 0.145 = 0.688 \text{ in.}$$

It is seen from Eq. (8) that the tensile parameter u depends, for a given material of the plate, upon the intensity of the load q and the

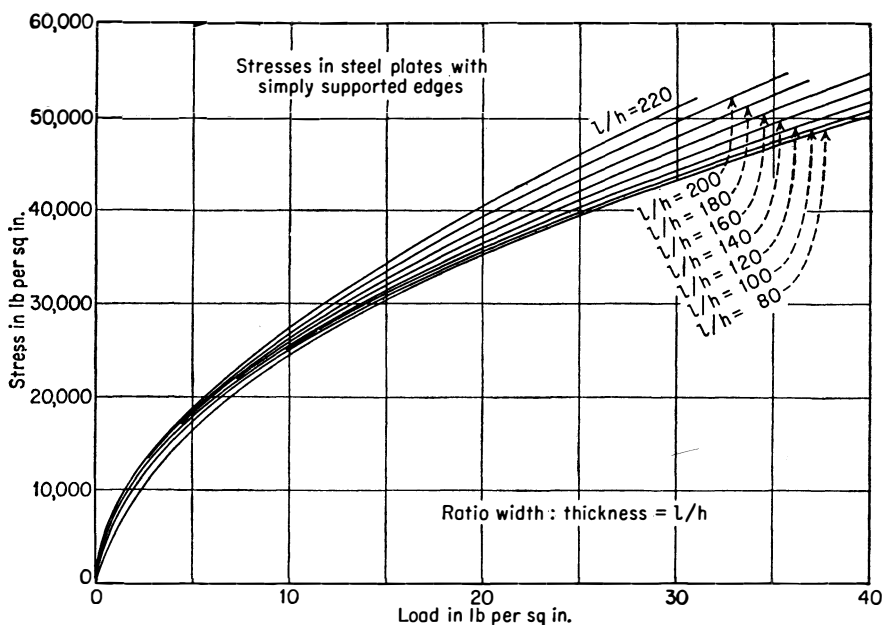


FIG. 6

ratio l/h of width to thickness of the plate. From Eqs. (10) and (11) we see that the stresses σ_1 and σ_2 are also functions of u , q , and l/h . Therefore, the maximum stress in the plate depends only on the load q and the ratio l/h . This means that we can plot a set of curves giving maximum stress in terms of q , each curve in the set corresponding to a particular value of l/h . Such curves are given in Fig. 6. It is seen that because of the presence of tensile forces S , which increase with the load, the maximum stress is not proportional to the load q ; and for large values of q this stress does not vary much with the thickness of the plate. By taking the curve marked $l/h = 100$ and assuming $q = 20$ psi, we obtain from the curve the value $\sigma_{\max}$ calculated before in the numerical example.

3. Cylindrical Bending of Uniformly Loaded Rectangular Plates with Built-in Edges. We assume that the longitudinal edges of the plate are fixed in such a manner that they cannot rotate. Taking an elemental strip of unit width in the same manner as before (Fig. 1) and denoting by M_0 the bending moment per unit length acting on the longitudinal edges of the plate, the forces acting on the strip will be as shown in Fig. 7. The bending moment at any cross section of the strip is

$$M = \frac{ql}{2}x - \frac{qx^2}{2} - Sw + M_0$$

Substituting this expression in Eq. (4), we obtain

$$\frac{d^2w}{dx^2} - \frac{S}{D}w = -\frac{qlx}{2D} + \frac{qx^2}{2D} - \frac{M_0}{D} \quad (a)$$

The general solution of this equation, using notation (5), will be represented in the following form:

$$w = C_1 \sinh \frac{2ux}{l} + C_2 \cosh \frac{2ux}{l} + \frac{ql^3x}{8u^2D} - \frac{ql^2x^2}{8u^2D} - \frac{ql^4}{16u^4D} + \frac{M_0l^2}{4u^2D} \quad (b)$$

Observing that the deflection curve is symmetrical with respect to the middle of the strip, we determine the constants of integration C_1 , C_2 , and

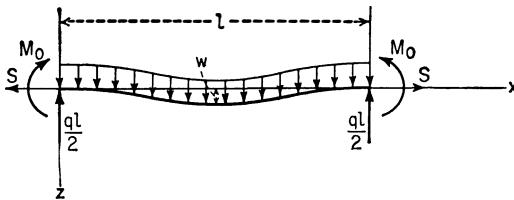


FIG. 7

the moment M_0 from the following three conditions:

$$\begin{aligned} \frac{dw}{dx} &= 0 & \text{for } x = 0 \text{ and } x = \frac{l}{2} \\ w &= 0 & \text{for } x = 0 \end{aligned} \quad (c)$$

Substituting expression (b) for w , we obtain from these conditions

$$\begin{aligned} C_1 &= -\frac{ql^4}{16u^3D} & C_2 &= \frac{ql^4}{16u^3D} \coth u \\ M_0 &= \frac{ql^2}{4u^2} - \frac{ql^2}{4u} \coth u = -\frac{ql^2}{12} \psi_1(u) \end{aligned} \quad (13)$$

where

$$\psi_1(u) = \frac{3(u - \tanh u)}{u^2 \tanh u}$$

The deflection w is therefore given by the expression

$$w = -\frac{ql^4}{16u^3D} \sinh \frac{2ux}{l} + \frac{ql^4}{16u^3D} \coth u \cosh \frac{2ux}{l} + \frac{ql^3x}{8u^2D} - \frac{ql^2x^2}{8u^2D} - \frac{ql^4}{16u^3D} \coth u$$

This can be further simplified and finally put in the following form:

$$w = \frac{ql^4}{16u^3D \tanh u} \left\{ \frac{\cosh \left[u \left(1 - \frac{2x}{l} \right) \right]}{\cosh u} - 1 \right\} + \frac{ql^2(l-x)x}{8u^2D} \quad (14)$$

For calculating the parameter u we proceed as in the previous article and use Eq. (d) of that article. Substituting in it expression (14) for w and performing the integration, we obtain

$$\frac{S(1-\nu^2)l}{hE} = \frac{q^2l^7}{D^2} \left(-\frac{3}{256u^5 \tanh u} - \frac{1}{256u^4 \sinh^2 u} + \frac{1}{64u^6} + \frac{1}{384u^4} \right)$$

Substituting S from Eq. (5) and expression (3) for D , the equation for calculating u finally becomes

$$\frac{E^2h^8}{(1-\nu^2)^2q^2l^8} = -\frac{81}{16u^7 \tanh u} - \frac{27}{16u^6 \sinh^2 u} + \frac{27}{4u^8} + \frac{9}{8u^6} \quad (15)$$

To simplify the solution of this equation we use the curve in Fig. 8, in which the parameter u is taken as abscissa and the ordinates are equal to $\log_{10} (10^4 \sqrt{U_1})$, where U_1 denotes the right-hand side of Eq. (15). For any given plate we begin by calculating the square root of the left-hand side of Eq. (15), equal to $Eh^4/[(1-\nu^2)ql^4]$, which gives us $\sqrt{U_1}$. The quantity $\log_{10} (10^4 \sqrt{U_1})$ then gives the ordinate of the curve in Fig. 8, and the corresponding abscissa gives the required value of u .

Having u , we can begin calculating the maximum stresses in the plate. The total stress at any point of a cross section of the strip consists of the constant tensile stress σ_1 and the bending stress. The maximum bending stress σ_2 will act at the built-in edges where the bending moment is the largest. Using Eq. (10) to calculate σ_1 and Eq. (13) to calculate the bending moment M_0 , we obtain

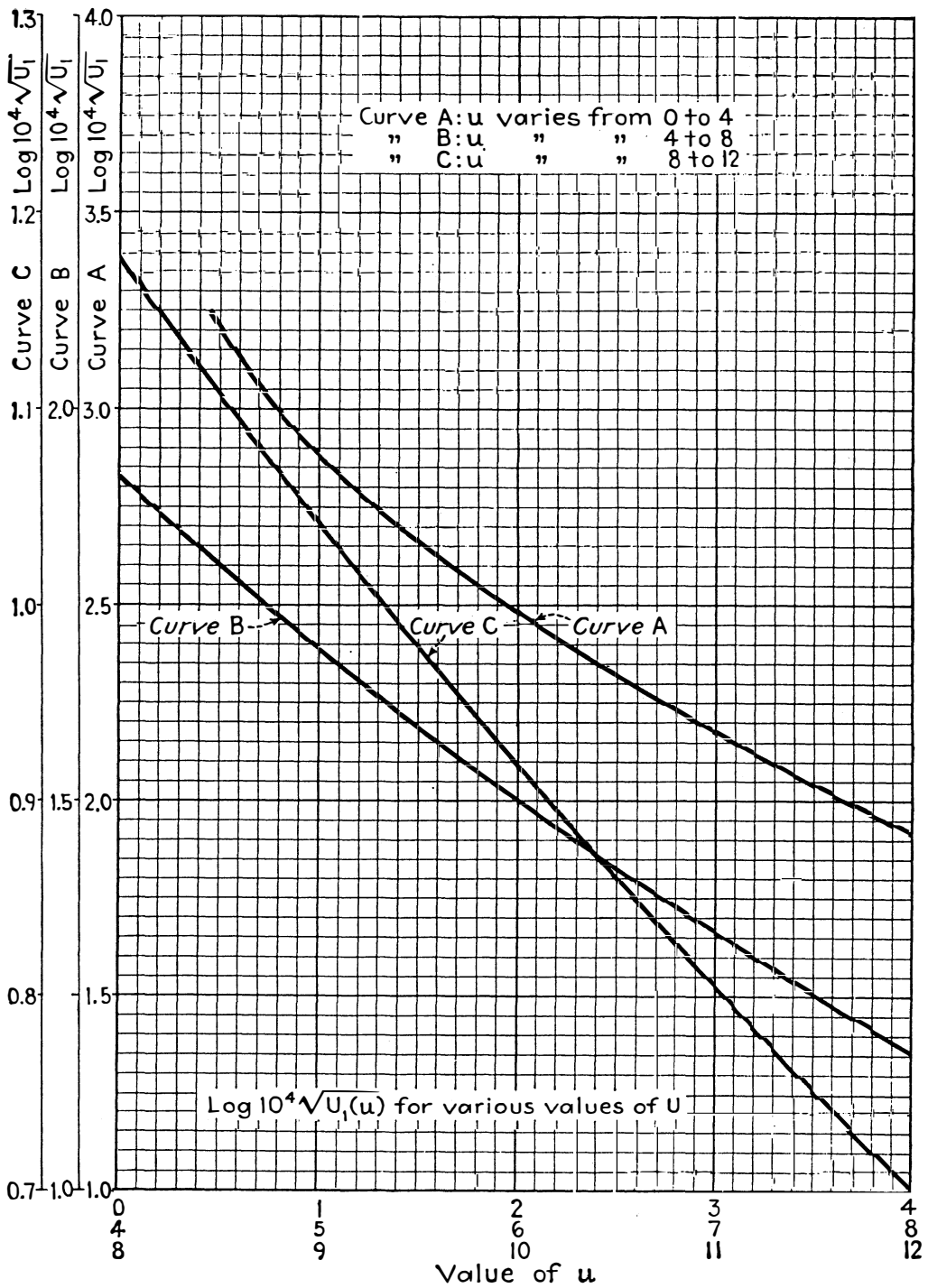
$$\sigma_1 = \frac{Eu^2}{3(1-\nu^2)} \left(\frac{h}{l} \right)^2 \quad (16)$$

$$\sigma_2 = -\frac{6M_0}{h^2} = \frac{q}{2} \left(\frac{l}{h} \right)^2 \psi_1(u) \quad (17)$$

$$\sigma_{\max} = \sigma_1 + \sigma_2$$

To simplify the calculation of the stress σ_2 , the values of the function $\psi_1(u)$ are given by a curve in Fig. 5.

The maximum deflection is at the middle of the strip and is obtained by



substituting $x = l/2$ in Eq. (14), from which

$$w_{\max} = \frac{ql^4}{384D} f_1(u) \quad (18)$$

where $f_1(u) = \frac{24}{u^4} \left(\frac{u^2}{2} + \frac{u}{\sinh u} - \frac{u}{\tanh u} \right)$

The function $f_1(u)$ is also given by a curve in Fig. 5.



The use of the curves in Figs. 5 and 8 will now be illustrated numerical example. A long rectangular steel plate has the dimensions $l = 50$ in., $h = \frac{1}{2}$ in., and $q = 10$ psi. In such a case we have

$$\sqrt{U_1} = \frac{E}{(1 - \nu^2)q} \left(\frac{h}{l} \right)^4 = \frac{30 \cdot 10^6}{(1 - 0.3^2)10 \cdot 10^4} = 0.032966$$

$$\log_{10} 10^4 \sqrt{U_1} = 2.5181$$

From Fig. 8 we now find $u = 1.894$; and from Fig. 5, $\psi_1 = 0.8212$. Substituting these values in Eqs. (16) and (17), we find

$$\sigma_1 = \frac{30 \cdot 10^6 \cdot 1.894^2}{3(1 - 0.3^2)10^4} = 3,940 \text{ psi}$$

$$\sigma_2 = \frac{1}{2} \cdot 10 \cdot 10^4 \cdot 0.8212 = 41,060 \text{ psi}$$

$$\sigma_{\max} = \sigma_1 + \sigma_2 = 45,000 \text{ psi}$$

Comparing these stress values with the maximum stresses obtained on a plate of the same size, but with twice the load, on the assumption

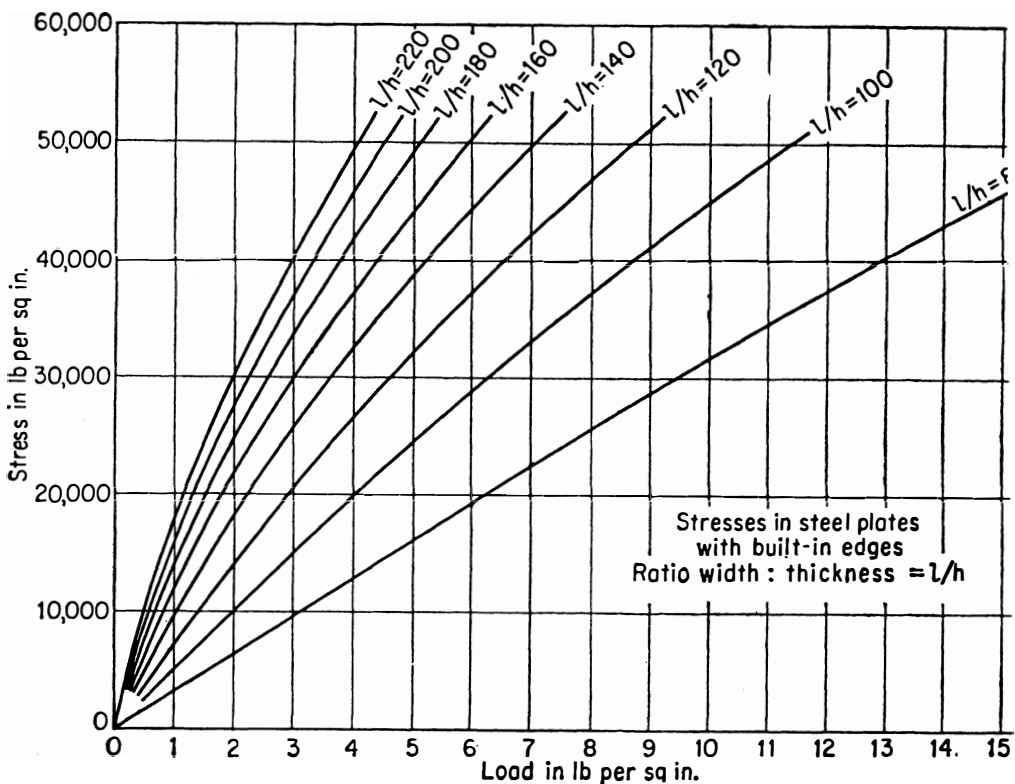


FIG. 9

simply supported edges (see page 11), it can be concluded that, owing to clamping of the edges, the direct tensile stress decreases considerably; whereas the maximum bending stress increases several times, so finally the maximum total stress in the case of clamped edges becomes larger than in the case of simply supported edges.

Proceeding as in the previous article it can be shown that the maximum stress in a plate depends only on the load q and the ratio l/h , and we can plot a set of curves giving maximum stress in terms of q , each curve in the set corresponding to a particular value of l/h . Such curves are given in Fig. 9. It is seen that for small values of the intensity of the load q , when the effect of the axial force on the deflections of the strip is small, the maximum stress increases approximately in the same ratio as q increases. But for larger values of q the relation between the load and the maximum stress becomes nonlinear.

In conclusion, we give in Table 1 the numerical values of all the functions plotted in Figs. 4, 5, and 8. This table can be used instead of the curves in calculating maximum stresses and maximum deflections of long, uniformly loaded rectangular plates.

4. Cylindrical Bending of Uniformly Loaded Rectangular Plates with Elastically Built-in Edges. Let us assume that when bending occurs, the longitudinal edges of the plate rotate through an angle proportional to the bending moment at the edges. In such a case the forces acting on an elemental strip will again be of the type shown in Fig. 7, and we shall obtain expression (b) of the previous article for the deflections w . However, the conditions at the edges, from which the constants of integration and the moment M_0 are determined, are different; *viz.*, the slope of the deflection curve at the ends of the strip is no longer zero but is proportional to the magnitude of the moment M_0 , and we have

$$\left(\frac{dw}{dx}\right)_{x=0} = -\beta M_0 \quad (a)$$

where β is a factor depending on the rigidity of restraint along the edges. If this restraint is very flexible, the quantity β is large, and the conditions at the edges approach those of simply supported edges. If the restraint is very rigid, the quantity β becomes small, and the edge conditions approach those of absolutely built-in edges. The remaining two end conditions are the same as in the previous article. Thus we have

$$\begin{aligned} \left(\frac{dw}{dx}\right)_{x=0} &= -\beta M_0 & \left(\frac{dw}{dx}\right)_{x=l/2} &= 0 \\ (w)_{x=0} &= 0 \end{aligned} \quad (b)$$

Using these conditions, we find both the constants of integration and the magnitude of M_0 in expression (b) of the previous article. Owing to flexibility of the boundary, the end moments M_0 will be smaller than those given by Eq. (13) for absolutely built-in edges, and the final result can be put in the form

$$M_0 = -\gamma \frac{ql^2}{12} \psi_1(u)$$

TABLE 1

u	$\log_{10} 10^4 \sqrt{U_0}$	$\log_{10} 10^4 \sqrt{U_1}$	$\log_{10} 10^4 \sqrt{U_2}$	$f_0(u)$	$f_1(u)$	$\psi_0(u)$	$\psi_1(u)$	u
0	∞	∞	∞	1.000	1.000	1.000	1.000	0
0.5	3.889 406	3.217 331	3.801 425	0.908	0.976	0.905	0.984	0.5
1.0	3.483 310	2.886 223	3.376 336	0.711	0.909	0.704	0.939	1.0
1.5	3.173 262	2.663 182	3.040 292	0.532	0.817	0.511	0.876	1.5
2.0	2.911 227	2.481 161	2.748 257	0.380	0.715	0.367	0.806	2.0
2.5	2.684 198	2.320 146	2.491 228	0.281	0.617	0.268	0.736	2.5
3.0	2.486 175	2.174 134	2.263 202	0.213	0.529	0.200	0.672	3.0
3.5	2.311 156	2.040 124	2.061 180	0.166	0.453	0.153	0.614	3.5
4.0	2.155 141	1.916 115	1.881 163	0.132	0.388	0.120	0.563	4.0
4.5	2.014 128	1.801 107	1.718 148	0.107	0.335	0.097	0.519	4.5
5.0	1.886 118	1.694 100	1.570 135	0.088	0.291	0.079	0.480	5.0
5.5	1.768 108	1.594 93	1.435 124	0.074	0.254	0.066	0.446	5.5
6.0	1.660 100	1.501 88	1.311 115	0.063	0.223	0.055	0.417	6.0
6.5	1.560 93	1.413 82	1.196 107	0.054	0.197	0.047	0.391	6.5
7.0	1.467 87	1.331 78	1.089 100	0.047	0.175	0.041	0.367	7.0
7.5	1.380 82	1.253 74	0.989 94	0.041	0.156	0.036	0.347	7.5
8.0	1.298 77	1.179 70	0.895 89	0.036	0.141	0.031	0.328	8.0
8.5	1.221 73	1.109 67	0.806 83	0.032	0.127	0.028	0.311	8.5
9.0	1.148 69	1.042 63	0.723 80	0.029	0.115	0.025	0.296	9.0
9.5	1.079 65	0.979 61	0.643 75	0.026	0.105	0.022	0.283	9.5
10.0	1.014 63	0.918 58	0.568 72	0.024	0.096	0.020	0.270	10.0
10.5	0.951 59	0.860 55	0.496 69	0.021	0.088	0.018	0.259	10.5
11.0	0.892 57	0.805 54	0.427 65	0.020	0.081	0.017	0.248	11.0
11.5	0.835 55	0.751 51	0.362 63	0.018	0.075	0.015	0.238	11.5
12.0	0.780	0.700	0.299	0.016	0.069	0.014	0.229	12.0

where γ is a numerical factor smaller than unity and given by the formula

$$\gamma = \frac{\tanh u}{\frac{2\beta}{l} Du + \tanh u}$$

It is seen that the magnitude of the moments M_0 at the edges depends upon the magnitude of the coefficient β defining the rigidity of the restraint. When β is very small, the coefficient γ approaches unity, and the moment M_0 approaches the value (13) calculated for absolutely built-in edges. When β is very large, the coefficient γ and the moment M_0 become small, and the edge conditions approach those of simply supported edges.

The deflection curve in the case under consideration can be represented in the following form:

$$w = \frac{ql^4}{16u^4D} \frac{\tanh u - \gamma(\tanh u - u)}{\tanh u} \left\{ \frac{\cosh \left[u \left(1 - \frac{2x}{l} \right) \right]}{\cosh u} - 1 \right\} + \frac{ql^2}{8u^2D} x(l - x) \quad (20)$$

For $\gamma = 1$ this expression reduces to expression (14) for deflections of a plate with absolutely built-in edges. For $\gamma = 0$ we obtain expression (6) for a plate with simply supported edges.

In calculating the tensile parameter u we proceed as in the previous cases and determine the tensile force S from the condition that the extension of the elemental strip is equal to the difference between the length of the arc along the deflection curve and the chord length l . Hence

$$\frac{S(1 - \nu^2)l}{hE} = \frac{1}{2} \int_0^l \left(\frac{dw}{dx} \right)^2 dx$$

Substituting expression (20) in this equation and performing the integration, we obtain

$$\frac{E^2 h^8}{(1 - \nu^2)^2 q^2 l^8} = (1 - \gamma)U_0 + \gamma U_1 - \gamma(1 - \gamma)U_2 \quad (21)$$

where U_0 and U_1 denote the right-hand sides of Eqs. (8) and (15), respectively, and

$$U_2 = \frac{27}{16} \frac{(u - \tanh u)^2}{u^9 \tanh^2 u} (u \tanh^2 u - u + \tanh u)$$

The values of $\log_{10} (10^4 \sqrt{U_2})$ are given in Table 1. By using this table, Eq. (21) can be readily solved by the trial-and-error method. For any particular plate we first calculate the left-hand side of the equation and,

by using the curves in Figs. 4 and 8, determine the values of the parameter u (1) for simply supported edges and (2) for absolutely built-in edges. Naturally u for elastically built-in edges must have a value intermediate between these two. Assuming one such value for u , we calculate U_0 , U_1 , and U_2 by using Table 1 and determine the value of the right-hand side of Eq. (21). Generally this value will be different from the value of the left-hand side calculated previously, and a new trial calculation with a new assumed value for u must be made. Two such trial calculations will usually be sufficient to determine by interpolation the value of u satisfying Eq. (21). As soon as the parameter u is determined, the bending moments M_0 at the ends may be calculated from Eq. (19). We can also calculate the moment at the middle of the strip and find the maximum stress. This stress will occur at the ends or at the middle, depending on the degree of rigidity of the constraints at the edges.

5. The Effect on Stresses and Deflections of Small Displacements of Longitudinal Edges in the Plane of the Plate. It was assumed in the previous discussion that, during bending, the longitudinal edges of the plate have no displacement in the plane of the plate. On the basis of this assumption the tensile force S was calculated in each particular case. Let us assume now that the edges of the plate undergo a displacement toward each other specified by Δ . Owing to this displacement the extension of the elemental strip will be diminished by the same amount, and the equation for calculating the tensile force S becomes

$$\frac{Sl(1 - \nu^2)}{hE} = \frac{1}{2} \int_0^l \left(\frac{dw}{dx} \right)^2 dx - \Delta \quad (a)$$

At the same time Eqs. (6), (14), and (20) for the deflection curve hold true regardless of the magnitude of the tensile force S . They may be differentiated and substituted under the integral sign in Eq. (a). After evaluating this integral and substituting $S = 4u^2D/l^2$, we obtain for simply supported edges

$$\frac{E^2h^8}{q^2(1 - \nu^2)^2l^8} \frac{u^2 + \frac{3l\Delta}{h^2}}{u^2} = U_0 \quad (22)$$

and for built-in edges

$$\frac{E^2h^8}{q^2(1 - \nu^2)^2l^8} \frac{u^2 + \frac{3l\Delta}{h^2}}{u^2} = U_1 \quad (23)$$

If Δ is made zero, Eqs. (22) and (23) reduce to Eqs. (8) and (15), obtained previously for immovable edges.

The simplest case is obtained by placing compression bars on the longitudinal sides of the boundary to prevent free motion of one edge of

the plate toward the other during bending. Tensile forces S in the plate produce contraction of these bars, which results in a displacement Δ proportional to S .* If k is the factor of proportionality depending on the elasticity and cross-sectional area of the bars, we obtain

$$S = k\Delta$$

or, substituting $S = 4u^2D/l^2$, we obtain

$$\Delta = \frac{1}{k} \frac{Eu^2h^3}{3l^2(1-\nu^2)}$$

and

$$\frac{u^2 + \frac{3l\Delta}{h^2}}{u^2} = 1 + \frac{Eh}{kl(1-\nu^2)}$$

Thus the second factor on the left-hand side of Eqs. (22) and (23) is a constant that can be readily calculated if the dimensions and the elastic properties of the structure are known. Having the magnitude of this factor, the solution of Eqs. (22) and (23) can be accomplished in exactly the same manner as used for immovable edges.

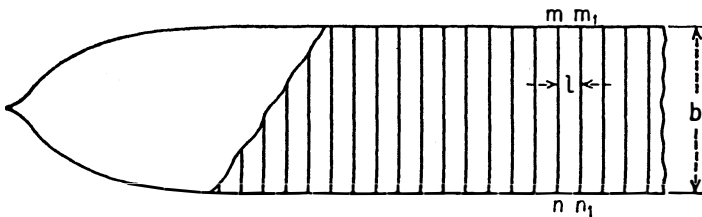


FIG. 10

In the general case the second factor on the left-hand side of Eqs. (22) and (23) may depend on the magnitude of the load acting on the structure, and the determination of the parameter u can be accomplished only by the trial-and-error method. This procedure will now be illustrated by an example that is encountered in analyzing stresses in the hull of a ship when it meets a wave. The bottom plates in the hull of a ship are subjected to a uniformly distributed water pressure and also to forces in the plane of the plates due to bending of the hull as a beam. Let b be the width of the ship at a cross section mn (Fig. 10) and l be the frame spacing at the bottom. When the hollow of a wave is amidships (Fig. 11b), the buoyancy is decreased there and increased at the ends. The effect of this change on the structure is that a sagging bending moment is produced and the normal distance l between the frames at the bottom is increased by a certain amount. To calculate this displacement accurately we must consider not only the action of the bending moment M on the hull but also the effect on this bending of a certain change in

* The edge support is assumed to be such that Δ is uniform along the edges.

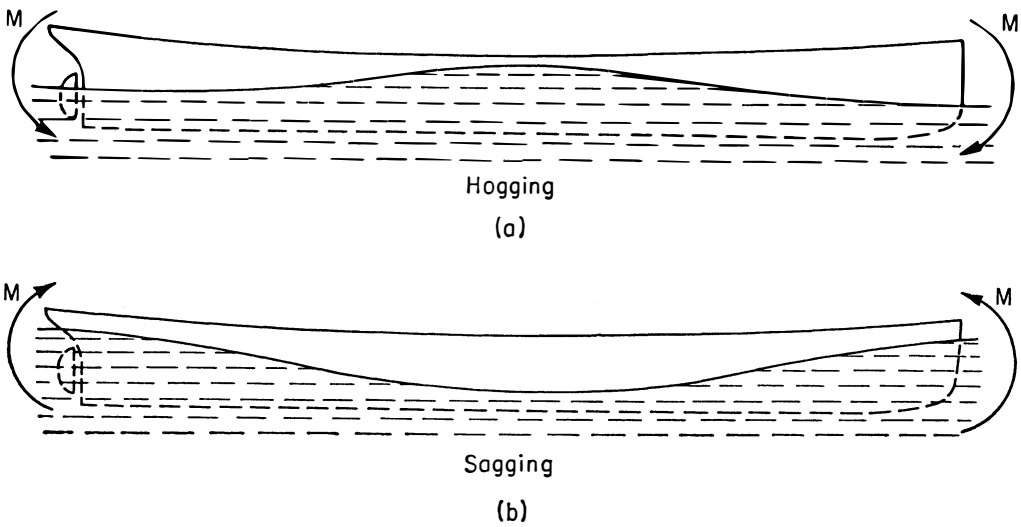


FIG. 11

tensile forces S distributed along the edges mn and m_1n_1 of the bottom plate mnm_1n_1 (Fig. 10), which will be considered as a long rectangular plate uniformly loaded by water pressure. Owing to the fact that the plates between the consecutive frames are equally loaded, there will be no rotation at the longitudinal edges of the plates, and they may be considered as absolutely built in along these edges.

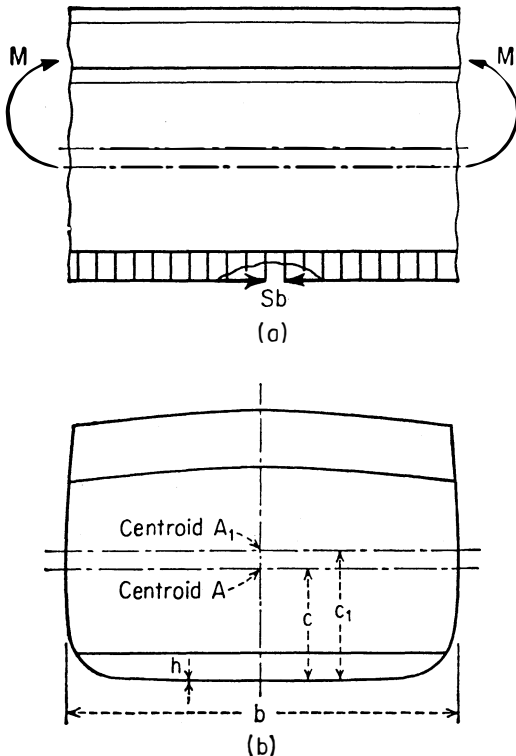


FIG. 12

To determine the value of Δ , which denotes, as before, the displacement of the edge mn toward the edge m_1n_1 in Fig. 10 and which is produced by the hull bending moment M and the tensile reactions S per unit length along the edges mn and m_1n_1 of the bottom plate, let us imagine that the plate mnm_1n_1 is removed and replaced by uniformly distributed forces S so that the total force along mn and m_1n_1 is Sb (Fig. 12a). We can then say that the displacement Δ of one frame relative to another is due to the bending moment M and to the

eccentric load Sb applied to the hull without bottom plating.

If A , I , and c are the cross-sectional area, the centroidal moment of inertia, and the distance from the bottom plate to the neutral axis of the

complete hull section, and if A_1 , I_1 , and c_1 are the corresponding quantities for the hull section without bottom plates, the latter set of quantities can be derived from the former by the relations

$$\begin{aligned} A_1 &= A - bh \\ c_1 &= \frac{Ac}{A_1} \\ I_1 &= I - bhc^2 - A_1(c_1 - c)^2 \end{aligned} \quad (b)$$

The relative displacement Δ_1 produced by the eccentrically applied forces Sb is

$$\Delta_1 = \frac{l(1 - \nu^2)}{E} \left(\frac{Sb}{A_1} + \frac{Sbc_1^2}{I_1} \right)$$

in which the factor $1 - \nu^2$ must be introduced if one neglects the lateral strain. The displacement due to the bending moment M is

$$\Delta_2 = - \frac{Mc_1 l}{EI_1}$$

Hence the total displacement is

$$\Delta = \Delta_1 + \Delta_2 = \frac{l(1 - \nu^2)}{E} \left[\frac{Sb}{A_1} + \frac{Sbc_1}{I_1} - \frac{Mc_1}{I_1(1 - \nu^2)} \right] \quad (c)$$

Substituting in this expression

$$S = \frac{4u^2 D}{l^2} = \frac{Eu^2 h^3}{3l^2(1 - \nu^2)}$$

we finally obtain

$$\Delta = \frac{u^2 h^3}{3l} \left(\frac{b}{A_1} + \frac{bc_1^2}{I_1} \right) - \frac{Mlc_1}{EI_1} \quad (d)$$

This quantity must be substituted in Eq. (23) for determining the tensile parameter u .

Let us apply this theory to a numerical example. Assume $b = 54$ ft, $I = 1,668$ ft⁴, $A = 13.5$ ft², $c = 12.87$ ft, $h = 0.75$ in. = 0.0625 ft, $l = 45$ in. = 3.75 ft, $q = 10$ psi, $M = 123,500$ ft-tons. From Eqs. (b) we obtain

$$A_1 = 13.5 - 0.0625 \cdot 54 = 10.125 \text{ ft}^2$$

$$c_1 = \frac{13.5 \cdot 12.87}{10.125} = 17.16 \text{ ft}$$

$$I_1 = 1,668 - 559.0 - 10.125(17.16 - 12.87)^2 = 922.7 \text{ ft}^4$$

Substituting these values in expression (d), we calculate Δ and finally obtain

$$\frac{3\Delta l}{h^2} = 1.410u^2 - 11.48$$

Equation (23) then becomes

$$\frac{E^2 h^8}{q^2(1 - \nu^2)^2 l^8} \frac{u^2 + 1.410u^2 - 11.48}{u^2} = U_1$$

or

$$\frac{1.552Eh^4}{q(1 - \nu^2)l^4} \sqrt{\frac{u^2 - 4.763}{u^2}} = \sqrt{U_1}$$

Substituting numerical values and taking logarithms of both sides, we obtain

$$3.597 + \log_{10} \sqrt{\frac{u^2 - 4.763}{u^2}} = \log_{10} (10^4 \sqrt{U_1})$$

Using the curve in Fig. 8, this equation can be readily solved by the trial-and-error method, and we obtain $u = 2.187$ and, from Fig. 5, $\psi_1(u) = 0.780$. The maximum stress is now calculated by using Eqs. (16) and (17), from which

$$\begin{aligned}\sigma_1 &= \frac{30 \cdot 10^6 \cdot 4.783}{3 \cdot 0.91 \cdot 60^2} = 14,600 \text{ psi} \\ \sigma_2 &= \frac{1}{2} \cdot 10 \cdot 60^2 \cdot 0.780 = 14,040 \text{ psi} \\ \sigma_{\max} &= \sigma_1 + \sigma_2 = 28,640 \text{ psi}\end{aligned}$$

If the bending stress in the plate due to water pressure were neglected and if the bottom plate stress were calculated from the formula $\sigma = Mc/I$, we would arrive at a figure of only 13,240 psi.

6. An Approximate Method of Calculating the Parameter u . In calculating the parameter u for plates in which the longitudinal edges do not move in the plane of the plate, we used the equation

$$\frac{Sl(1 - \nu^2)}{hE} = \frac{1}{2} \int_0^l \left(\frac{dw}{dx} \right)^2 dx \quad (a)$$

which states that the extension of an elemental strip produced by the forces S is equal to the difference between the length of the arc along the deflection curve of the strip and the chord length l . In the particular cases considered in the previous articles, exact expressions for the deflections w were derived, and numerical tables and curves for the right-hand side of Eq. (a) were given. When such tables are not at hand, the solution of the equation becomes complicated, and to simplify the problem recourse should be had to an approximate method. From the discussion of bending of beams it is known¹ that, in the case of simply supported ends with all lateral loads acting in the same direction, the deflection curve of an elemental strip produced by a combination of a lateral load and an axial tensile force S (Fig. 3) can be represented with sufficient

¹ See Timoshenko, "Strength of Materials," part II, 3d ed., p. 52, 1956.

accuracy by the equation

$$w = \frac{w_0}{1 + \alpha} \sin \frac{\pi x}{l} \quad (b)$$

in which w_0 denotes the deflection at the middle of the strip produced by the lateral load alone, and the quantity α is given by the equation

$$\alpha = \frac{S}{S_{cr}} = \frac{Sl^2}{\pi^2 D} \quad (c)$$

Thus, α represents the ratio of the axial force S to the Euler critical load for the elemental strip.

Substituting expression (b) in Eq. (a) and integrating, we obtain

$$\frac{Sl(1 - \nu^2)}{hE} = \frac{\pi^2 w_0^2}{4l(1 + \alpha)^2}$$

Now, using notation (c) and substituting for D its expression (3), we finally obtain

$$\alpha(1 + \alpha)^2 = \frac{3w_0^2}{h^2} \quad (24)$$

From this equation the quantity α can be calculated in each particular case, and the parameter u is now determined from the equation

$$u^2 = \frac{S}{D} \frac{l^2}{4} = \frac{\pi^2 \alpha}{4} \quad (d)$$

To show the application of the approximate Eq. (24) let us take a numerical example. A long rectangular steel plate with simply supported edges and of dimensions $l = 50$ in. and $h = \frac{1}{2}$ in. is loaded with a uniformly distributed load $q = 20$ psi. In such a case

$$w_0 = \frac{5}{384} \frac{ql^4}{D}$$

and, after substituting numerical values, Eq. (24) becomes

$$\alpha(1 + \alpha)^2 = 269.56$$

The solution of the equation can be simplified by letting

$$1 + \alpha = x \quad (e)$$

Then

$$x^3 - x^2 = 269.56$$

i.e., the quantity x is such that the difference between its cube and its square has a known value. Thus x can be readily determined from a slide rule or a suitable table, and we find in our case

$$x = 6.8109$$

and

$$\alpha = 5.8109$$

Then, from Eq. (d)

$$u = 3.7865$$

and from the formula (e) (see page 9)

$$\psi_0 = 0.13316$$

For calculating direct stress and maximum bending stress we use Eqs. (10) and (11). In this way we find

$$\begin{aligned}\sigma_1 &= 15,759 \text{ psi} \\ \sigma_2 &= 19,974 \text{ psi} \\ \sigma_{\max} &= \sigma_1 + \sigma_2 = 35,733 \text{ psi}\end{aligned}$$

The calculations made in Art. 2 (page 11) give, for this example,

$$\sigma_{\max} = 35,760 \text{ psi}$$

Thus the accuracy of the approximate Eq. (24) is in this case very high. In general, this accuracy depends on the magnitude of u . The error increases with increase of u . Calculations show that for $u = 1.44$ the error in the maximum stress is only 0.065 of 1 per cent and that for $u = 12.29$, which corresponds to very flexible plates, it is about 0.30 of 1 per cent. These values of u will cover the range ordinarily encountered in practice, and we conclude that Eq. (24) can be used with sufficient accuracy in all practical cases of uniformly loaded plates with simply supported edges.

It can also be used when the load is not uniformly distributed, as in the case of a hydrostatic pressure nonuniformly distributed along the elemental strip. If the longitudinal force is found by using the approximate Eq. (24), the deflections may be obtained from Eq. (b), and the bending moment at any cross section may be found as the algebraic sum of the moment produced by the lateral load and the moment due to the longitudinal force.¹

In the case of built-in edges the approximate expression for the deflection curve of an elemental strip can be taken in the form

$$w = \frac{w_0}{1 + \alpha/4} \frac{1}{2} \left(1 - \cos \frac{2\pi x}{l} \right) \quad (f)$$

in which w_0 is the deflection of the built-in beam under the lateral load acting alone and α has the same meaning as before. Substituting this expression in Eq. (a) and integrating, we obtain for determining α the equation

¹ More accurate values for the deflections and for the bending moments can be obtained by substituting the approximate value of the longitudinal force in Eq. (4) and integrating this equation, which gives Eqs. (12) and (9).

$$\alpha \left(1 + \frac{\alpha}{4}\right)^2 = \frac{3w_0^2}{h^2} \quad (25)$$

which can be solved in each particular case by the method suggested for solving Eq. (24).

When α is found, the parameter u is determined from Eq. (d); the maximum stress can be calculated by using Eqs. (16) and (17); and the maximum deflection, by using Eq. (18).

If, during bending, one edge moves toward the other by an amount Δ , the equation

$$\frac{Sl(1 - \nu^2)}{hE} = \frac{1}{2} \int_0^l \left(\frac{dw}{dx}\right)^2 dx - \Delta \quad (g)$$

must be used instead of Eq. (a). Substituting expression (b) in this equation, we obtain for determining α in the case of simply supported edges the equation

$$\alpha(1 + \alpha)^2 \frac{\alpha + 12 \frac{\Delta l}{\pi^2 h^2}}{\alpha} = \frac{3w_0^2}{h^2} \quad (26)$$

In the case of built-in edges we use expression (f). Then for determining α we obtain

$$\alpha \left(1 + \frac{\alpha}{4}\right)^2 \frac{\alpha + 12 \frac{\Delta l}{\pi^2 h^2}}{\alpha} = \frac{3w_0^2}{h^2} \quad (27)$$

If the dimensions of the plate and the load q are given, and the displacement Δ is known, Eqs. (26) and (27) can both be readily solved in the same manner as before. If the displacement Δ is proportional to the tensile force S , the second factor on the left-hand sides of Eqs. (26) and (27) is a constant and can be determined as explained in the previous article (see page 21). Thus again the equations can be readily solved.

7. Long Uniformly Loaded Rectangular Plates Having a Small Initial Cylindrical Curvature. It is seen from the discussions in Arts. 2 and 3 that the tensile forces S contribute to the strength of the plates by counteracting the bending produced by lateral load. This action increases with an increase in deflection. A further reduction of maximum stress can be accomplished by giving a suitable initial curvature to a plate. The effect on stresses and deflections of such an initial curvature can be investigated¹ by using the approximate method developed in the previous article.

Let us consider the case of a long rectangular plate with simply supported edges (Fig. 13), the initial curvature of which is given by the equation

¹ See S. Timoshenko's paper in "Festschrift zum siebenzigsten Geburtstage August Föppl," p. 74, Berlin, 1923.

$$w_1 = \delta \sin \frac{\pi x}{l} \quad (a)$$

If tensile forces S are applied to the edges of the plate, the initial deflections (a) will be reduced in the ratio $1/(1 + \alpha)$, where α has the same meaning as in the previous article¹ (page 25). The lateral load in combination with the forces S will produce deflections that can be expressed approximately by Eq. (b) of the previous article. Thus the total deflection of the plate, indicated in Fig. 13 by the dashed line, is

$$w = \frac{\delta}{1 + \alpha} \sin \frac{\pi x}{l} + \frac{w_0}{1 + \alpha} \sin \frac{\pi x}{l} = \frac{\delta + w_0}{1 + \alpha} \sin \frac{\pi x}{l} \quad (b)$$

Assuming that the longitudinal edges of the plate do not move in the plane of the plate, the tensile force S is found from the condition that the extension of the elemental strip produced by the forces S is equal to

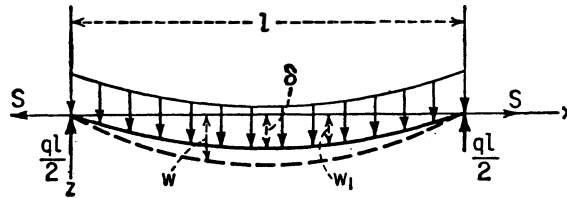


FIG. 13

the difference between the length of the arc along the deflection curve of the elemental strip and the initial length of the strip. This difference, in the case of small deflections, is given by the equation

$$\lambda = \frac{1}{2} \int_0^l \left(\frac{dw}{dx} \right)^2 dx - \frac{1}{2} \int_0^l \left(\frac{dw_1}{dx} \right)^2 dx \quad (c)$$

Substituting expressions (a) and (b) for w and w_1 and integrating, we obtain

$$\lambda = \frac{\pi^2}{4l} \left[\left(\frac{\delta + w_0}{1 + \alpha} \right)^2 - \delta^2 \right]$$

Putting λ equal to the extension of the strip $Sl(1 - \nu^2)/hE$, we finally obtain

$$\alpha(1 + \alpha)^2 = \frac{3(\delta + w_0)^2}{h^2} - \frac{3\delta^2(1 + \alpha)^2}{h^2} \quad (28)$$

If we take $\delta = 0$, this equation reduces to Eq. (24) for a plate without initial curvature.

To show the effect of the initial curvature on the maximum stress in a plate, let us apply Eq. (28) to a numerical example. Assume that a steel plate having $l = 45$ in. and $h = \frac{3}{8}$ in. is submitted to the action of

¹ See Timoshenko, "Strength of Materials," part II, 3d ed., p. 52, 1956.

a uniformly distributed load $q = 10$ psi. If there is no initial deflection, $\delta = 0$ and Eq. (28) becomes

$$\alpha(1 + \alpha)^2 = 290$$

from which

$$\alpha = 5.97 \quad \text{and} \quad u = \frac{\pi}{2} \sqrt{\alpha} = 3.83$$

From Eq. (10) we then obtain

$$\sigma_1 = 11,300 \text{ psi}$$

and from Eq. (11)

$$\sigma_2 = 14,200 \text{ psi}$$

The maximum stress in the plate is

$$\sigma_{\max} = \sigma_1 + \sigma_2 = 25,500 \text{ psi}$$

Let us now assume that there is an initial deflection in the plate such that $\delta = h = \frac{3}{8}$ in. In such a case Eq. (28) gives

$$\alpha(1 + \alpha)^2 = 351.6 - 3(1 + \alpha)^2$$

Letting

$$1 + \alpha = x$$

we obtain

$$x^3 + 2x^2 = 351.6$$

from which

$$x = 6.45 \quad \alpha = 5.45 \quad u = \frac{\pi}{2} \sqrt{\alpha} = 3.67$$

The tensile stress, from Eq. (10), is

$$\sigma_1 = 10,200 \text{ psi}$$

In calculating the bending stress we must consider only the change in deflections

$$w - w_1 = \frac{w_0}{1 + \alpha} \sin \frac{\pi x}{l} - \frac{\alpha \delta}{1 + \alpha} \sin \frac{\pi x}{l} \quad (d)$$

The maximum bending stress, corresponding to the first term on the right-hand side of Eq. (d), is the same as for a flat plate with $u = 3.67$. From Table 1 we find $\psi_0 = 0.142$ and from Eq. (11)

$$\sigma'_2 = 15,300 \text{ psi}$$

The bending moment corresponding to the second term in Eq. (d) is

$$-D \frac{d^2}{dx^2} \left(-\frac{\alpha \delta}{1 + \alpha} \sin \frac{\pi x}{l} \right) = -\frac{\alpha \pi^2 \delta D}{(1 + \alpha) l^2} \sin \frac{\pi x}{l}$$

This moment has a negative sign, and a corresponding maximum stress of

$$\sigma_2'' = \frac{6}{h^2} \frac{\alpha \pi^2 \delta D}{(1 + \alpha) l^2} = 9,500 \text{ psi}$$

must be subtracted from the bending stress σ_2' calculated above. Hence the maximum stress for the plate with the initial deflection is

$$\sigma_{\max} = 10,200 + 15,300 - 9,500 = 16,000 \text{ psi}$$

Comparison of this result with that obtained for the plane plate shows that the effect of the initial curvature is to reduce the maximum stress from 25,500 to 16,000 psi. This result is obtained assuming the initial deflection equal to the thickness of the plate. By increasing the initial deflection, the maximum stress can be reduced still further.

8. Cylindrical Bending of a Plate on an Elastic Foundation. Let us consider the problem of bending of a long uniformly loaded rectangular plate supported over the entire surface by an elastic foundation and rigidly supported along the edges (Fig. 14).

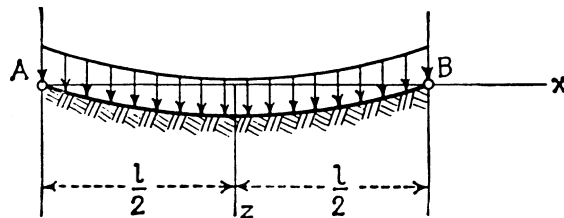


FIG. 14

Cutting out from the plate an elemental strip, as before, we may consider it as a beam on an elastic foundation. Assuming that the reaction of the foundation at any point is proportional to the deflection w at that point, and using Eq. (4), we obtain by double differentiation of that equation¹

$$D \frac{d^4 w}{dx^4} = q - kw \quad (29)$$

where q is the intensity of the load acting on the plate and k is the reaction of the foundation per unit area for a deflection equal to unity. Introducing the notation

$$\beta = \frac{l}{2} \sqrt[4]{\frac{k}{4D}} \quad (30)$$

the general solution of Eq. (29) can be written as follows:

$$w = \frac{q}{k} + C_1 \sin \frac{2\beta x}{l} \sinh \frac{2\beta x}{l} + C_2 \sin \frac{2\beta x}{l} \cosh \frac{2\beta x}{l} + C_3 \cos \frac{2\beta x}{l} \sinh \frac{2\beta x}{l} + C_4 \cos \frac{2\beta x}{l} \cosh \frac{2\beta x}{l} \quad (a)$$

The four constants of integration must now be determined from the conditions at the ends of the strip. In the case under consideration the deflection is symmetrical with respect to the middle of the strip. Thus, taking the coordinate axes as shown in Fig.

¹ *Ibid.*, p. 21.

14, we conclude¹ that $C_2 = C_3 = 0$. The constants C_1 and C_4 are found from the conditions that the deflection and the bending moment of the strip are zero at the end ($x = l/2$). Hence

$$\begin{aligned} (w)_{x=l/2} &= 0 \\ \left(\frac{d^2w}{dx^2}\right)_{x=l/2} &= 0 \end{aligned} \quad (b)$$

Substituting expression (a) for w and observing that $C_2 = C_3 = 0$, we obtain

$$\begin{aligned} \frac{q}{k} + C_1 \sin \beta \sinh \beta + C_4 \cos \beta \cosh \beta &= 0 \\ C_1 \cos \beta \cosh \beta - C_4 \sin \beta \sinh \beta &= 0 \end{aligned} \quad (c)$$

from which we find

$$\begin{aligned} C_1 &= -\frac{q}{k} \frac{\sin \beta \sinh \beta}{\sin^2 \beta \sinh^2 \beta + \cos^2 \beta \cosh^2 \beta} = -\frac{q}{k} \frac{2 \sin \beta \sinh \beta}{\cos 2\beta + \cosh 2\beta} \\ C_4 &= -\frac{q}{k} \frac{\cos \beta \cosh \beta}{\sin^2 \beta \sinh^2 \beta + \cos^2 \beta \cosh^2 \beta} = -\frac{q}{k} \frac{2 \cos \beta \cosh \beta}{\cos 2\beta + \cosh 2\beta} \end{aligned}$$

Substituting these values of the constants in expression (a) and using Eq. (30), we finally represent the deflection of the strip by the equation

$$\begin{aligned} w = \frac{ql^4}{64D\beta^4} \left(1 - \frac{2 \sin \beta \sinh \beta}{\cos 2\beta + \cosh 2\beta} \sin \frac{2\beta x}{l} \sinh \frac{2\beta x}{l} \right. \\ \left. - \frac{2 \cos \beta \cosh \beta}{\cos 2\beta + \cosh 2\beta} \cos \frac{2\beta x}{l} \cosh \frac{2\beta x}{l} \right) \end{aligned} \quad (d)$$

The deflection at the middle is obtained by substituting $x = 0$, which gives

$$(w)_{x=0} = \frac{5ql^4}{384D} \varphi(\beta) \quad (31)$$

where

$$\varphi(\beta) = \frac{6}{5\beta^4} \left(1 - \frac{2 \cos \beta \cosh \beta}{\cos 2\beta + \cosh 2\beta} \right)$$

To obtain the angles of rotation of the edges of the plate, we differentiate expression (d) with respect to x and put $x = -l/2$. In this way we obtain

$$\left(\frac{dw}{dx}\right)_{x=-l/2} = \frac{ql^3}{24D} \varphi_1(\beta) \quad (32)$$

where

$$\varphi_1(\beta) = \frac{3}{4\beta^3} \frac{\sinh 2\beta - \sin 2\beta}{\cosh 2\beta + \cos 2\beta}$$

The bending moment at any cross section of the strip is obtained from the equation

$$M = -D \frac{d^2w}{dx^2}$$

Substituting expression (d) for w , we find for the middle of the strip

$$(M)_{x=0} = \frac{ql^2}{8} \varphi_2(\beta) \quad (33)$$

where

$$\varphi_2(\beta) = \frac{2}{\beta^2} \frac{\sinh \beta \sin \beta}{\cosh 2\beta + \cos 2\beta}$$

¹ It is seen that the terms with coefficients C_2 and C_3 change sign when x is replaced by $-x$.

To simplify the calculation of deflections and stresses, numerical values of functions φ , φ_1 , and φ_2 are given in Table 2. For small values of β , that is, for a yielding foundation, the functions φ and φ_2 do not differ greatly from unity. Thus the maximum deflection and bending stresses are close to those for a simply supported strip without an elastic foundation. With an increase in β , the effect of the foundation becomes more and more important.

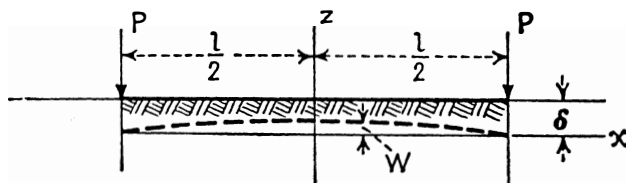


FIG. 15

Conditions similar to those represented in Fig. 14 are obtained if a long rectangular plate of width l is pressed into an elastic foundation by loads uniformly distributed along the edges and of the amount P per unit length (Fig. 15). The plate will be

TABLE 2

β	φ	φ_1	φ_2	β	φ	φ_1	φ_2
0.1	1.000	1.000	1.000	1.6	0.186	0.200	0.164
0.2	0.999	0.999	0.999	1.7	0.151	0.166	0.129
0.3	0.996	0.995	0.995	1.8	0.124	0.138	0.101
0.4	0.984	0.983	0.983	1.9	0.102	0.116	0.079
0.5	0.961	0.961	0.959	2.0	0.084	0.099	0.062
0.6	0.921	0.923	0.919	2.2	0.058	0.072	0.037
0.7	0.863	0.866	0.859	2.4	0.042	0.055	0.021
0.8	0.787	0.791	0.781	2.6	0.029	0.043	0.011
0.9	0.698	0.702	0.689	2.8	0.022	0.034	0.005
1.0	0.602	0.609	0.591	3.0	0.016	0.028	0.002
1.1	0.508	0.517	0.494	3.2	0.012	0.023	0.000
1.2	0.421	0.431	0.405	3.4	0.010	0.019	-0.001
1.3	0.345	0.357	0.327	3.6	0.007	0.016	-0.002
1.4	0.281	0.294	0.262	3.8	0.006	0.014	-0.002
1.5	0.228	0.242	0.208	4.0	0.005	0.012	-0.002

pressed into the elastic foundation and bent, as shown by the dashed line. If δ denotes the deflection at the edges of the plate, the reaction of the foundation at any point is

$$k(\delta - w) = k\delta - kw$$

where w is given by Eq. (d) with $q = k\delta$. The magnitude of δ is then obtained from the condition that the load is balanced by the reaction of the foundation. Hence

$$P = \frac{k\delta l}{2} - k \int_0^{l/2} w dx$$

Plates on elastic foundation with other conditions at the longitudinal edges can also be discussed in a similar manner.

CHAPTER 2

PURE BENDING OF PLATES

9. Slope and Curvature of Slightly Bent Plates. In discussing small deflections of a plate we take the *middle plane* of the plate, before bending occurs, as the xy plane. During bending, the particles that were in the xy plane undergo small displacements w perpendicular to the xy plane and form the *middle surface* of the plate. These displacements of the middle surface are called *deflections* of a plate in our further discussion.

Taking a normal section of the plate parallel to the xz plane (Fig. 16a), we find that the slope of the middle surface in the x direction is $i_x = \partial w / \partial x$. In the same manner the slope in the y direction is $i_y = \partial w / \partial y$. Taking now any direction an in the xy plane (Fig. 16b) making an angle α with the x axis, we find that the difference in the deflections of the two adjacent points a and a_1 in the an direction is

$$dw = \frac{\partial w}{\partial x} dx + \frac{\partial w}{\partial y} dy$$

and that the corresponding slope is

$$\frac{\partial w}{\partial n} = \frac{\partial w}{\partial x} \frac{dx}{dn} + \frac{\partial w}{\partial y} \frac{dy}{dn} = \frac{\partial w}{\partial x} \cos \alpha + \frac{\partial w}{\partial y} \sin \alpha \quad (a)$$

To find the direction α_1 for which the slope is a maximum we equate to zero the derivative with respect to α of expression (a). In this way we obtain

$$\tan \alpha_1 = \frac{\partial w}{\partial y} / \frac{\partial w}{\partial x} \quad (b)$$

Substituting the corresponding values of $\sin \alpha_1$ and $\cos \alpha_1$ in (a), we obtain for the maximum slope the expression

$$\left(\frac{\partial w}{\partial n} \right)_{\max} = \sqrt{\left(\frac{\partial w}{\partial x} \right)^2 + \left(\frac{\partial w}{\partial y} \right)^2} \quad (c)$$

By setting expression (a) equal to zero we obtain the direction for which

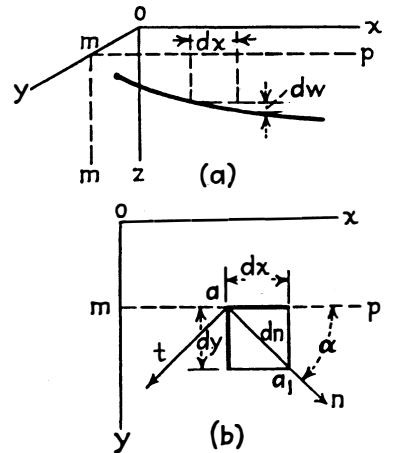


FIG. 16

the slope of the surface is zero. The corresponding angle α_2 is determined from the equation

$$\tan \alpha_2 = - \frac{\partial w}{\partial x} / \frac{\partial w}{\partial y} \quad (d)$$

From Eqs. (b) and (d) we conclude that

$$\tan \alpha_1 \tan \alpha_2 = -1$$

which shows that the directions of zero slope and of maximum slope are perpendicular to each other.

In determining the curvature of the middle surface of the plate we observe that the deflections of the plate are very small. In such a case the slope of the surface in any direction can be taken equal to the angle that the tangent to the surface in that direction makes with the xy plane, and the square of the slope may be neglected compared to unity. The curvature of the surface in a plane parallel to the xz plane (Fig. 16) is then numerically equal to

$$\frac{1}{r_x} = - \frac{\partial}{\partial x} \left(\frac{\partial w}{\partial x} \right) = - \frac{\partial^2 w}{\partial x^2} \quad (e)$$

We consider a curvature positive if it is convex downward. The minus sign is taken in Eq. (e), since for the deflection convex downward, as shown in the figure, the second derivative $\partial^2 w / \partial x^2$ is negative.

In the same manner we obtain for the curvature in a plane parallel to the yz plane

$$\frac{1}{r_y} = - \frac{\partial}{\partial y} \left(\frac{\partial w}{\partial y} \right) = - \frac{\partial^2 w}{\partial y^2} \quad (f)$$

These expressions are similar to those used in discussing the curvature of a bent beam.

In considering the curvature of the middle surface in any direction an (Fig. 16) we obtain

$$\frac{1}{r_n} = - \frac{\partial}{\partial n} \left(\frac{\partial w}{\partial n} \right)$$

Substituting expression (a) for $\partial w / \partial n$ and observing that

$$\frac{\partial}{\partial n} = \frac{\partial}{\partial x} \cos \alpha + \frac{\partial}{\partial y} \sin \alpha$$

we find

$$\begin{aligned} \frac{1}{r_n} &= - \left(\frac{\partial}{\partial x} \cos \alpha + \frac{\partial}{\partial y} \sin \alpha \right) \left(\frac{\partial w}{\partial x} \cos \alpha + \frac{\partial w}{\partial y} \sin \alpha \right) \\ &= - \left(\frac{\partial^2 w}{\partial x^2} \cos^2 \alpha + 2 \frac{\partial^2 w}{\partial x \partial y} \sin \alpha \cos \alpha + \frac{\partial^2 w}{\partial y^2} \sin^2 \alpha \right) \\ &= \frac{1}{r_x} \cos^2 \alpha - \frac{1}{r_{xy}} \sin 2\alpha + \frac{1}{r_y} \sin^2 \alpha \end{aligned}$$

It is seen that the curvature in any direction n at a point of the middle surface can be calculated if we know at that point the curvatures

$$\frac{1}{r_x} = -\frac{\partial^2 w}{\partial x^2} \quad \frac{1}{r_y} = -\frac{\partial^2 w}{\partial y^2}$$

and the quantity

$$\frac{1}{r_{xy}} = \frac{\partial^2 w}{\partial x \partial y} \quad (h)$$

which is called the *twist of the surface* with respect to the x and y axes.

If instead of the direction an (Fig. 16b) we take the direction at perpendicular to an , the curvature in this new direction will be obtained from expression (g) by substituting $\pi/2 + \alpha$ for α . Thus we obtain

$$\frac{1}{r_t} = \frac{1}{r_x} \sin^2 \alpha + \frac{1}{r_{xy}} \sin 2\alpha + \frac{1}{r_y} \cos^2 \alpha \quad (i)$$

Adding expressions (g) and (i), we find

$$\frac{1}{r_n} + \frac{1}{r_t} = \frac{1}{r_x} + \frac{1}{r_y} \quad (34)$$

which shows that at any point of the middle surface the sum of the curvatures in two perpendicular directions such as n and t is independent of the angle α . This sum is usually called the *average curvature* of the surface at a point.

The twist of the surface at a with respect to the an and at directions is

$$\frac{1}{r_{nt}} = \frac{d}{dt} \left(\frac{dw}{dn} \right)$$

In calculating the derivative with respect to t , we observe that the direction at is perpendicular to an . Thus we obtain the required derivative by substituting $\pi/2 + \alpha$ for α in Eq. (a). In this manner we find

$$\begin{aligned} \frac{1}{r_{nt}} &= \left(\frac{\partial}{\partial x} \cos \alpha + \frac{\partial}{\partial y} \sin \alpha \right) \left(-\frac{\partial w}{\partial x} \sin \alpha + \frac{\partial w}{\partial y} \cos \alpha \right) \\ &= \frac{1}{2} \sin 2\alpha \left(-\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right) + \cos 2\alpha \frac{\partial^2 w}{\partial x \partial y} \\ &= \frac{1}{2} \sin 2\alpha \left(\frac{1}{r_x} - \frac{1}{r_y} \right) + \cos 2\alpha \frac{1}{r_{xy}} \end{aligned} \quad (j)$$

In our further discussion we shall be interested in finding in terms of α the directions in which the curvature of the surface is a maximum or a minimum and in finding the corresponding values of the curvature. We obtain the necessary equation for determining α by equating the derivative of expression (g) with respect to α to zero, which gives

$$\frac{1}{r_x} \sin 2\alpha + \frac{2}{r_{xy}} \cos 2\alpha - \frac{1}{r_y} \sin 2\alpha = 0 \quad (k)$$

whence

$$\tan 2\alpha = -\frac{\frac{2}{r_{xy}}}{\frac{1}{r_x} - \frac{1}{r_y}} \quad (35)$$

From this equation we find two values of α , differing by $\pi/2$. Substituting these in Eq. (g) we find two values of $1/r_n$, one representing the maximum and the other the minimum curvature at a point a of the surface. These two curvatures are called the *principal curvatures* of the surface; and the corresponding planes naz and taz , the *principal planes of curvature*.

Observing that the left-hand side of Eq. (k) is equal to the doubled value of expression (j), we conclude that, if the directions an and at (Fig. 16) are in the principal planes, the corresponding twist $1/r_{nt}$ is equal to zero.

We can use a circle, similar to Mohr's circle representing combined stresses, to show how the curvature and the twist of a surface vary with the angle α .* To simplify the discussion we assume that the coordinate planes xz and yz are taken parallel to the principal planes of curvature at the point a . Then

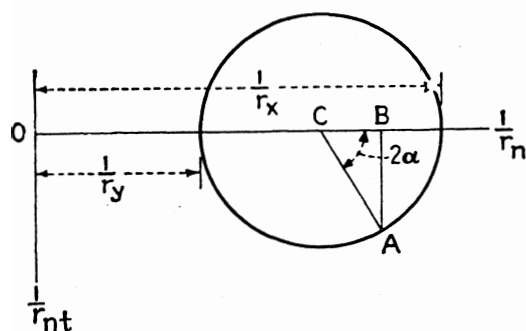


FIG. 17

$$\frac{1}{r_{xy}} = 0$$

and we obtain from Eqs. (g) and (j) for any angle α

$$\begin{aligned} \frac{1}{r_n} &= \frac{1}{r_x} \cos^2 \alpha + \frac{1}{r_y} \sin^2 \alpha \\ \frac{1}{r_{nt}} &= \frac{1}{2} \left(\frac{1}{r_x} - \frac{1}{r_y} \right) \sin 2\alpha \end{aligned} \quad (36)$$

Taking the curvatures as abscissas and the twists as ordinates and constructing a circle on the diameter $1/r_x - 1/r_y$, as shown in Fig. 17, we see that the point A defined by the angle 2α has the abscissa

$$\begin{aligned} \overline{OB} &= \overline{OC} + \overline{CB} = \frac{1}{2} \left(\frac{1}{r_x} + \frac{1}{r_y} \right) + \frac{1}{2} \left(\frac{1}{r_x} - \frac{1}{r_y} \right) \cos 2\alpha \\ &= \frac{1}{r_x} \cos^2 \alpha + \frac{1}{r_y} \sin^2 \alpha \end{aligned}$$

and the ordinate

$$\overline{AB} = \frac{1}{2} \left(\frac{1}{r_x} - \frac{1}{r_y} \right) \sin 2\alpha$$

Comparing these results with formulas (36), we conclude that the coordi-

* See S. Timoshenko, "Strength of Materials," part I, 3d ed., p. 40, 1955.

nates of the point A define the curvature and the twist of the surface for any value of the angle α . It is seen that the maximum twist, represented by the radius of the circle, takes place when $\alpha = \pi/4$, i.e., when we take two perpendicular directions bisecting the angles between the principal planes.

In our example the curvature in any direction is positive; hence the surface is bent convex downward. If the curvatures $1/r_x$ and $1/r_y$ are both negative, the curvature in any direction is also negative, and we have a bending of the plate convex upward. Surfaces in which the curvatures in all planes have like signs are called *synclastic*. Sometimes we shall deal with surfaces in which the two principal curvatures have opposite signs. A saddle is a good example. Such surfaces are called *anticlastic*. The circle in Fig. 18 represents a particular case of such surfaces when

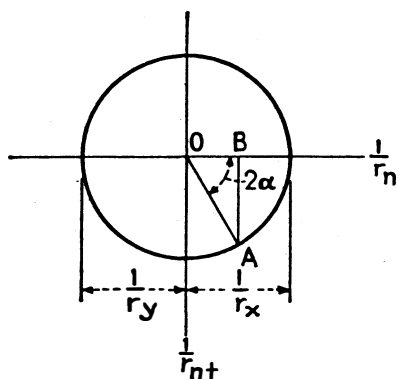


FIG. 18

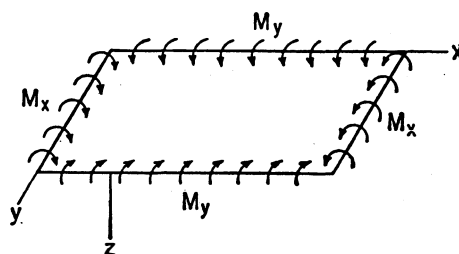


FIG. 19

$1/r_y = -1/r_x$. It is seen that in this case the curvature becomes zero for $\alpha = \pi/4$ and for $\alpha = 3\pi/4$, and the twist becomes equal to $\pm 1/r_x$.

10. Relations between Bending Moments and Curvature in Pure Bending of Plates. In the case of pure bending of prismatic bars a rigorous solution for stress distribution is obtained by assuming that cross sections of the bar remain plane during bending and rotate only with respect to their neutral axes so as to be always normal to the deflection curve. Combination of such bending in two perpendicular directions brings us to pure bending of plates. Let us begin with pure bending of a rectangular plate by moments that are uniformly distributed along the edges of the plate, as shown in Fig. 19. We take the xy plane to coincide with the middle plane of the plate before deflection and the x and y axes along the edges of the plate as shown. The z axis, which is then perpendicular to the middle plane, is taken positive downward. We denote by M_x the bending moment per unit length acting on the edges parallel to the y axis and by M_y the moment per unit length acting on the edges parallel to the x axis. These moments we consider positive when they are directed as shown in the figure, i.e., when they produce compression

in the upper surface of the plate and tension in the lower. The thickness of the plate we denote, as before, by h and consider it small in comparison with other dimensions.

Let us consider an element cut out of the plate by two pairs of planes parallel to the xz and yz planes, as shown in Fig. 20. Since the case shown in Fig. 19 represents the combination of two uniform bendings, the stress conditions are identical in all elements, as shown in Fig. 20, and we have

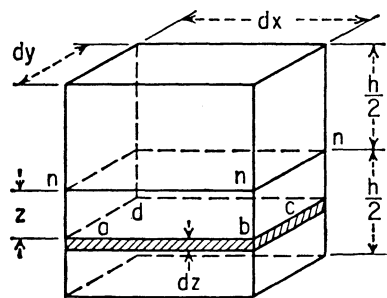


FIG. 20

a uniform bending of the plate. Assuming that during bending of the plate the lateral sides of the element remain plane and rotate about the neutral axes nn so as to remain normal to the deflected middle surface of the plate, it can be concluded that the middle plane of the plate does not undergo any extension during this bending, and the middle surface is therefore the *neutral surface*.¹ Let

$1/r_x$ and $1/r_y$ denote, as before, the curvatures of this neutral surface in sections parallel to the xz and yz planes, respectively. Then the unit elongations in the x and y directions of an elemental lamina $abcd$ (Fig. 20), at a distance z from the neutral surface, are found, as in the case of a beam, and are equal to

$$\epsilon_x = \frac{z}{r_x} \quad \epsilon_y = \frac{z}{r_y} \quad (a)$$

Using now Hooke's law [Eq. (1), page 5], the corresponding stresses in the lamina $abcd$ are

$$\begin{aligned} \sigma_x &= \frac{Ez}{1 - \nu^2} \left(\frac{1}{r_x} + \nu \frac{1}{r_y} \right) \\ \sigma_y &= \frac{Ez}{1 - \nu^2} \left(\frac{1}{r_y} + \nu \frac{1}{r_x} \right) \end{aligned} \quad (b)$$

These stresses are proportional to the distance z of the lamina $abcd$ from the neutral surface and depend on the magnitude of the curvatures of the bent plate.

The normal stresses distributed over the lateral sides of the element in Fig. 20 can be reduced to couples, the magnitudes of which per unit length evidently must be equal to the external moments M_x and M_y . In this way we obtain the equations

$$\begin{aligned} \int_{-h/2}^{h/2} \sigma_x z \, dy \, dz &= M_x \, dy \\ \int_{-h/2}^{h/2} \sigma_y z \, dx \, dz &= M_y \, dx \end{aligned} \quad (c)$$

¹ It will be shown in Art. 13 that this conclusion is accurate enough if the deflections of the plate are small in comparison with the thickness h .